FORMALIZED MATHEMATICS
Vol. 25, No. 3, Pages 217–225, 2017
DOI: 10.1515/forma-2017-0021

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Gauge Integral

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Summary. Some authors have formalized the integral in the Mizar Mathematical Library (MML). The first article in a series on the Darboux/Riemann integral was written by Noboru Endou and Artur Korniłowicz: [6]. The Lebesgue integral was formalized a little later [13] and recently the integral of Riemann-Stieltjes was introduced in the MML by Keiko Narita, Kazuhisa Nakasho and Yasunari Shidama [12].

A presentation of definitions of integrals in other proof assistants or proof checkers (ACL2, COQ, Isabelle/HOL, HOL4, HOL Light, PVS, ProofPower) may be found in [10] and [4].

Using the Mizar system [1], we define the Gauge integral (Henstock-Kurzweil) of a real-valued function on a real interval $[a, b]$ (see [2], [3], [15], [14], [11]). In the next section we formalize that the Henstock-Kurzweil integral is linear.

In the last section, we verified that a real-valued bounded integrable (in sense Darboux/Riemann [6, 7, 8]) function over a interval a, b is Gauge integrable.

Note that, in accordance with the possibilities of the MML [9], we reuse a large part of demonstrations already present in another article. Instead of rewriting the proof already contained in [7] (MML Version: 5.42.1290), we slightly modified this article in order to use directly the expected results.

MSC: 26A39 26A42 03B35

Keywords: Gauge integral; Henstock-Kurzweil integral; generalized Riemann integral

MML identifier: COUSIN2, version: 8.1.06 5.44.1305

1. Preliminaries

From now on a, b, c, d, e denote real numbers.

Now we state the propositions:

ISSN 1426–2630(Print), 1898-9934(Online)

(1) If $a - b \leqslant c$ and $b \leqslant a$, then $|b - a| \leqslant c$.

(2) If $b - a \leqslant c$ and $a \leqslant b$, then $|b - a| \leqslant c$.

(3) If $a \leqslant b \leqslant c$ and $|a - d| \leqslant e$ and $|c - d| \leqslant e$, then $|b - d| \leqslant e$.

(4) If for every c such that $0 < c$ holds $|a - b| \leqslant c$, then $a = b$.

(5) Let us consider non negative real numbers b, c, d. Suppose $d < \frac{e}{2 \cdot b \cdot |c|}$. Then

 (i) b is positive, and

 (ii) c is positive.

(6) If $a \neq 0$, then $a \cdot \frac{b}{2 \cdot a} = \frac{b}{2}$.

(7) Let us consider non negative real numbers b, c, d. Suppose $a \leqslant b \cdot c \cdot d$ and $d < \frac{e}{2 \cdot b \cdot |c|}$. Then $a \leqslant \frac{e}{2}$. The theorem is a consequence of (5) and (6).

2. Vector Lattice / Riesz Space

Let X be a non empty set and f, g be functions from X into $\mathbb{R}$. The functor $\min(f, g)$ yielding a function from X into $\mathbb{R}$ is defined by

(Def. 1) for every element x of X, $it(x) = \min(f(x), g(x))$.

One can verify that the functor is commutative. The functor $\max(f, g)$ yielding a function from X into $\mathbb{R}$ is defined by

(Def. 2) for every element x of X, $it(x) = \max(f(x), g(x))$.

Note that the functor is commutative.

Let f, g be positive yielding functions from X into $\mathbb{R}$. One can check that $\min(f, g)$ is positive yielding and $\max(f, g)$ is positive yielding.

Let f, g be functions from X into $\mathbb{R}$. We say that $f \leqslant g$ if and only if

(Def. 3) for every element x of X, $f(x) \leqslant g(x)$.

Now we state the proposition:

(8) Let us consider a non empty set X, and functions f, g from X into $\mathbb{R}$. Then $\min(f, g) \leqslant f$.

Let us consider a non empty, real-membered set X. Now we state the propositions:

(9) If for every real number r such that $r \in X$ holds $\sup X = r$, then there exists a real number r such that $X = \{r\}$.

(10) If for every real number r such that $r \in X$ holds $\inf X = r$, then there exists a real number r such that $X = \{r\}$.

(11) Let us consider a non empty, lower bounded, upper bounded, real-membered set X. Suppose $\sup X = \inf X$. Then there exists a real number r such that $X = \{r\}$. The theorem is a consequence of (9).

