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About Supergraphs. Part III

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Summary. The previous articles [5] and [6] introduced formalizations of the step-by-step operations we use to construct finite graphs by hand. That implicitly showed that any finite graph can be constructed from the trivial edgeless graph K_1 by applying a finite sequence of these basic operations. In this article that claim is proven explicitly with Mizar[4].

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0. Introduction

In the literature a mutual understanding how the graphical representation of graphs is to be translated into a description fitting the set-theoretic definition is usually assumed (cf. [9], [3], [8], [2]), but in Mizar we need explicit operations, which were provided in [5] and [6].

The rather extensive preliminaries contain many theorems that would fit well into earlier articles of the GLIB series, for example:

- In a simple graph, the degree of a vertex equals the cardinality of its neighbors.
- The operations of removing a vertex or an edge in a graph commute.
- Every finite connected graph has a spanning tree.

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- Endvertices are no cut vertices.

Graphs without edges are rigorously introduced in the following section. Wilson calls those *null graphs* ([9]). Bondy and Murty call them *empty graphs* ([3]), while naming the graph without vertices *the null graph*. Both notations are common in the literature. To avoid confusion those graphs are simply introduced as `edgeless` here.

To describe the construction of finite graphs starting from the trivial edgeless K_1, finite sequences yielding graphs are needed, which are introduced in the next section expanding the notation from [7], [1].

The last section contains the formalizations of the main results:

- Adding n vertices to a graph can be done by adding one vertex after another.
- Any finite edgeless graph can be constructed from K_1 by adding one vertex at a time.
- Any finite (connected) graph can be reconstructed from a spanning (connected) subgraph by adding one edge at a time.
- Any finite graph can be constructed from K_1 by adding one vertex or one edge at a time.
- Any finite tree can be constructed from K_1 by adding one vertex and an edge incident with that vertex at a time.
- Any finite connected graph can be constructed from K_1 by adding one edge or one vertex and an edge incident with that vertex at a time.
- Adding a vertex to a graph and connecting it to a (possibly empty) subset of the vertices of said graph can be done by first adding the new vertex and then adding one edge at a time.
- Any finite simple graph can be constructed from K_1 by adding one vertex connecting it to a (possibly empty) subset of the vertices of the previous contruction step at a time.
- If the finite simple graph is also connected, the subset of adjacent vertices can be guarantied to be non empty.

The number of operations needed is given for each process in terms of order and size of the involved graphs. Some proof schemes are presented to make use of these constructions.

1. Preliminaries

Let G be a graph and v be a vertex of G. Let us observe that every subgraph of G induced by $\{v\}$ is trivial.

Let us consider a graph G, a set X, and a vertex v of G. Now we state the propositions:

(1) G.edgesBetween$(X \setminus \{v\}) = G$.edgesBetween$(X) \setminus v$.edgesInOut().

(2) If v is isolated, then G.edgesBetween$(X \setminus \{v\}) = G$.edgesBetween(X). The theorem is a consequence of (1).

Let us consider a non-directed-multi graph G and a vertex v of G. Now we state the propositions:

(3) v.inDegree() $= \overline{\overline{v\text{.inNeighbors()}}}$.

Proof: Define $\mathcal{P}$[object, object] $\equiv$ $\$_2$ joins $\$_1$ to v in G. Consider f being a function such that $\operatorname{dom} f = v$.inNeighbors() and for every object x such that $x \in v$.inNeighbors() holds $\mathcal{P}[x, f(x)]$. f is a bijection between v.inNeighbors() and v.edgesIn(). □

(4) v.outDegree() $= \overline{\overline{v\text{.outNeighbors()}}}$.

Proof: Define $\mathcal{P}$[object, object] $\equiv$ $\$_2$ joins v to $\$_1$ in G. Consider f being a function such that $\operatorname{dom} f = v$.outNeighbors() and for every object x such that $x \in v$.outNeighbors() holds $\mathcal{P}[x, f(x)]$. f is a bijection between v.outNeighbors() and v.edgesOut(). □

(5) Let us consider a simple graph G, and a vertex v of G. Then v.degree() $= \overline{\overline{v\text{.allNeighbors()}}}$.

Proof: v.inNeighbors() $\cap$ v.outNeighbors() $= \emptyset$. □

(6) Let us consider a graph G. Then G is loopless if and only if for every vertex v of G, $v \notin v$.allNeighbors().

Proof: For every object v, there exists no object e such that e joins v and v in G. □

(7) Let us consider a graph G, and a vertex v of G. Then v is isolated if and only if v.allNeighbors() $= \emptyset$.

(8) Let us consider a graph G_1, a set v, and a subgraph G_2 of G_1 with vertex v removed. Suppose G_1 is trivial or $v \notin$ the vertices of G_1. Then $G_1 \approx G_2$.

(9) Let us consider graphs G_1, G_2, and a set v. Suppose $G_1 \approx G_2$ and (G_1 is trivial or $v \notin$ the vertices of G_1). Then G_2 is a subgraph of G_1 with vertex v removed.

(10) Let us consider a graph G. Suppose there exist vertices v_1, v_2 of G such that $v_1 \neq v_2$. Then G is not trivial.

Proof: $\overline{\overline{\alpha}} \neq 1$, where α is the vertices of G. □

Let G be a non trivial graph and X be a set. Let us note that every subgraph of G with edges X removed is non trivial. Now we state the propositions:

(11) Let us consider a finite graph G_1, and a subgraph G_2 of G_1. Then G_2 is spanning if and only if G_1.order() = G_2.order().

(12) Let us consider a graph G_1, and a spanning subgraph G_2 of G_1. Suppose the edges of G_1 = the edges of G_2. Then $G_1 \approx G_2$.

(13) Let us consider a finite graph G_1, and a spanning subgraph G_2 of G_1. If G_1.size() = G_2.size(), then $G_1 \approx G_2$. The theorem is a consequence of (12).

(14) Let us consider a graph G_1, a set V, and a subgraph G_2 of G_1 induced by V. If G_2 is spanning, then $G_1 \approx G_2$.

Let us consider a graph G. Now we state the propositions:

(15) G is not trivial if and only if there exists a subgraph H of G such that H is not spanning.

(16) If there exists a vertex v of G such that v is endvertex, then G is not trivial.

PROOF: Consider e being an object such that v.edgesInOut() $= \{e\}$ and e does not join v and v in G. For every vertex u of G, the vertices of $G \neq \{u\}$. □

(17) Let us consider a graph G_1, sets v, e, a subgraph G_2 of G_1 with vertex v removed, and a subgraph G_3 of G_1 with edge e removed. Then every subgraph of G_2 with edge e removed is a subgraph of G_3 with vertex v removed. The theorem is a consequence of (1), (8), and (9).

(18) Let us consider a graph G_1, sets v, e, a subgraph G_2 of G_1 with edge e removed, and a subgraph G_3 of G_1 with vertex v removed. Then every subgraph of G_2 with vertex v removed is a subgraph of G_3 with edge e removed. The theorem is a consequence of (1) and (8).

Let G be a finite, connected graph. Note that there exists a subgraph of G which is spanning, tree-like, connected, and acyclic.

Now we state the propositions:

(19) Let us consider a connected graph G_1, and a subgraph G_2 of G_1. Suppose the edges of $G_1 \subseteq$ the edges of G_2. Then $G_1 \approx G_2$.

PROOF: The vertices of G_1 = the vertices of G_2. □

(20) Let us consider a finite, connected graph G_1, and a subgraph G_2 of G_1. If G_1.size() = G_2.size(), then $G_1 \approx G_2$. The theorem is a consequence of (19).

(21) Let us consider a finite, tree-like graph G_1, and a spanning, tree-like subgraph G_2 of G_1. Then $G_1 \approx G_2$. The theorem is a consequence of (11)

and (13).

