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Propositional Linear Temporal Logic with Initial Validity Semantics[1]

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Summary. In the article [10] a formal system for Propositional Linear Temporal Logic (in short LTLB) with normal semantics is introduced. The language of this logic consists of "until" operator in a very strict version. The very strict "until" operator enables to express all other temporal operators.

In this article we construct a formal system for LTLB with the initial semantics [12]. Initial semantics means that we define the validity of the formula in a model as satisfaction in the initial state of model while normal semantics means that we define the validity as satisfaction in all states of model. We prove the Deduction Theorem, and the soundness and completeness of the introduced formal system. We also prove some theorems to compare both formal systems, i.e., the one introduced in the article [10] and the one introduced in this article.

Formal systems for temporal logics are applied in the verification of computer programs. In order to carry out the verification one has to derive an appropriate formula within a selected formal system. The formal systems introduced in [10] and in this article can be used to carry out such verifications in Mizar [4].

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The notation and terminology used in this paper have been introduced in the following articles: [13], [3], [9], [5], [6], [11], [14], [10], [16], [1], [2], [7], [17], [15], and [8].

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1. Preliminaries

Now we state the proposition:

(1) Let us consider a set X, a finite sequence f of elements of X, and a natural number i. If $1 \leqslant i \leqslant \operatorname{len} f$, then $f(i) = f_i$.

From now on A, B, C, p, q, r denote elements of LTLB-WFF, F, G, X denote subsets of LTLB-WFF, M denotes a LTL Model, i, j, n denote elements of $\mathbb{N}$, and f, f_1, f_2, g denote finite sequences of elements of LTLB-WFF.

Now we state the propositions:

(2) If $F \subseteq G$ and $F \vdash A$, then $G \vdash A$.

(3) $A \Rightarrow B \Rightarrow (B \Rightarrow C \Rightarrow (A \Rightarrow C))$ is tautologically valid.

(4) $A \Rightarrow (B \Rightarrow C) \Rightarrow (A \Rightarrow B \Rightarrow (A \Rightarrow C))$ is tautologically valid.

(5) $F \vdash \mathcal{G}\, A \Rightarrow A$.

(6) $\{A\} \models \mathcal{G}\, \mathcal{X}\, A$.

(7) $F \vdash \mathcal{G}\, A \Rightarrow \mathcal{G}\, \mathcal{X}\, A$. The theorem is a consequence of (6) and (2).

(8) $F \vdash \mathcal{G}(A \Rightarrow B) \Rightarrow (\mathcal{G}(A \Rightarrow \mathcal{X}\, A) \Rightarrow \mathcal{G}(A \Rightarrow \mathcal{G}\, B))$.

2. Initial Validity Semantics - Definitions

Let us consider M and A. We say that $M \overset{0}{\models} A$ if and only if

(Def. 1) $\mathrm{SAT}_M(\langle 0, A \rangle) = 1$.

Let us consider F. We say that $M \overset{0}{\models} F$ if and only if

(Def. 2) for every A such that $A \in F$ holds $M \overset{0}{\models} A$.

Let us consider A. We say that $F \overset{0}{\models} A$ if and only if

(Def. 3) for every M such that $M \overset{0}{\models} F$ holds $M \overset{0}{\models} A$.

3. The Connections between Normal Semantics and Initial Semantics

Now we state the propositions:

(9) If $M \models F$, then $M \overset{0}{\models} F$.

(10) $M \models A$ if and only if $M \overset{0}{\models} \mathcal{G}\, A$.

(11) If $F \overset{0}{\models} A$, then $F \models A$. The theorem is a consequence of (9).

Let us consider F. The functor $\mathcal{G}\, F$ yielding a subset of LTLB-WFF is defined by the term

(Def. 4) $\{\mathcal{G}\,A$, where A is an element of LTLB-WFF : $A \in F\}$.

Now we state the propositions:

(12) $M \models F$ if and only if $M \overset{0}{\models} \mathcal{G}\,F$. The theorem is a consequence of (10).