3. Some Properties of the χ Function

In the sequel X, Y denote sets, Z denotes a non empty set, r denotes a real number, s denotes an extended real, A denotes a subset of $\mathbb{R}$, and f denotes a real-valued function.

Now we state the propositions:

(12) $\chi_{X,Y}$ is a function from Y into $\mathbb{R}$.

(13) If $A \subseteq]r, s[$, then A is lower bounded.

(14) If $A \subseteq]s, r[$, then A is upper bounded.

(15) If $\operatorname{rng} f \subseteq [a, b]$, then f is bounded.

(16) If $a \leqslant b$, then $\{a, b\} \subseteq [a, b]$.

(17) $\chi_{X,Y}$ is bounded. The theorem is a consequence of (16) and (15).

(18) If X misses Y, then for every element x of X, $\chi_{Y,X}(x) = 0$.

(19) Let us consider a function f from Z into $\mathbb{R}$. Then f is constant if and only if there exists a real number r such that $f = r \cdot \chi_{Z,Z}$.

(20) $\chi_{X,X}$ is positive yielding.

4. Refinement of Tagged Partition

In the sequel I denotes a non empty, closed interval subset of $\mathbb{R}$, T_1 denotes a tagged partition of I, D denotes a partition of I, T denotes an element of the set of tagged partitions of D, and f denotes a partial function from I to $\mathbb{R}$.

Now we state the propositions:

(21) If f is lower integrable, then $\text{lower\_sum}(f, D) \leqslant \text{lower\_integral}\, f$.

(22) If f is upper integrable, then $\text{upper\_integral}\, f \leqslant \text{upper\_sum}(f, D)$.

Let A be a non empty, closed interval subset of $\mathbb{R}$. The functor tagged-divs(A) yielding a set is defined by

(Def. 4) for every set x, $x \in \mathit{it}$ iff x is a tagged partition of A.

One can check that tagged-divs(A) is non empty.

Let T_1 be a tagged partition of A. The functor T_1-tags yielding a non empty, non-decreasing finite sequence of elements of $\mathbb{R}$ is defined by

(Def. 5) there exists a partition D of A and there exists an element T of the set of tagged partitions of D such that $\mathit{it} = T$ and $T_1 = \langle D, T \rangle$.

Now we state the propositions:

(23) If $T_1 = \langle D, T \rangle$, then $T = T_1$-tags and $D = T_1$-partition.

(24) $\operatorname{len}(T_1\text{-tags}) = \operatorname{len}(T_1\text{-partition})$. The theorem is a consequence of (23).

Let A be a non empty, closed interval subset of $\mathbb{R}$ and T_1 be a tagged partition of A. The functor $\operatorname{len} T_1$ yielding an element of $\mathbb{N}$ is defined by the term

(Def. 6) $\operatorname{len}(T_1\text{-partition})$.

The functor $\operatorname{dom} T_1$ yielding a set is defined by the term

(Def. 7) $\operatorname{dom}(T_1\text{-partition})$.

Now we state the propositions:

(25) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, a partition D of I, and an element T of the set of tagged partitions of D. Then $\operatorname{rng} T \subseteq I$.

(26) Let us consider a non empty, closed interval subset I of $\mathbb{R}$, positive yielding functions j_1, j_2 from I into $\mathbb{R}$, and a j_1-fine tagged partition T_1 of I. If $j_1 \leqslant j_2$, then T_1 is a j_2-fine tagged partition of I. The theorem is a consequence of (23), (24), and (25).

5. Definition of the Gauge Integral on a Real Bounded Interval

Let I be a non empty, closed interval subset of $\mathbb{R}$, f be a partial function from I to $\mathbb{R}$, and T_1 be a tagged partition of I. The functor tagged-volume(f, T_1) yielding a finite sequence of elements of $\mathbb{R}$ is defined by

(Def. 8) $\operatorname{len} it = \operatorname{len} T_1$ and for every natural number i such that $i \in \operatorname{dom} T_1$ holds $it(i) = f((T_1\text{-tags})(i)) \cdot \operatorname{vol}(\operatorname{divset}(T_1\text{-partition}, i))$.

The functor tagged-sum(f, T_1) yielding a real number is defined by the term

(Def. 9) $\sum(\text{tagged-volume}(f, T_1))$.

Now we state the proposition:

(27) If $Y \subseteq X$, then $\chi_{X,Y} = \chi_{Y,Y}$.