Let G be a non trivial graph. Observe that there exists a subgraph of G which is non spanning, trivial, and connected.

Now we state the propositions:

(22) Let us consider a graph G, and vertices v_1, v_2 of G. Suppose $v_1 \notin G$.reachableFrom(v_2).
Then G.reachableFrom(v_1) misses G.reachableFrom(v_2).

(23) Let us consider a graph G. Then G.componentSet() is a partition of the vertices of G.
Proof: Set $V =$ the vertices of G. For every subset A of V such that $A \in G$.componentSet() holds $A \neq \emptyset$ and for every subset B of V such that $B \in G$.componentSet() holds $A = B$ or A misses B. □

(24) Let us consider a graph G, a partition C of the vertices of G, and a vertex v of G. If $C = G$.componentSet(),
then EqClass$(v, C) = G$.reachableFrom(v).

(25) Let us consider a graph G_1, vertices v_0, v_1 of G_1, a subgraph G_2 of G_1 with vertex v_0 removed, and a vertex v_2 of G_2. Suppose v_0 is endvertex and $v_1 = v_2$ and $v_1 \in G_1$.reachableFrom(v_0). Then G_2.reachableFrom$(v_2) =$ $(G_1$.reachableFrom$(v_1)) \setminus \{v_0\}$.
Proof: G_1 is not trivial. For every object w, $w \in G_2$.reachableFrom(v_2) iff $w \in G_1$.reachableFrom(v_1) and $w \notin \{v_0\}$. □

(26) Let us consider a non trivial graph G_1, vertices v_0, v_1 of G_1, a subgraph G_2 of G_1 with vertex v_0 removed, and a vertex v_2 of G_2. Suppose $v_1 = v_2$ and $v_1 \notin G_1$.reachableFrom(v_0). Then G_2.reachableFrom$(v_2) =$ G_1.reachableFrom(v_1).
Proof: For every object w such that $w \in G_1$.reachableFrom(v_1) holds $w \in G_2$.reachableFrom(v_2). □

(27) Let us consider a non trivial, finite, tree-like graph G, and a vertex v of G. If G.order() $= 2$, then v is endvertex.

Let G be a non trivial, connected graph and v be a vertex of G. Observe that v.allNeighbors() is non empty.

Now we state the propositions:

(28) Let us consider a tree T, and a vertex a of T. Then T.pathBetween$(a, a) =$ T.walkOf(a).

(29) Let us consider a tree T, vertices a, b of T, and an object e. If e joins a and b in T, then T.pathBetween$(a, b) = T$.walkOf(a, e, b).

(30) Let us consider a non trivial, finite tree T, and a vertex v of T. Then there exist vertices v_1, v_2 of T such that

(i) $v_1 \neq v_2$, and

(ii) v_1 is endvertex, and

(iii) v_2 is endvertex, and

(iv) $v \in (T.\text{pathBetween}(v_1, v_2)).\text{vertices}()$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every non trivial, finite tree T for every vertex v of T such that $T.\text{order}() = \$_1 + 2$ there exist vertices v_1, v_2 of T such that $v_1 \neq v_2$ and v_1 is endvertex and v_2 is endvertex and $v \in (T.\text{pathBetween}(v_1, v_2)).\text{vertices}()$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. Consider k being a natural number such that $T.\text{order}() = 2 + k$. $\square$

(31) Let us consider a non trivial, finite, tree-like graph G_1, and a non spanning, connected subgraph G_2 of G_1. Then there exists a vertex v of G_1 such that

(i) v is endvertex, and

(ii) $v \notin$ the vertices of G_2.

The theorem is a consequence of (30).

(32) Let us consider graphs G_2, G_3, a set V, and a supergraph G_1 of G_2 extended by the vertices from V. Suppose $G_2 \approx G_3$. Then G_1 is a supergraph of G_3 extended by the vertices from V.

(33) Let us consider a graph G_2, and a supergraph G_1 of G_2. Suppose the edges of $G_1 =$ the edges of G_2. Then G_1 is a supergraph of G_2 extended by the vertices from (the vertices of G_1) $\setminus$ (the vertices of G_2).

(34) Let us consider a finite graph G_1, and a subgraph G_2 of G_1. Suppose $G_1.\text{size}() = G_2.\text{size}()$. Then G_1 is a supergraph of G_2 extended by the vertices from (the vertices of G_1) $\setminus$ (the vertices of G_2). The theorem is a consequence of (33).

(35) Let us consider a non trivial graph G_1, a vertex v of G_1, and a subgraph G_2 of G_1 with vertex v removed. If v is isolated, then G_1 is a supergraph of G_2 extended by v. The theorem is a consequence of (2).

(36) Let us consider graphs G_2, G_3, objects v_1, e, v_2, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Suppose $G_2 \approx G_3$. Then G_1 is a supergraph of G_3 extended by e between vertices v_1 and v_2.

(37) Let us consider a graph G_1, a set e, and a subgraph G_2 of G_1 with edge e removed. Suppose $e \in$ the edges of G_1. Then G_1 is a supergraph of G_2 extended by e between vertices (the source of G_1)(e) and (the target of G_1)(e).

PROOF: Set $u =$ (the source of G_1)(e). Set $w =$ (the target of G_1)(e). For every object e_0 such that $e_0 \in \text{dom}$(the source of G_1) holds (the source of

$G_1)(e_0) = ((\text{the source of } G_2)+\cdot(e \dot{\longmapsto} u))(e_0)$. For every object e_0 such that $e_0 \in \text{dom}(\text{the target of } G_1)$ holds $(\text{the target of } G_1)(e_0) = ((\text{the target of } G_2)+\cdot(e \dot{\longmapsto} w))(e_0)$. $\square$

(38) Let us consider a non trivial graph G_1, a vertex v of G_1, an object e, and a subgraph G_2 of G_1 with vertex v removed. Suppose $\{e\} = v.\text{edgesInOut}()$ and e does not join v and v in G_1. Then G_1 is supergraph of G_2 extended by $v.\text{adj}(e)$, v and e between them or supergraph of G_2 extended by v, $v.\text{adj}(e)$ and e between them. The theorem is a consequence of (1).

(39) Let us consider a graph G_2, vertices v_1, v_2 of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2, a vertex w of G_1, and a vertex v of G_2. Suppose $v_2 \in G_2.\text{reachableFrom}(v_1)$ and $v = w$. Then $G_1.\text{reachableFrom}(w) = G_2.\text{reachableFrom}(v)$.

(40) Let us consider a graph G_2, vertices v_1, v_2 of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Suppose $v_2 \in G_2.\text{reachableFrom}(v_1)$. Then $G_1.\text{componentSet}() = G_2.\text{componentSet}()$. The theorem is a consequence of (39).

(41) Let us consider a graph G_2, vertices v_1, v_2 of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2, and vertices w_1, w_2 of G_1. Suppose $e \notin$ the edges of G_2 and $w_1 = v_1$ and $w_2 = v_2$. Then $w_2 \in G_1.\text{reachableFrom}(w_1)$.

(42) Let us consider a graph G_2, vertices v_1, v_2 of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2, and a vertex w_1 of G_1. Suppose $e \notin$ the edges of G_2 and $w_1 = v_1$. Then $G_1.\text{reachableFrom}(w_1) = (G_2.\text{reachableFrom}(v_1)) \cup (G_2.\text{reachableFrom}(v_2))$.