(13) $F \models A$ if and only if $\mathcal{G}\,F \overset{0}{\models} A$.

PROOF: $F \models A$ by [10, (29)], (12), [10, (28)]. □

(14) (i) $\{\mathrm{prop}\,n\} \models \mathcal{X}\,\mathrm{prop}\,n$, and

(ii) $\{\mathrm{prop}\,n\} \overset{0}{\not\models} \mathcal{X}\,\mathrm{prop}\,n$.

PROOF: $\{\mathrm{prop}\,n\} \models \mathcal{X}\,\mathrm{prop}\,n$ by [10, (23), (9)]. $\{\mathrm{prop}\,n\} \overset{0}{\not\models} \mathcal{X}\,\mathrm{prop}\,n$ by [8, (31)], [10, (9)]. □

(15) There exists F and there exists A such that $F \models A$ and $F \overset{0}{\not\models} A$. The theorem is a consequence of (14).

(16) If $F \overset{0}{\models} \mathcal{G}\,A$, then $F \models A$.

(17) (i) $\{\mathrm{prop}\,i\} \models \mathrm{prop}\,i$, and

(ii) $\{\mathrm{prop}\,i\} \overset{0}{\not\models} \mathcal{G}\,\mathrm{prop}\,i$.

The theorem is a consequence of (14).

(18) There exists F and there exists A such that $F \models A$ and $F \overset{0}{\not\models} \mathcal{G}\,A$. The theorem is a consequence of (17).

(19) $M \overset{0}{\models} F$ and $M \overset{0}{\models} G$ if and only if $M \overset{0}{\models} F \cup G$.

(20) $M \overset{0}{\models} A$ if and only if $M \overset{0}{\models} \{A\}$.

(21) $F \cup \{A\} \overset{0}{\models} B$ if and only if $F \overset{0}{\models} A \Rightarrow B$. The theorem is a consequence of (20) and (19).

(22) $\mathcal{G}\,\emptyset_{\mathrm{LTLB\text{-}WFF}} = \emptyset_{\mathrm{LTLB\text{-}WFF}}$.

(23) If $F \models A$ and for every B such that $B \in F$ holds $\emptyset_{\mathrm{LTLB\text{-}WFF}} \models B$, then $\emptyset_{\mathrm{LTLB\text{-}WFF}} \models A$.

(24) Suppose $F \models A$ and for every B such that $B \in F$ holds $\emptyset_{\mathrm{LTLB\text{-}WFF}} \overset{0}{\models} B$. Then $\emptyset_{\mathrm{LTLB\text{-}WFF}} \overset{0}{\models} A$. The theorem is a consequence of (13), (22), and (23).

(25) If $\emptyset_{\mathrm{LTLB\text{-}WFF}} \overset{0}{\models} A$, then $\emptyset_{\mathrm{LTLB\text{-}WFF}} \overset{0}{\models} \mathcal{X}\,A$. The theorem is a consequence of (24).

4. A Formal System (Hilbert-like) for LTLB with Initial Semantics

The functor LTL_0-axioms yielding a subset of LTLB-WFF is defined by the term

(Def. 5) $\mathcal{G}\, AX_{\mathrm{LTL}}$.

Let us consider p and q. We say that $p\, \mathrm{REFL}_0\text{-rule}\, q$ if and only if

(Def. 6) $p = \mathcal{G}\, q$.

We say that $p\, \mathrm{NEX}_0\text{-rule}\, q$ if and only if

(Def. 7) there exists A such that $p = \mathcal{G}\, A$ and $q = \mathcal{G}\, \mathcal{X}\, A$.

Let us consider r. We say that $p, q\, \mathrm{MP}_0\text{-rule}\, r$ if and only if

(Def. 8) there exists A and there exists B such that $p = \mathcal{G}\, A$ and $q = \mathcal{G}(A \Rightarrow B)$ and $r = \mathcal{G}\, B$.

We say that $p, q\, \mathrm{IND}_0\text{-rule}\, r$ if and only if

(Def. 9) there exists A and there exists B such that $p = \mathcal{G}(A \Rightarrow B)$ and $q = \mathcal{G}(A \Rightarrow \mathcal{X}\, A)$ and $r = \mathcal{G}(A \Rightarrow \mathcal{G}\, B)$.