From now on f denotes a function from I into $\mathbb{R}$.

Now we state the propositions:

(28) If I is non empty and trivial, then $\operatorname{vol}(I) = 0$.

(29) Let us consider a real number r. If $I = \{r\}$, then for every partition D of I, $D = \langle r \rangle$.

Let I be a non empty, closed interval subset of $\mathbb{R}$ and f be a function from I into $\mathbb{R}$. We say that f is HK-integrable if and only if

(Def. 10) there exists a real number J such that for every real number ε such that $\varepsilon > 0$ there exists a positive yielding function j from I into $\mathbb{R}$ such that for every tagged partition T_1 of I such that T_1 is j-fine holds $|\text{tagged-sum}(f, T_1) - J| \leqslant \varepsilon$.

Assume f is HK-integrable. The functor HK-integral(f) yielding a real number is defined by

(Def. 11) for every real number ε such that $\varepsilon > 0$ there exists a positive yielding function j from I into $\mathbb{R}$ such that for every tagged partition T_1 of I such that T_1 is j-fine holds $|\text{tagged-sum}(f, T_1) - it| \leqslant \varepsilon$.

Now we state the propositions:

(30) Let us consider a function f from I into $\mathbb{R}$. Suppose I is trivial. Then

(i) f is HK-integrable, and

(ii) HK-integral$(f) = 0$.

The theorem is a consequence of (20), (12), and (29).

(31) If A misses I and $f = \chi_{A,I}$, then tagged-sum$(f, T_1) = 0$.
Proof: For every natural number i such that $i \in \operatorname{dom} T_1$ holds (tagged-volume$(f, T_1))(i) = 0$. □

(32) If A misses I and $f = \chi_{A,I}$, then f is HK-integrable and HK-integral$(f) = 0$. The theorem is a consequence of (12) and (31).

(33) If $I \subseteq A$ and $f = \chi_{A,I}$, then f is HK-integrable and HK-integral$(f) = \text{vol}(I)$. The theorem is a consequence of (12) and (27).

Let I be a non empty, closed interval subset of $\mathbb{R}$. One can check that there exists a function from I into $\mathbb{R}$ which is HK-integrable.

6. The Linearity Property of the Gauge Integral

In the sequel f, g denote HK-integrable functions from I into $\mathbb{R}$ and r denotes a real number.

Now we state the propositions:

(34) Let us consider a natural number i. Suppose $i \in \operatorname{dom} T_1$.
Then (tagged-volume$(r \cdot f, T_1))(i) =$
$r \cdot f((T_1\text{-tags})(i)) \cdot \text{vol}(\text{divset}(T_1\text{-partition}, i))$.

(35) tagged-volume$(r \cdot f, T_1) = r \cdot (\text{tagged-volume}(f, T_1))$.
Proof: For every natural number i such that $i \in \operatorname{dom}(\text{tagged-volume}(r \cdot f, T_1))$ holds (tagged-volume$(r \cdot f, T_1))(i) = (r \cdot (\text{tagged-volume}(f, T_1)))(i)$. □

(36) Let us consider a natural number i. Suppose $i \in \operatorname{dom} T_1$. Then (tagged-volume$(f + g, T_1))(i) = f((T_1\text{-tags})(i)) \cdot \text{vol}(\text{divset}(T_1\text{-partition}, i)) + (g((T_1\text{-tags})(i)) \cdot \text{vol}(\text{divset}(T_1\text{-partition}, i)))$. The theorem is a consequence of (23), (24), and (25).

(37) tagged-volume$(f + g, T_1) =$
(tagged-volume$(f, T_1)) + (\text{tagged-volume}(g, T_1))$.
Proof: For every natural number i such that $i \in \operatorname{dom}$(tagged-volume

$(f+g, T_1))$ holds $(\text{tagged-volume}(f+g, T_1))(i) = ((\text{tagged-volume}(f, T_1)) + (\text{tagged-volume}(g, T_1)))(i)$. □

(38) Suppose f is HK-integrable. Then

(i) $r \cdot f$ is an HK-integrable function from I into $\mathbb{R}$, and

(ii) $\text{HK-integral}(r \cdot f) = r \cdot \text{HK-integral}(f)$.