Proof: For every object x such that $x \in G_1.\text{reachableFrom}(w_1)$ holds $x \in (G_2.\text{reachableFrom}(v_1)) \cup (G_2.\text{reachableFrom}(v_2))$. $G_2.\text{reachableFrom}(v_2) \subseteq G_1.\text{reachableFrom}(w_1)$. $\square$

(43) Let us consider a graph G_2, vertices v_1, v_2 of G_2, an object e, a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2, a vertex w of G_1, and a vertex v of G_2. Suppose $e \notin$ the edges of G_2 and $v = w$ and $v \notin G_2.\text{reachableFrom}(v_1)$ and $v \notin G_2.\text{reachableFrom}(v_2)$. Then $G_1.\text{reachableFrom}(w) = G_2.\text{reachableFrom}(v)$.

Proof: For every object x such that $x \in G_1.\text{reachableFrom}(w)$ holds $x \in G_2.\text{reachableFrom}(v)$. $\square$

(44) Let us consider a graph G_2, vertices v_1, v_2 of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Suppose $e \notin$ the edges of G_2. Then $G_1.\text{componentSet}() = (G_2.\text{componentSet}() \setminus \{G_2.\text{reachableFrom}(v_1), G_2.\text{reachableFrom}(v_2)\}) \cup \{(G_2.\text{reachableFrom}(v_1)) \cup (G_2.\text{reachableFrom}(v_2))\}$.

(45) Let us consider a graph G_1, a vertex v of G_1, and a subgraph G_2 of G_1 with vertex v removed. If v is endvertex, then G_1.numComponents() = G_2.numComponents().

PROOF: G_1 is not trivial. There exists a function f such that f is one-to-one and $\operatorname{dom} f = G_1$.componentSet() and $\operatorname{rng} f = G_2$.componentSet(). □

Let G be a graph. One can check that every vertex of G which is endvertex is also non cut-vertex. Now we state the propositions:

(46) Let us consider a non trivial, finite, connected graph G_1, and a non spanning, connected subgraph G_2 of G_1. Then there exists a vertex v of G_1 such that

(i) v is not cut-vertex, and

(ii) $v \notin$ the vertices of G_2.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every non trivial, finite, connected graph G_1 for every non spanning, connected subgraph G_2 of G_1 such that G_1.order() + $\$_1$ = G_1.size() + 1 there exists a vertex v of G_1 such that v is not cut-vertex and $v \notin$ the vertices of G_2. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(47) Let us consider a non trivial, simple graph G_1, a vertex v of G_1, and a subgraph G_2 of G_1 with vertex v removed. Then G_1 is a supergraph of G_2 extended by vertex v and edges between v and v.allNeighbors() of G_2.

2. Edgeless and Non Edgeless Graphs

Let G be a graph. We say that G is edgeless if and only if

(Def. 1) the edges of $G = \emptyset$.

Let us consider a graph G. Now we state the propositions:

(48) G is edgeless if and only if $\overline{\overline{\alpha}} = 0$, where α is the edges of G.

(49) G is edgeless if and only if G.size() = 0.

Let G be a graph. Observe that every subgraph of G with edges the edges of G removed is edgeless and there exists a graph which is edgeless and there exists a subgraph of G which is edgeless and spanning and there exists a subgraph of G which is edgeless and trivial.

Let G be an edgeless graph. One can check that the edges of G is empty and every graph which is edgeless is also non-multi, non-directed-multi, loopless, simple, and directed-simple and every graph which is trivial and loopless is also edgeless.

Let V be a non empty set and S, T be functions from $\emptyset$ into V. One can check that createGraph$(V, \emptyset, S, T)$ is edgeless.

Now we state the propositions:

(50) Let us consider an edgeless graph G, and objects e, v_1, v_2. Then

(i) e does not join v_1 and v_2 in G, and

(ii) e does not join v_1 to v_2 in G.

(51) Let us consider an edgeless graph G, an object e, and sets X, Y. Then

(i) e does not join a vertex from X and a vertex from Y in G, and

(ii) e does not join a vertex from X to a vertex from Y in G.

(52) Let us consider graphs G_1, G_2. If $G_1 \approx G_2$, then if G_1 is edgeless, then G_2 is edgeless.

Let G be an edgeless graph. Let us observe that every walk of G is trivial and every subgraph of G is edgeless.

Let X be a set. Note that G.edgesInto(X) is empty and G.edgesOutOf(X) is empty and G.edgesInOut(X) is empty and G.edgesBetween(X) is empty and G.set(WeightSelector, X) is edgeless and G.set(ELabelSelector, X) is edgeless and G.set(VLabelSelector, X) is edgeless and every supergraph of G extended by the vertices from X is edgeless and every graph given by reversing directions of the edges X of G is edgeless.

Let Y be a set. Let us note that G.edgesBetween(X, Y) is empty and G.edgesDBetween(X, Y) is empty and every graph which is edgeless is also acyclic and chordal and every graph which is trivial and edgeless is also tree-like and every graph which is non trivial and edgeless is also non connected, non tree-like, and non complete and every graph which is connected and edgeless is also trivial.

Now we state the propositions:

(53) Let us consider an edgeless graph G_1, and a subgraph G_2 of G_1. Then G_1 is a supergraph of G_2 extended by the vertices from (the vertices of G_1) $\setminus$ (the vertices of G_2). The theorem is a consequence of (33).

(54) Let us consider a graph G_2, vertices v_1, v_2 of G_2, an object e, and a supergraph G_1 of G_2 extended by e between vertices v_1 and v_2. Suppose $e \notin$ the edges of G_2. Then G_1 is not edgeless.

(55) Let us consider a graph G_2, a vertex v_1 of G_2, objects e, v_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose $v_2 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2. Then G_1 is not edgeless.

(56) Let us consider a graph G_2, objects v_1, e, a vertex v_2 of G_2, and a supergraph G_1 of G_2 extended by v_1, v_2 and e between them. Suppose

$v_1 \notin$ the vertices of G_2 and $e \notin$ the edges of G_2. Then G_1 is not edgeless.

(57) Let us consider a graph G_2, an object v, a non empty subset V of the vertices of G_2, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $v \notin$ the vertices of G_2. Then G_1 is not edgeless.

Let G be a graph. Let us observe that every supergraph of G extended by vertex the vertices of G and edges from the vertices of G to the vertices of G is non edgeless and every supergraph of G extended by vertex the vertices of G and edges from the vertices of G to the vertices of G is non edgeless and every supergraph of G extended by vertex the vertices of G and edges between the vertices of G and the vertices of G is non edgeless.

Let v be a vertex of G. Let us note that every supergraph of G extended by v, the vertices of G and the edges of G between them is non edgeless and every supergraph of G extended by the vertices of G, v and the edges of G between them is non edgeless.

Let w be a vertex of G. Let us note that every supergraph of G extended by the edges of G between vertices v and w is non edgeless.

Let G be an edgeless graph. Note that every component of G is trivial and v.edgesIn() is empty and v.edgesOut() is empty and v.edgesInOut() is empty and every vertex of G is isolated, non cut-vertex, and non endvertex and v.inDegree() is empty and v.outDegree() is empty and v.inNeighbors() is empty and v.outNeighbors() is empty and v.degree() is empty and v.allNeighbors() is empty and there exists a graph which is trivial, finite, and edgeless and there exists a graph which is non trivial, finite, and edgeless and there exists a graph which is trivial, finite, and non edgeless and there exists a graph which is non trivial, finite, and non edgeless.

Let G be a non edgeless graph. One can check that the edges of G is non empty and every supergraph of G is non edgeless.

Let X be a set. One can verify that every graph given by reversing directions of the edges X of G is non edgeless and $G.\mathrm{set}(\mathrm{WeightSelector}, X)$ is non edgeless and $G.\mathrm{set}(\mathrm{ELabelSelector}, X)$ is non edgeless and $G.\mathrm{set}(\mathrm{VLabelSelector}, X)$ is non edgeless.