Let i be a natural number. Let us consider f and X. We say that $\mathrm{prc}_0 f, X, i$ if and only if

(Def. 10) $f(i) \in \mathrm{LTL}_0$-axioms or $f(i) \in X$ or there exist natural numbers j, k such that $1 \leqslant j < i$ and $1 \leqslant k < i$ and $(\mathrm{MP}(f_j, f_k, f_i)$ or $f_j, f_k\, \mathrm{MP}_0\text{-rule}\, f_i$ or $f_j, f_k\, \mathrm{IND}_0\text{-rule}\, f_i)$ or there exists a natural number j such that $1 \leqslant j < i$ and $(f_j\, \mathrm{NEX}_0\text{-rule}\, f_i$ or $f_j\, \mathrm{REFL}_0\text{-rule}\, f_i)$.

Now we state the propositions:

(26) Let us consider natural numbers i, n. Suppose $n + \mathrm{len}\, f \leqslant \mathrm{len}\, f_2$ and for every natural number k such that $1 \leqslant k \leqslant \mathrm{len}\, f$ holds $f(k) = f_2(k+n)$ and $1 \leqslant i \leqslant \mathrm{len}\, f$. If $\mathrm{prc}_0 f, X, i$, then $\mathrm{prc}_0 f_2, X, i+n$. The theorem is a consequence of (1).

(27) Suppose $f_2 = f \frown f_1$ and $1 \leqslant \mathrm{len}\, f$ and $1 \leqslant \mathrm{len}\, f_1$ and for every natural number i such that $1 \leqslant i \leqslant \mathrm{len}\, f$ holds $\mathrm{prc}_0 f, X, i$ and for every natural number i such that $1 \leqslant i \leqslant \mathrm{len}\, f_1$ holds $\mathrm{prc}_0 f_1, X, i$. Let us consider a natural number i. If $1 \leqslant i \leqslant \mathrm{len}\, f_2$, then $\mathrm{prc}_0 f_2, X, i$. The theorem is a consequence of (1) and (26).

Let us consider X and p. We say that $X \vdash^{0} p$ if and only if

(Def. 11) there exists f such that $f(\mathrm{len}\, f) = p$ and $1 \leqslant \mathrm{len}\, f$ and for every natural number i such that $1 \leqslant i \leqslant \mathrm{len}\, f$ holds $\mathrm{prc}_0 f, X, i$.

(28) Suppose $f = f_1 \frown \langle p \rangle$ and $1 \leqslant \mathrm{len}\, f_1$ and for every natural number i such that $1 \leqslant i \leqslant \mathrm{len}\, f_1$ holds $\mathrm{prc}_0 f_1, X, i$ and $\mathrm{prc}_0 f, X, \mathrm{len}\, f$. Then

(i) for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} f$ holds $\operatorname{prc}_0 f, X, i$, and

(ii) $X \overset{0}{\vdash} p$.

The theorem is a consequence of (26).

5. Soundness Theorem for LTLB with Initial Semantics

Now we state the propositions:

(29) If $A \in \mathrm{LTL}_0$-axioms, then $F \overset{0}{\models} A$. The theorem is a consequence of (13) and (22).

(30) If $F \overset{0}{\models} A$ and $F \overset{0}{\models} A \Rightarrow B$, then $F \overset{0}{\models} B$.

(31) Suppose $F \overset{0}{\models} \mathcal{G}\, A$ and $F \overset{0}{\models} \mathcal{G}(A \Rightarrow B)$. Then $F \overset{0}{\models} \mathcal{G}\, B$.

Let us assume that $F \overset{0}{\models} \mathcal{G}\, A$. Now we state the propositions:

(32) $F \overset{0}{\models} \mathcal{G}\, \mathcal{X}\, A$.

(33) $F \overset{0}{\models} A$.