PROOF: Consider J being a real number such that for every real number ε such that $\varepsilon > 0$ there exists a positive yielding function j from I into $\mathbb{R}$ such that for every tagged partition T_1 of I such that T_1 is j-fine holds $|\text{tagged-sum}(f, T_1) - J| \leqslant \varepsilon$. For every real number ε such that $\varepsilon > 0$ there exists a positive yielding function j from I into $\mathbb{R}$ such that for every tagged partition T_1 of I such that T_1 is j-fine holds $|\text{tagged-sum}(r \cdot f, T_1) - (r \cdot J)| \leqslant \varepsilon$. □

(39) (i) $f+g$ is an HK-integrable function from I into $\mathbb{R}$, and

(ii) $\text{HK-integral}(f+g) = \text{HK-integral}(f) + \text{HK-integral}(g)$.

PROOF: Consider J_1 being a real number such that for every real number ε such that $\varepsilon > 0$ there exists a positive yielding function j from I into $\mathbb{R}$ such that for every tagged partition T_1 of I such that T_1 is j-fine holds $|\text{tagged-sum}(f, T_1) - J_1| \leqslant \varepsilon$. Consider J_2 being a real number such that for every real number ε such that $\varepsilon > 0$ there exists a positive yielding function j from I into $\mathbb{R}$ such that for every tagged partition T_1 of I such that T_1 is j-fine holds $|\text{tagged-sum}(g, T_1) - J_2| \leqslant \varepsilon$. For every real number ε such that $\varepsilon > 0$ there exists a positive yielding function j from I into $\mathbb{R}$ such that for every tagged partition T_1 of I such that T_1 is j-fine holds $|\text{tagged-sum}(f+g, T_1) - (J_1 + J_2)| \leqslant \varepsilon$. □

(40) Let us consider a function f from I into $\mathbb{R}$. Suppose f is constant. Then

(i) f is HK-integrable, and

(ii) there exists a real number r such that $f = r \cdot \chi_{I,I}$ and $\text{HK-integral}(f) = r \cdot \text{vol}(I)$.

The theorem is a consequence of (19), (12), (33), and (38).

7. Riemann Integrability and Gauge Integrability

Let I be a non empty, closed interval subset of $\mathbb{R}$. Note that there exists a function from I into $\mathbb{R}$ which is integrable.

Let X be a non empty set. Observe that there exists a function from X into $\mathbb{R}$ which is bounded.

Now we state the proposition:

(41) Let us consider a bounded function f from I into $\mathbb{R}$.
Then $|\sup \operatorname{rng} f - \inf \operatorname{rng} f| = 0$ if and only if f is constant. The theorem is a consequence of (11).

Let I be a non empty, closed interval subset of $\mathbb{R}$. Observe that there exists an integrable function from I into $\mathbb{R}$ which is bounded.

Let us consider a partial function f from I to $\mathbb{R}$. Now we state the propositions:

(42) If f is upper integrable, then there exists a real number r such that for every partition D of I, $r <$ upper_sum(f, D).

(43) If f is lower integrable, then there exists a real number r such that for every partition D of I, lower_sum$(f, D) < r$.

(44) Let us consider a function f from I into $\mathbb{R}$, and partitions D, D_1 of I. Suppose $D(1) = \inf I$ and $D_1 = D_{\restriction 1}$. Then

(i) upper_sum$(f, D_1) =$ upper_sum(f, D), and

(ii) lower_sum$(f, D_1) =$ lower_sum(f, D).

Proof: (upper_volume(f, D))$(1) = 0$ by [5, (50)]. (lower_volume(f, D))$(1) = 0$ by [5, (50)]. □

In the sequel f denotes a bounded, integrable function from I into $\mathbb{R}$.

Now we state the propositions:

(45) Let us consider a natural number i. Suppose $i \in \operatorname{dom} T_1$. Then (lower_volume$(f, T_1$-partition$)$)$(i) \leqslant$ (tagged-volume(f, T_1))$(i) \leqslant$ (upper_volume$(f, T_1$-partition$)$)(i). The theorem is a consequence of (23).

(46) $\sum$ lower_volume$(f, T_1$-partition$) \leqslant \sum($tagged-volume$(f, T_1)) \leqslant$ $\sum$ upper_volume$(f, T_1$-partition$)$. The theorem is a consequence of (45).

(47) Let us consider a real number ε. Suppose I is not trivial and $0 < \varepsilon$. Then there exists a partition D of I such that

(i) $D(1) \neq \inf I$, and

(ii) upper_sum$(f, D) <$ integral $f + \frac{\varepsilon}{2}$, and

(iii) integral $f - \frac{\varepsilon}{2} <$ lower_sum(f, D), and

(iv) upper_sum$(f, D) -$ lower_sum$(f, D) < \varepsilon$.