An edge of G is an element of the edges of G. Now we state the proposition:

(58) Let us consider a finite, edgeless graph G_1, and a subgraph G_2 of G_1. If $G_1.\mathrm{order}() = G_2.\mathrm{order}()$, then $G_1 \approx G_2$.

Let F be a graph-yielding function. We say that F is edgeless if and only if

(Def. 2) for every object x such that $x \in \mathrm{dom}\, F$ there exists a graph G such that $F(x) = G$ and G is edgeless.

Let F be a non empty, graph-yielding function. Note that F is edgeless if and only if the condition (Def. 3) is satisfied.

(Def. 3) for every element x of $\operatorname{dom} F$, $F(x)$ is edgeless.

Let S be a graph sequence. Let us note that S is edgeless if and only if the condition (Def. 4) is satisfied.

(Def. 4) for every natural number n, $S(n)$ is edgeless.

Let us observe that every graph-yielding function which is trivial and loopless is also edgeless and every graph-yielding function which is edgeless is also non-multi, non-directed-multi, loopless, simple, directed-simple, and acyclic.

Let F be an edgeless, non empty, graph-yielding function and x be an element of $\operatorname{dom} F$. Observe that $F(x)$ is edgeless.

Let S be an edgeless graph sequence and x be a natural number. Observe that $S(x)$ is edgeless.

3. Finite Graph Sequences

Let G be a graph. Note that $\langle G \rangle$ is graph-yielding.

Let G be a finite graph. Let us note that $\langle G \rangle$ is finite.

Let G be a loopless graph. Observe that $\langle G \rangle$ is loopless.

Let G be a trivial graph. Let us observe that $\langle G \rangle$ is trivial.

Let G be a non trivial graph. Let us observe that $\langle G \rangle$ is nontrivial.

Let G be a non-multi graph. One can verify that $\langle G \rangle$ is non-multi.

Let G be a non-directed-multi graph. One can check that $\langle G \rangle$ is non-directed-multi.

Let G be a simple graph. Note that $\langle G \rangle$ is simple.

Let G be a directed-simple graph. Let us note that $\langle G \rangle$ is directed-simple.

Let G be a connected graph. Observe that $\langle G \rangle$ is connected.

Let G be an acyclic graph. Let us observe that $\langle G \rangle$ is acyclic.

Let G be a tree-like graph. One can verify that $\langle G \rangle$ is tree-like.

Let G be an edgeless graph. One can check that $\langle G \rangle$ is edgeless and there exists a finite sequence which is empty and graph-yielding and there exists a finite sequence which is non empty and graph-yielding.

Let p be a non empty, graph-yielding finite sequence. Note that $p(1)$ is function-like and relation-like and $p(\operatorname{len} p)$ is function-like and relation-like and $p(1)$ is finite and $\mathbb{N}$-defined and $p(\operatorname{len} p)$ is finite and $\mathbb{N}$-defined and $p(1)$ is graph-like and $p(\operatorname{len} p)$ is graph-like and there exists a graph-yielding finite sequence which is non empty, finite, loopless, trivial, non-multi, non-directed-multi, simple, directed-simple, connected, acyclic, tree-like, and edgeless and there exists a graph-yielding finite sequence which is non empty, finite, loopless, nontrivial, non-multi, non-directed-multi, simple, directed-simple, connected, acyclic, and tree-like.

Let p be a graph-yielding finite sequence and n be a natural number. Let us observe that $p{\restriction}n$ is graph-yielding and $p_{\lfloor n}$ is graph-yielding.

Let m be a natural number. Note that $\mathrm{smid}(p, m, n)$ is graph-yielding and $\langle p(m), \ldots, p(n)\rangle$ is graph-yielding.

Let p be a finite, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is finite and $p_{\lfloor n}$ is finite and $\mathrm{smid}(p, m, n)$ is finite and $\langle p(m), \ldots, p(n)\rangle$ is finite.

Let p be a loopless, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is loopless and $p_{\lfloor n}$ is loopless and $\mathrm{smid}(p, m, n)$ is loopless and $\langle p(m), \ldots, p(n)\rangle$ is loopless.

Let p be a trivial, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is trivial and $p_{\lfloor n}$ is trivial and $\mathrm{smid}(p, m, n)$ is trivial and $\langle p(m), \ldots, p(n)\rangle$ is trivial.

Let p be a nontrivial, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is nontrivial and $p_{\lfloor n}$ is nontrivial and $\mathrm{smid}(p, m, n)$ is nontrivial and $\langle p(m), \ldots, p(n)\rangle$ is nontrivial.

Let p be a non-multi, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is non-multi and $p_{\lfloor n}$ is non-multi and $\mathrm{smid}(p, m, n)$ is non-multi and $\langle p(m), \ldots, p(n)\rangle$ is non-multi.

Let p be a non-directed-multi, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is non-directed-multi and $p_{\lfloor n}$ is non-directed-multi and $\mathrm{smid}(p, m, n)$ is non-directed-multi and $\langle p(m), \ldots, p(n)\rangle$ is non-directed-multi.

Let p be a simple, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is simple and $p_{\lfloor n}$ is simple and $\mathrm{smid}(p, m, n)$ is simple and $\langle p(m), \ldots, p(n)\rangle$ is simple.

Let p be a directed-simple, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is directed-simple and $p_{\lfloor n}$ is directed-simple and $\mathrm{smid}(p, m, n)$ is directed-simple and $\langle p(m), \ldots, p(n)\rangle$ is directed-simple.

Let p be a connected, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is connected and $p_{\lfloor n}$ is connected and $\mathrm{smid}(p, m, n)$ is connected and $\langle p(m), \ldots, p(n)\rangle$ is connected.

Let p be an acyclic, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is acyclic and $p_{\lfloor n}$ is acyclic and $\mathrm{smid}(p, m, n)$ is acyclic and $\langle p(m), \ldots, p(n)\rangle$ is acyclic.

Let p be a tree-like, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is tree-like and $p_{\lfloor n}$ is tree-like and $\mathrm{smid}(p, m, n)$ is tree-like and $\langle p(m), \ldots, p(n)\rangle$ is tree-like.

Let p be an edgeless, graph-yielding finite sequence. One can verify that $p{\restriction}n$ is edgeless and $p_{\lfloor n}$ is edgeless and $\mathrm{smid}(p, m, n)$ is edgeless and $\langle p(m), \ldots, p(n)\rangle$

is edgeless.

Let p, q be graph-yielding finite sequences. Let us note that $p \frown q$ is graph-yielding and $p \frown\!\frown q$ is graph-yielding.

Let p, q be finite, graph-yielding finite sequences. Let us observe that $p \frown q$ is finite and $p \frown\!\frown q$ is finite.

Let p, q be loopless, graph-yielding finite sequences. Let us observe that $p \frown q$ is loopless and $p \frown\!\frown q$ is loopless.

Let p, q be trivial, graph-yielding finite sequences. Let us observe that $p \frown q$ is trivial and $p \frown\!\frown q$ is trivial.

Let p, q be nontrivial, graph-yielding finite sequences. Let us observe that $p \frown q$ is nontrivial and $p \frown\!\frown q$ is nontrivial.

Let p, q be non-multi, graph-yielding finite sequences. Observe that $p \frown q$ is non-multi and $p \frown\!\frown q$ is non-multi.

Let p, q be non-directed-multi, graph-yielding finite sequences. Observe that $p \frown q$ is non-directed-multi and $p \frown\!\frown q$ is non-directed-multi.

Let p, q be simple, graph-yielding finite sequences. Observe that $p \frown q$ is simple and $p \frown\!\frown q$ is simple.

Let p, q be directed-simple, graph-yielding finite sequences. One can verify that $p \frown q$ is directed-simple and $p \frown\!\frown q$ is directed-simple.