(34) Suppose $F \overset{0}{\models} \mathcal{G}(A \Rightarrow B)$ and $F \overset{0}{\models} \mathcal{G}(A \Rightarrow \mathcal{X}\, A)$. Then $F \overset{0}{\models} \mathcal{G}(A \Rightarrow \mathcal{G}\, B)$.

(35) Soundness Theorem for LTLB with initial semantics:
If $F \overset{0}{\vdash} A$, then $F \overset{0}{\models} A$.
Proof: Consider f such that $f(\operatorname{len} f) = A$ and $1 \leqslant \operatorname{len} f$ and for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} f$ holds $\operatorname{prc}_0 f, F, i$. Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} f$, then $F \overset{0}{\models} f_{\$_1}$. For every natural number i such that for every natural number j such that $j < i$ holds $\mathcal{P}[j]$ holds $\mathcal{P}[i]$ by [1, (14)], (1), (29), (30). For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 4]. $f_{\operatorname{len} f} = A$. □

6. Weak Completeness Theorem for LTLB with Initial Semantics

Now we state the proposition:

(36) If $A \in \mathrm{LTL}_0$-axioms or $A \in F$, then $F \overset{0}{\vdash} A$.
Proof: Define $\mathcal{S}$[set, set] $\equiv \$_2 = A$. Consider g such that $\operatorname{dom} g = \operatorname{Seg} 1$ and for every natural number k such that $k \in \operatorname{Seg} 1$ holds $\mathcal{S}[k, g(k)]$ from [3, Sch. 5]. For every natural number j such that $1 \leqslant j \leqslant \operatorname{len} g$ holds $\operatorname{prc}_0 g, F, j$. □

Let us assume that $F \overset{0}{\vdash} \mathcal{G}\, A$. Now we state the propositions:

(37) $F \overset{0}{\vdash} A$. The theorem is a consequence of (1) and (28).

(38) $F \overset{0}{\vdash} \mathcal{G}\,\mathcal{X}\,A$. The theorem is a consequence of (1) and (28).

(39) If $F \overset{0}{\vdash} A$ and $F \overset{0}{\vdash} A \Rightarrow B$, then $F \overset{0}{\vdash} B$. The theorem is a consequence of (27), (1), and (28).

(40) If $F \overset{0}{\vdash} \mathcal{G}\,A$ and $F \overset{0}{\vdash} \mathcal{G}(A \Rightarrow B)$, then $F \overset{0}{\vdash} \mathcal{G}\,B$. The theorem is a consequence of (27), (1), and (28).

(41) Suppose $F \overset{0}{\vdash} \mathcal{G}(A \Rightarrow B)$ and $F \overset{0}{\vdash} \mathcal{G}(A \Rightarrow \mathcal{X}\,A)$. Then $F \overset{0}{\vdash} \mathcal{G}(A \Rightarrow \mathcal{G}\,B)$. The theorem is a consequence of (27), (1), and (28).

(42) If $A \in AX_{\mathrm{LTL}}$, then $F \overset{0}{\vdash} A$. The theorem is a consequence of (36) and (37).

(43) If $A \in \mathrm{LTL}_0$-axioms, then $F \vdash A$.

(44) If $\emptyset_{\text{LTLB-WFF}} \vdash A$, then $\emptyset_{\text{LTLB-WFF}} \overset{0}{\vdash} A$.

Proof: Consider f such that $f(\operatorname{len} f) = A$ and $1 \leqslant \operatorname{len} f$ and for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} f$ holds $\operatorname{prc}(f, \emptyset_{\text{LTLB-WFF}}, i)$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} f$, then $\emptyset_{\text{LTLB-WFF}} \overset{0}{\vdash} \mathcal{G}\,f_{\$_1}$. For every natural number i such that for every natural number j such that $j < i$ holds $\mathcal{P}[j]$ holds $\mathcal{P}[i]$ by [1, (14)], (1), (36), (40). For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 4]. $A = f_{\operatorname{len} f}$. □

(45) (i) $\{\operatorname{prop} i\} \vdash \mathcal{X} \operatorname{prop} i$, and

(ii) $\{\operatorname{prop} i\} \overset{0}{\nvdash} \mathcal{X} \operatorname{prop} i$.