The theorem is a consequence of (44).

From now on j denotes a positive yielding function from I into $\mathbb{R}$.

(48) If $j = r \cdot \chi_{I,I}$, then $0 < r$.

In the sequel D denotes a tagged partition of I. Now we state the proposition:

(49) If $j = r \cdot \chi_{I,I}$ and D is j-fine, then $\delta_{D\text{-partition}} \leqslant r$.

Proof: Reconsider $g = \chi_{I,I}$ as a function from I into $\mathbb{R}$. For every natural number i such that $i \in \mathrm{dom}(D\text{-partition})$ holds $(\mathrm{upper\_volume}(g, D\text{-partition}))(i) \leqslant r$. $\delta_{D\text{-partition}} \leqslant r$. □

From now on r_1, r_2, s denote real numbers, D, D_1 denote partitions of I, and f_1 denotes a function from I into $\mathbb{R}$. Now we state the propositions:

(50) There exists a natural number i such that

(i) $i \in \mathrm{dom}\, D$, and

(ii) $\min \mathrm{rng}\, \mathrm{upper\_volume}(f_1, D) = (\mathrm{upper\_volume}(f_1, D))(i)$.

(51) Let us consider a function f from I into $\mathbb{R}$, and a real number ε. Suppose $f_1 = \chi_{I,I}$ and $r_1 = \min \mathrm{rng}\, \mathrm{upper\_volume}(f_1, D_1)$ and $r_2 = \frac{\varepsilon}{2 \cdot \mathrm{len}\, D_1 \cdot |\sup \mathrm{rng}\, f - \inf \mathrm{rng}\, f|}$ and $0 < r_1$ and $0 < r_2$ and $s = \frac{\min(r_1, r_2)}{2}$ and $j = s \cdot f_1$ and T_1 is j-fine. Then

(i) $\delta_{T_1\text{-partition}} < \min \mathrm{rng}\, \mathrm{upper\_volume}(f_1, D_1)$, and

(ii) $\delta_{T_1\text{-partition}} < \frac{\varepsilon}{2 \cdot \mathrm{len}\, D_1 \cdot |\sup \mathrm{rng}\, f - \inf \mathrm{rng}\, f|}$.

The theorem is a consequence of (49).

(52) Let us consider a finite sequence p of elements of $\mathbb{R}$. Suppose for every natural number i such that $i \in \mathrm{dom}\, p$ holds $r \leqslant p(i)$ and there exists a natural number i_0 such that $i_0 \in \mathrm{dom}\, p$ and $p(i_0) = r$. Then $r = \inf \mathrm{rng}\, p$.

(53) Suppose $f_1 = \chi_{I,I}$. Then

(i) $0 \leqslant \min \mathrm{rng}\, \mathrm{upper\_volume}(f_1, D)$, and

(ii) $0 = \min \mathrm{rng}\, \mathrm{upper\_volume}(f_1, D)$ iff $\mathrm{divset}(D, 1) = [D(1), D(1)]$.

Proof: Consider i_0 being a natural number such that $i_0 \in \mathrm{dom}\, D$ and $\min \mathrm{rng}\, \mathrm{upper\_volume}(f_1, D) = (\mathrm{upper\_volume}(f_1, D))(i_0)$. $0 = \min \mathrm{rng}\, \mathrm{upper\_volume}(f_1, D)$ iff $\mathrm{divset}(D, 1) = [D(1), D(1)]$. □

(54) If $\mathrm{divset}(D, 1) = [D(1), D(1)]$, then $D(1) = \inf I$.

(55) Let us consider a bounded, integrable function f from I into $\mathbb{R}$. Then

(i) f is HK-integrable, and

(ii) $\mathrm{HK\text{-}integral}(f) = \mathrm{integral}\, f$.

The theorem is a consequence of (40), (12), (17), (28), (30), (47), (53), (54), (41), (20), (46), (51), (21), (22), (7), (1), (2), and (3).

Let I be a non empty, closed interval subset of $\mathbb{R}$. Note that every function from I into $\mathbb{R}$ which is bounded and integrable is also HK-integrable.

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Received September 3, 2017

The English version of this volume of Formalized Mathematics was financed under agreement 548/P-DUN/2016 with the funds from the Polish Minister of Science and Higher Education for the dissemination of science.