Let p, q be connected, graph-yielding finite sequences. Note that $p \frown q$ is connected and $p \frown\!\frown q$ is connected.

Let p, q be acyclic, graph-yielding finite sequences. Note that $p \frown q$ is acyclic and $p \frown\!\frown q$ is acyclic.

Let p, q be tree-like, graph-yielding finite sequences. Note that $p \frown q$ is tree-like and $p \frown\!\frown q$ is tree-like.

Let p, q be edgeless, graph-yielding finite sequences. Observe that $p \frown q$ is edgeless and $p \frown\!\frown q$ is edgeless.

Let G_1, G_2 be graphs. Note that $\langle G_1, G_2 \rangle$ is graph-yielding.

Let G_3 be a graph. Let us note that $\langle G_1, G_2, G_3 \rangle$ is graph-yielding.

Let G_1, G_2 be finite graphs. Let us observe that $\langle G_1, G_2 \rangle$ is finite.

Let G_3 be a finite graph. One can verify that $\langle G_1, G_2, G_3 \rangle$ is finite.

Let G_1, G_2 be loopless graphs. Note that $\langle G_1, G_2 \rangle$ is loopless.

Let G_3 be a loopless graph. Let us note that $\langle G_1, G_2, G_3 \rangle$ is loopless.

Let G_1, G_2 be trivial graphs. Let us observe that $\langle G_1, G_2 \rangle$ is trivial.

Let G_3 be a trivial graph. One can verify that $\langle G_1, G_2, G_3 \rangle$ is trivial.

Let G_1, G_2 be non trivial graphs. One can check that $\langle G_1, G_2 \rangle$ is nontrivial.

Let G_3 be a non trivial graph. One can check that $\langle G_1, G_2, G_3 \rangle$ is nontrivial.

Let G_1, G_2 be non-multi graphs. Let us note that $\langle G_1, G_2 \rangle$ is non-multi.

Let G_3 be a non-multi graph. Observe that $\langle G_1, G_2, G_3 \rangle$ is non-multi.

Let G_1, G_2 be non-directed-multi graphs. One can verify that $\langle G_1, G_2 \rangle$ is non-directed-multi.

Let G_3 be a non-directed-multi graph. One can check that $\langle G_1, G_2, G_3 \rangle$ is non-directed-multi.

Let G_1, G_2 be simple graphs. Let us note that $\langle G_1, G_2 \rangle$ is simple.

Let G_3 be a simple graph. Observe that $\langle G_1, G_2, G_3 \rangle$ is simple.

Let G_1, G_2 be directed-simple graphs. One can verify that $\langle G_1, G_2 \rangle$ is directed-simple.

Let G_3 be a directed-simple graph. One can check that $\langle G_1, G_2, G_3 \rangle$ is directed-simple.

Let G_1, G_2 be connected graphs. Let us note that $\langle G_1, G_2 \rangle$ is connected.

Let G_3 be a connected graph. Observe that $\langle G_1, G_2, G_3 \rangle$ is connected.

Let G_1, G_2 be acyclic graphs. One can verify that $\langle G_1, G_2 \rangle$ is acyclic.

Let G_3 be an acyclic graph. One can check that $\langle G_1, G_2, G_3 \rangle$ is acyclic.

Let G_1, G_2 be tree-like graphs. Let us note that $\langle G_1, G_2 \rangle$ is tree-like.

Let G_3 be a tree-like graph. Observe that $\langle G_1, G_2, G_3 \rangle$ is tree-like.

Let G_1, G_2 be edgeless graphs. One can verify that $\langle G_1, G_2 \rangle$ is edgeless.

Let G_3 be an edgeless graph. One can check that $\langle G_1, G_2, G_3 \rangle$ is edgeless.

4. Construction of Finite Graphs

Now we state the propositions:

(59) Let us consider a graph G_2, a finite set V, and a supergraph G_1 of G_2 extended by the vertices from V. Then there exists a non empty, graph-yielding finite sequence p such that

- (i) $p(1) \approx G_2$, and
- (ii) $p(\mathrm{len}\, p) = G_1$, and
- (iii) $\mathrm{len}\, p = \overline{\overline{V \setminus \alpha}} + 1$, and
- (iv) for every element n of $\mathrm{dom}\, p$ such that $n \leqslant \mathrm{len}\, p - 1$ there exists a vertex v of G_1 such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$,

where α is the vertices of G_2.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite set V for every supergraph G_1 of G_2 extended by the vertices from V such that $\overline{\overline{V \setminus (\text{the vertices of } G_2)}} = \$_1$ there exists a non empty, graph-yielding finite sequence p such that $p(1) \approx G_2$ and $p(\mathrm{len}\, p) = G_1$ and $\mathrm{len}\, p = \overline{\overline{V \setminus (\text{the vertices of } G_2)}} + 1$ and for every element n of $\mathrm{dom}\, p$ such that

$n \leqslant \operatorname{len} p - 1$ there exists a vertex v of G_1 such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$.

$\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(60) Let us consider a finite graph G, and a subgraph H of G. Suppose G.size() = H.size(). Then there exists a non empty, finite, graph-yielding finite sequence p such that

- (i) $p(1) \approx H$, and
- (ii) $p(\operatorname{len} p) = G$, and
- (iii) $\operatorname{len} p = G.\text{order}() - H.\text{order}() + 1$, and
- (iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists a vertex v of G such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$.

Proof: Set V = (the vertices of G)$\setminus$(the vertices of H). G is a supergraph of H extended by the vertices from V. Consider p being a non empty, graph-yielding finite sequence such that $p(1) \approx H$ and $p(\operatorname{len} p) = G$ and $\operatorname{len} p = \overline{\overline{V \setminus \alpha}} + 1$, where α is the vertices of H and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists a vertex v of G such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$.

Define $\mathcal{P}$[natural number] $\equiv$ for every element n of $\operatorname{dom} p$ such that $\$_1 = n$ holds $p(n)$ is finite. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every non zero natural number k, $\mathcal{P}[k]$. For every element x of $\operatorname{dom} p$, $p(x)$ is finite. □

(61) Let us consider a finite, edgeless graph G, and a subgraph H of G. Then there exists a non empty, finite, edgeless, graph-yielding finite sequence p such that

- (i) $p(1) \approx H$, and
- (ii) $p(\operatorname{len} p) = G$, and
- (iii) $\operatorname{len} p = G.\text{order}() - H.\text{order}() + 1$, and
- (iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists a vertex v of G such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$.

Proof: G.size() = 0. Consider p being a non empty, finite, graph-yielding finite sequence such that $p(1) \approx H$ and $p(\operatorname{len} p) = G$ and $\operatorname{len} p = G.\text{order}() - H.\text{order}() + 1$ and for every element n of $\operatorname{dom} p$ such that

$n \leqslant \operatorname{len} p - 1$ there exists a vertex v of G such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$. Define $\mathcal{P}$[natural number] $\equiv$ for every element n of $\operatorname{dom} p$ such that $\$_1 = n$ holds $p(n)$ is edgeless.

$\mathcal{P}[1]$. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every non zero natural number k, $\mathcal{P}[k]$. For every element x of $\operatorname{dom} p$, $p(x)$ is edgeless. □

(62) Let us consider a finite, edgeless graph G. Then there exists a non empty, finite, edgeless, graph-yielding finite sequence p such that

(i) $p(1)$ is trivial and edgeless, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G.\text{order}()$, and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists a vertex v of G such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$.

The theorem is a consequence of (61) and (52).

The scheme *FinEdgelessGraphs* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 1) For every finite, edgeless graph G, $\mathcal{P}[G]$

provided

- for every trivial, edgeless graph G, $\mathcal{P}[G]$ and
- for every finite, edgeless graph G_2 and for every object v and for every supergraph G_1 of G_2 extended by v such that $v \notin$ the vertices of G_2 and $\mathcal{P}[G_2]$ holds $\mathcal{P}[G_1]$.