The theorem is a consequence of (35) and (14).

(46) If $F \subseteq G$ and $F \overset{0}{\vdash} A$, then $G \overset{0}{\vdash} A$.

Let us consider f and A. The functor $\operatorname{implications}(f, A)$ yielding a finite sequence of elements of LTLB-WFF is defined by

(Def. 12) (i) $\operatorname{len} it = \operatorname{len} f$ and $it(1) = f_1 \Rightarrow A$ and for every i such that $1 \leqslant i < \operatorname{len} f$ holds $it(i+1) = f_{i+1} \Rightarrow it_i$, **if** $\operatorname{len} f > 0$,

(ii) $it = \varepsilon_{(\text{LTLB-WFF})}$, **otherwise**.

Now we state the proposition:

(47) Weak Completeness Theorem for LTLB with initial semantics:

Let us consider a finite subset F of LTLB-WFF. If $F \overset{0}{\models} A$, then $F \overset{0}{\vdash} A$. The theorem is a consequence of (13), (22), (44), (21), (36), (39), and (46).

7. DEDUCTION THEOREM

Now we state the propositions:

(48) If $F \cup \{A\} \overset{0}{\vdash} B$, then $F \overset{0}{\vdash} A \Rightarrow B$.

PROOF: Consider f such that $f(\operatorname{len} f) = B$ and $1 \leqslant \operatorname{len} f$ and for every natural number i such that $1 \leqslant i \leqslant \operatorname{len} f$ holds $\operatorname{prc}_0 f, F \cup \{A\}, i$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} f$, then $F \overset{0}{\vdash} A \Rightarrow f_{\$_1}$. For every natural number i such that for every natural number j such that $j < i$ holds $\mathcal{P}[j]$ holds $\mathcal{P}[i]$ by [1, (14)], (42), [10, (34)], (1). For every natural number i, $\mathcal{P}[i]$ from [1, Sch. 4]. $B = f_{\operatorname{len} f}$. □

(49) If $F \overset{0}{\vdash} A \Rightarrow B$, then $F \cup \{A\} \overset{0}{\vdash} B$. The theorem is a consequence of (36), (46), and (39).

8. THE CONNECTIONS BETWEEN DERIVABILITY IN THE FORMAL SYSTEM FOR LTLB WITH NORMAL SEMANTICS AND THE FORMAL SYSTEM FOR LTLB WITH INITIAL SEMANTICS

Let F be a finite subset of LTLB-WFF. Note that $\mathcal{G}\, F$ is finite.

Let us consider a finite subset F of LTLB-WFF. Now we state the propositions:

(50) $F \vdash A$ if and only if $\mathcal{G}\, F \overset{0}{\vdash} A$. The theorem is a consequence of (47), (13), and (35).

(51) If $F \overset{0}{\vdash} A$, then $F \vdash A$. The theorem is a consequence of (35) and (11).

Now we state the propositions:

(52) (i) $\{\operatorname{prop} i\} \vdash \mathcal{G} \operatorname{prop} i$, and

(ii) $\{\operatorname{prop} i\} \overset{0}{\nvdash} \mathcal{G} \operatorname{prop} i$.

PROOF: $\{\operatorname{prop} i\} \vdash \mathcal{G} \operatorname{prop} i$ by [10, (42), (54)]. $\{\operatorname{prop} i\} \overset{0}{\nvdash} \mathcal{G} \operatorname{prop} i$ by (35), (47), (45), [10, (10), (9)]. □

(53) Let us consider a finite subset F of LTLB-WFF. If $F \overset{0}{\vdash} \mathcal{G}\, A$, then $F \vdash A$. The theorem is a consequence of (35) and (16).

(54) (i) $\{\operatorname{prop} i\} \vdash \operatorname{prop} i$, and

(ii) $\{\operatorname{prop} i\} \overset{0}{\nvdash} \mathcal{G} \operatorname{prop} i$.

The theorem is a consequence of (35) and (17).

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