Now we state the propositions:

(63) Let us consider a non empty, graph-yielding finite sequence p. Suppose $p(1)$ is edgeless and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists an object v such that $p(n+1)$ is a supergraph of $p(n)$ extended by v. Then $p(\operatorname{len} p)$ is edgeless.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every non empty, graph-yielding finite sequence p such that $\operatorname{len} p = \$_1$ and $p(1)$ is edgeless and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists an object v such that $p(n+1)$ is a supergraph of $p(n)$ extended by v holds $p(\operatorname{len} p)$ is edgeless.

For every non zero natural number m such that $\mathcal{P}[m]$ holds $\mathcal{P}[m+1]$. For every non zero natural number m, $\mathcal{P}[m]$. □

(64) Let us consider a finite graph G, and a spanning subgraph H of G. Then there exists a non empty, finite, graph-yielding finite sequence p such that

(i) $p(1) \approx H$, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G.\text{size}() - H.\text{size}() + 1$, and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by e between vertices v_1 and v_2 and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and v_1, $v_2 \in$ the vertices of $p(n)$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ for every spanning subgraph H of G such that $G.\text{size}() - H.\text{size}() = \$_1$ there exists a non empty, finite, graph-yielding finite sequence p such that $p(1) \approx H$ and $p(\operatorname{len} p) = G$ and $\operatorname{len} p = G.\text{size}() - H.\text{size}() + 1$ and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p-1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by e between vertices v_1 and v_2 and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and v_1, $v_2 \in$ the vertices of $p(n)$.

$\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(65) Let us consider a finite graph G. Then there exists a non empty, finite, graph-yielding finite sequence p such that

(i) $p(1)$ is edgeless, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G.\text{size}() + 1$, and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by e between vertices v_1 and v_2 and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and v_1, $v_2 \in$ the vertices of $p(n)$.

The theorem is a consequence of (64), (52), and (49).

(66) Let us consider a finite, connected graph G, and a spanning, connected subgraph H of G. Then there exists a non empty, finite, connected, graph-yielding finite sequence p such that

(i) $p(1) \approx H$, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G.\text{size}() - H.\text{size}() + 1$, and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by e between vertices v_1 and v_2 and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and v_1, $v_2 \in$ the vertices of $p(n)$.

PROOF: Consider p being a non empty, finite, graph-yielding finite sequence such that $p(1) \approx H$ and $p(\operatorname{len} p) = G$ and $\operatorname{len} p = G.\text{size}() - H.\text{size}() + 1$ and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by e between vertices v_1 and v_2 and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and v_1, $v_2 \in$ the vertices of $p(n)$.

Define $\mathcal{P}$[natural number] $\equiv$ for every element n of $\operatorname{dom} p$ such that $\$_1 = n$ holds $p(n)$ is connected. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every non zero natural number k, $\mathcal{P}[k]$. For every element x of $\operatorname{dom} p$, $p(x)$ is connected. □

(67) Let us consider a finite graph G_1, and a subgraph H of G_1. Then there exists a spanning subgraph G_2 of G_1 and there exists a non empty, finite, graph-yielding finite sequence p such that $H.\text{size}() = G_2.\text{size}()$ and $p(1) \approx H$ and $p(\operatorname{len} p) = G_2$ and $\operatorname{len} p = G_1.\text{order}() - H.\text{order}() + 1$ and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists a vertex v of G_1 such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$.

PROOF: Set $V =$ (the vertices of G_1) $\setminus$ (the vertices of H). Set $G_2 =$ the supergraph of H extended by the vertices from V. Consider p being a non empty, graph-yielding finite sequence such that $p(1) \approx H$ and $p(\operatorname{len} p) = G_2$ and $\operatorname{len} p = \overline{\overline{V \setminus \alpha}} + 1$, where α is the vertices of H and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists a vertex v of G_2 such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$.

Define $\mathcal{P}$[natural number] $\equiv$ for every element n of $\operatorname{dom} p$ such that $\$_1 = n$ holds $p(n)$ is finite. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every non zero natural number k, $\mathcal{P}[k]$. For every element x of $\operatorname{dom} p$, $p(x)$ is finite. G_2 is a subgraph of G_1. Consider v being a vertex of G_2 such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$. □

(68) Let us consider a finite graph G, and a subgraph H of G. Then there exists a non empty, finite, graph-yielding finite sequence p such that

(i) $p(1) \approx H$, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\text{len}\, p = G.\text{order}() + G.\text{size}() - (H.\text{order}() + H.\text{size}()) + 1$, and

(iv) for every element n of $\text{dom}\, p$ such that $n \leqslant \text{len}\, p - 1$ holds there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by e between vertices v_1 and v_2 and $e \in (\text{the edges of } G) \setminus (\text{the edges of } p(n))$ and $v_1, v_2 \in$ the vertices of $p(n)$ or there exists a vertex v of G such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$.

The theorem is a consequence of (67), (64), (36), and (60).

(69) Let us consider a finite graph G. Then there exists a non empty, finite, graph-yielding finite sequence p such that

(i) $p(1)$ is trivial and edgeless, and

(ii) $p(\text{len}\, p) = G$, and

(iii) $\text{len}\, p = G.\text{order}() + G.\text{size}()$, and

(iv) for every element n of $\text{dom}\, p$ such that $n \leqslant \text{len}\, p - 1$ holds there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by e between vertices v_1 and v_2 and $e \in (\text{the edges of } G) \setminus (\text{the edges of } p(n))$ and $v_1, v_2 \in$ the vertices of $p(n)$ or there exists a vertex v of G such that $p(n+1)$ is a supergraph of $p(n)$ extended by v and $v \notin$ the vertices of $p(n)$.

The theorem is a consequence of (68), (52), and (49).

The scheme *FinGraphs* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 2) For every finite graph G, $\mathcal{P}[G]$

provided

- for every trivial, edgeless graph G, $\mathcal{P}[G]$ and
- for every finite graph G_2 and for every object v and for every supergraph G_1 of G_2 extended by v such that $v \notin$ the vertices of G_2 and $\mathcal{P}[G_2]$ holds $\mathcal{P}[G_1]$ and
- for every finite graph G_2 and for every vertices v_1, v_2 of G_2 and for every object e and for every supergraph G_1 of G_2 extended by e between vertices v_1 and v_2 such that $e \notin$ the edges of G_2 and $\mathcal{P}[G_2]$ holds $\mathcal{P}[G_1]$.

Now we state the propositions:

(70) Let us consider a non empty, graph-yielding finite sequence p. Suppose $p(1)$ is finite and for every element n of $\text{dom}\, p$ such that $n \leqslant \text{len}\, p - 1$ holds there exists an object v such that $p(n+1)$ is a supergraph of $p(n)$ extended by v or there exist objects v_1, e, v_2 such that $p(n+1)$ is a supergraph of $p(n)$ extended by e between vertices v_1 and v_2. Then $p(\text{len}\, p)$ is finite.

PROOF: Define $\mathcal{Q}$[natural number] $\equiv$ if $\$_1 \leqslant \operatorname{len} p$, then there exists an element k of $\operatorname{dom} p$ such that $\$_1 = k$ and $p(k)$ is finite. $\mathcal{Q}[1]$. For every non zero natural number m such that $\mathcal{Q}[m]$ holds $\mathcal{Q}[m+1]$.

For every non zero natural number m, $\mathcal{Q}[m]$. Consider k being an element of $\operatorname{dom} p$ such that $\operatorname{len} p = k$ and $p(k)$ is finite. □

(71) Let us consider a finite, tree-like graph G, and a connected subgraph H of G. Then there exists a non empty, finite, tree-like, graph-yielding finite sequence p such that

(i) $p(1) \approx H$, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G.\text{order}() - H.\text{order}() + 1$, and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and ($v_1 \in$ the vertices of $p(n)$ and $v_2 \notin$ the vertices of $p(n)$ or $v_1 \notin$ the vertices of $p(n)$ and $v_2 \in$ the vertices of $p(n)$).

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite, tree-like graph G for every connected subgraph H of G such that $\$_1 = G.\text{order}() - H.\text{order}()$ there exists a non empty, finite, tree-like, graph-yielding finite sequence p such that $p(1) \approx H$ and $p(\operatorname{len} p) = G$ and $\operatorname{len} p = G.\text{order}() - H.\text{order}() + 1$ and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and ($v_1 \in$ the vertices of $p(n)$ and $v_2 \notin$ the vertices of $p(n)$ or $v_1 \notin$ the vertices of $p(n)$ and $v_2 \in$ the vertices of $p(n)$).

$\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(72) Let us consider a finite, tree-like graph G. Then there exists a non empty, finite, tree-like, graph-yielding finite sequence p such that

(i) $p(1)$ is trivial and edgeless, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G.\text{order}()$, and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and ($v_1 \in$ the vertices of

$p(n)$ and $v_2 \notin$ the vertices of $p(n)$ or $v_1 \notin$ the vertices of $p(n)$ and $v_2 \in$ the vertices of $p(n)$).

The theorem is a consequence of (71) and (52).

The scheme *FinTrees* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 3) For every finite, tree-like graph G, $\mathcal{P}[G]$

provided

- for every trivial, edgeless graph G, $\mathcal{P}[G]$ and
- for every finite, tree-like graph G_2 and for every vertex v of G_2 and for every objects e, w such that $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2 and $\mathcal{P}[G_2]$ holds for every supergraph G_1 of G_2 extended by v, w and e between them, $\mathcal{P}[G_1]$ and for every supergraph G_1 of G_2 extended by w, v and e between them, $\mathcal{P}[G_1]$.

Now we state the propositions:

(73) Let us consider a non empty, graph-yielding finite sequence p. Suppose $p(1)$ is tree-like and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist objects v_1, e, v_2 such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them. Then $p(\operatorname{len} p)$ is tree-like.

Proof: Define $\mathcal{Q}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} p$, then there exists an element k of $\operatorname{dom} p$ such that $\$_1 = k$ and $p(k)$ is tree-like. $\mathcal{Q}[1]$.

For every non zero natural number m such that $\mathcal{Q}[m]$ holds $\mathcal{Q}[m+1]$. For every non zero natural number m, $\mathcal{Q}[m]$. Consider k being an element of $\operatorname{dom} p$ such that $\operatorname{len} p = k$ and $p(k)$ is tree-like. □

(74) Let us consider a finite, connected graph G. Then there exists a non empty, finite, connected, graph-yielding finite sequence p such that

(i) $p(1)$ is trivial and edgeless, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G.\text{size}() + 1$, and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ holds there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by v_1, v_2 and e between them and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and ($v_1 \in$ the vertices of $p(n)$ and $v_2 \notin$ the vertices of $p(n)$ or $v_1 \notin$ the vertices of $p(n)$ and $v_2 \in$ the vertices of $p(n)$) or there exist vertices v_1, v_2 of G and there exists an object e such that $p(n+1)$ is a supergraph of $p(n)$ extended by e between vertices v_1 and v_2 and $e \in$ (the edges of G) $\setminus$ (the edges of $p(n)$) and $v_1, v_2 \in$ the vertices of $p(n)$.

The theorem is a consequence of (72), (66), and (36).

The scheme *FinConnectedGraphs* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 4) For every finite, connected graph G, $\mathcal{P}[G]$

provided

- for every trivial, edgeless graph G, $\mathcal{P}[G]$ and
- for every finite, connected graph G_2 and for every vertex v of G_2 and for every objects e, w such that $e \notin$ the edges of G_2 and $w \notin$ the vertices of G_2 and $\mathcal{P}[G_2]$ holds for every supergraph G_1 of G_2 extended by v, w and e between them, $\mathcal{P}[G_1]$ and for every supergraph G_1 of G_2 extended by w, v and e between them, $\mathcal{P}[G_1]$ and
- for every finite, connected graph G_2 and for every vertices v_1, v_2 of G_2 and for every object e and for every supergraph G_1 of G_2 extended by e between vertices v_1 and v_2 such that $e \notin$ the edges of G_2 and $\mathcal{P}[G_2]$ holds $\mathcal{P}[G_1]$.

Now we state the propositions:

(75) Let us consider a non empty, graph-yielding finite sequence p. Suppose $p(1)$ is connected and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exist objects v_1, e, v_2 such that $p(n+1)$ is supergraph of $p(n)$ extended by v_1, v_2 and e between them or supergraph of $p(n)$ extended by e between vertices v_1 and v_2. Then $p(\operatorname{len} p)$ is connected.

PROOF: Define $\mathcal{Q}$[natural number] $\equiv$ if $\$_1 \leqslant \operatorname{len} p$, then there exists an element k of $\operatorname{dom} p$ such that $\$_1 = k$ and $p(k)$ is connected. $\mathcal{Q}[1]$. For every non zero natural number m such that $\mathcal{Q}[m]$ holds $\mathcal{Q}[m+1]$. For every non zero natural number m, $\mathcal{Q}[m]$. □

(76) Let us consider a graph G_2, an object v, a set V_1, a finite set V_2, and a supergraph G_1 of G_2 extended by vertex v and edges between v and $V_1 \cup V_2$ of G_2. Suppose $V_1 \cup V_2 \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and V_1 misses V_2. Then there exists a non empty, graph-yielding finite sequence p such that

(i) $p(1) = G_2$, and

(ii) $p(\operatorname{len} p) = G_1$, and

(iii) $\operatorname{len} p = \overline{\overline{V_2}} + 2$, and

(iv) $p(2)$ is a supergraph of G_2 extended by vertex v and edges between v and V_1 of G_2, and

(v) for every element n of $\operatorname{dom} p$ such that $2 \leqslant n \leqslant \operatorname{len} p - 1$ there exists a vertex w of G_2 and there exists an object e such that $e \in$ (the edges of G_1)$\setminus$(the edges of $p(n)$) and $p(n+1)$ is supergraph of $p(n)$ extended by e between vertices v and w or supergraph of $p(n)$ extended by e between vertices w and v.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite set V_2 for every supergraph G_1 of G_2 extended by vertex v and edges between v and $V_1 \cup V_2$ of G_2 such that $V_1 \cup V_2 \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2 and V_1 misses V_2 and $\overline{\overline{V_2}} = \$_1$ there exists a non empty, graph-yielding finite sequence p such that $p(1) = G_2$ and $p(\operatorname{len} p) = G_1$ and $\operatorname{len} p = \overline{\overline{V_2}} + 2$ and $p(2)$ is a supergraph of G_2 extended by vertex v and edges between v and V_1 of G_2 and for every element n of $\operatorname{dom} p$ such that $2 \leqslant n \leqslant \operatorname{len} p - 1$.

There exists a vertex w of G_2 and there exists an object e such that $e \in$ (the edges of G_1) $\setminus$ (the edges of $p(n)$) and $p(n+1)$ is supergraph of $p(n)$ extended by e between vertices v and w or supergraph of $p(n)$ extended by e between vertices w and v. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. $\square$

(77) Let us consider a graph G_2, an object v, a finite set V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then there exists a non empty, graph-yielding finite sequence p such that

(i) $p(1) = G_2$, and

(ii) $p(\operatorname{len} p) = G_1$, and

(iii) $\operatorname{len} p = \overline{\overline{V}} + 2$, and

(iv) $p(2)$ is a supergraph of G_2 extended by v, and

(v) for every element n of $\operatorname{dom} p$ such that $2 \leqslant n \leqslant \operatorname{len} p - 1$ there exists a vertex w of G_2 and there exists an object e such that $e \in$ (the edges of G_1)$\setminus$(the edges of $p(n)$) and $p(n+1)$ is supergraph of $p(n)$ extended by e between vertices v and w or supergraph of $p(n)$ extended by e between vertices w and v.

The theorem is a consequence of (76).

(78) Let us consider a graph G_2, an object v, a non empty, finite set V, and a supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2. Suppose $V \subseteq$ the vertices of G_2 and $v \notin$ the vertices of G_2. Then there exists a non empty, graph-yielding finite sequence p such that

(i) $p(1) = G_2$, and

(ii) $p(\operatorname{len} p) = G_1$, and

(iii) $\operatorname{len} p = \overline{\overline{V}} + 1$, and

(iv) there exists a vertex w of G_2 and there exists an object e such that $e \in$ (the edges of G_1) $\setminus$ (the edges of G_2) and $p(2)$ is supergraph of G_2 extended by v, w and e between them or supergraph of G_2 extended by w, v and e between them, and

(v) for every element n of $\operatorname{dom} p$ such that $2 \leqslant n \leqslant \operatorname{len} p - 1$ there exists a vertex w of G_2 and there exists an object e such that $e \in$ (the edges of G_1) $\setminus$ (the edges of $p(n)$) and $p(n+1)$ is supergraph of $p(n)$ extended by e between vertices v and w or supergraph of $p(n)$ extended by e between vertices w and v.

The theorem is a consequence of (76).

(79) Let us consider a finite, simple graph G, a set W, and a subgraph H of G induced by W. Then there exists a non empty, finite, simple, graph-yielding finite sequence p such that

(i) $p(1) \approx H$, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G.\mathrm{order}() - H.\mathrm{order}() + 1$, and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists an object v and there exists a finite set V such that $v \in$ (the vertices of G) $\setminus$ (the vertices of $p(n)$) and $V \subseteq$ the vertices of $p(n)$ and $p(n+1)$ is a supergraph of $p(n)$ extended by vertex v and edges between v and V of $p(n)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every finite, simple graph G for every set W for every subgraph H of G induced by W such that $G.\mathrm{order}() - H.\mathrm{order}() = \$_1$ there exists a non empty, finite, simple, graph-yielding finite sequence p such that $p(1) \approx H$ and $p(\operatorname{len} p) = G$ and $\operatorname{len} p = G.\mathrm{order}() - H.\mathrm{order}() + 1$ and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists an object v and there exists a finite set V such that $v \in$ (the vertices of G) $\setminus$ (the vertices of $p(n)$) and $V \subseteq$ the vertices of $p(n)$ and $p(n+1)$ is a supergraph of $p(n)$ extended by vertex v and edges between v and V of $p(n)$.

$\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$. □

(80) Let us consider a finite, simple graph G. Then there exists a non empty, finite, simple, graph-yielding finite sequence p such that

(i) $p(1)$ is trivial and edgeless, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G$.order(), and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists an object v and there exists a finite set V such that $v \in$ (the vertices of G) $\setminus$ (the vertices of $p(n)$) and $V \subseteq$ the vertices of $p(n)$ and $p(n+1)$ is a supergraph of $p(n)$ extended by vertex v and edges between v and V of $p(n)$.

The theorem is a consequence of (79) and (52).

The scheme *FinSimpleGraphs* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 5) For every finite, simple graph G, $\mathcal{P}[G]$

provided

- for every trivial, edgeless graph G, $\mathcal{P}[G]$ and
- for every finite, simple graph G_2 and for every object v and for every finite set V and for every supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2 such that $v \notin$ the vertices of G_2 and $V \subseteq$ the vertices of G_2 and $\mathcal{P}[G_2]$ holds $\mathcal{P}[G_1]$.

Now we state the propositions:

(81) Let us consider a non empty, graph-yielding finite sequence p. Suppose $p(1)$ is simple and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists an object v and there exists a set V such that $p(n + 1)$ is a supergraph of $p(n)$ extended by vertex v and edges between v and V of $p(n)$. Then $p(\operatorname{len} p)$ is simple.

Proof: Define $\mathcal{Q}$[natural number] $\equiv$ if $\$_1 \leqslant \operatorname{len} p$, then there exists an element k of $\operatorname{dom} p$ such that $\$_1 = k$ and $p(k)$ is simple. $\mathcal{Q}[1]$. For every non zero natural number m such that $\mathcal{Q}[m]$ holds $\mathcal{Q}[m+1]$. For every non zero natural number m, $\mathcal{Q}[m]$. $\square$

(82) Let us consider a finite, simple, connected graph G. Then there exists a non empty, finite, simple, connected, graph-yielding finite sequence p such that

(i) $p(1)$ is trivial and edgeless, and

(ii) $p(\operatorname{len} p) = G$, and

(iii) $\operatorname{len} p = G$.order(), and

(iv) for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists an object v and there exists a non empty, finite set V such that $v \in$ (the vertices of G) $\setminus$ (the vertices of $p(n)$) and $V \subseteq$ the vertices of $p(n)$ and $p(n + 1)$ is a supergraph of $p(n)$ extended by vertex v and edges between v and V of $p(n)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every finite, simple, connected graph G such that G.order() $= \$_1$ there exists a non empty, finite, simple, connected, graph-yielding finite sequence p such that $p(1)$ is trivial and edgeless and $p(\operatorname{len} p) = G$ and $\operatorname{len} p = G$.order() and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists an object v and there exists a non empty, finite set V such that $v \in$ (the vertices of G) $\setminus$ (the vertices of $p(n)$) and $V \subseteq$ the vertices of $p(n)$ and $p(n+1)$ is a supergraph of $p(n)$ extended by vertex v and edges between v and V of $p(n)$.

$\mathcal{P}[1]$. For every non zero natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every non zero natural number k, $\mathcal{P}[k]$. □

The scheme *FinSimpleConnectedGraphs* deals with a unary predicate $\mathcal{P}$ and states that

(Sch. 6) For every finite, simple, connected graph G, $\mathcal{P}[G]$

provided

- for every trivial, edgeless graph G, $\mathcal{P}[G]$ and
- for every finite, simple, connected graph G_2 and for every object v and for every non empty, finite set V and for every supergraph G_1 of G_2 extended by vertex v and edges between v and V of G_2 such that $v \notin$ the vertices of G_2 and $V \subseteq$ the vertices of G_2 and $\mathcal{P}[G_2]$ holds $\mathcal{P}[G_1]$.

Now we state the proposition:

(83) Let us consider a non empty, graph-yielding finite sequence p. Suppose $p(1)$ is simple and connected and for every element n of $\operatorname{dom} p$ such that $n \leqslant \operatorname{len} p - 1$ there exists an object v and there exists a non empty set V such that $p(n+1)$ is a supergraph of $p(n)$ extended by vertex v and edges between v and V of $p(n)$. Then $p(\operatorname{len} p)$ is simple and connected.

PROOF: Define $\mathcal{Q}$[natural number] $\equiv$ if $\$_1 \leqslant \operatorname{len} p$, then there exists an element k of $\operatorname{dom} p$ such that $\$_1 = k$ and $p(k)$ is simple and connected. $\mathcal{Q}[1]$.

For every non zero natural number m such that $\mathcal{Q}[m]$ holds $\mathcal{Q}[m+1]$. For every non zero natural number m, $\mathcal{Q}[m]$. □

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