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Topological Manifolds[1]

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Summary. Let us recall that a topological space M is a topological manifold if M is second-countable Hausdorff and locally Euclidean, i.e. each point has a neighborhood that is homeomorphic to an open ball of $\mathcal{E}^n$ for some n. However, if we would like to consider a topological manifold with a boundary, we have to extend this definition. Therefore, we introduce here the concept of a locally Euclidean space that covers both cases (with and without a boundary), i.e. where each point has a neighborhood that is homeomorphic to a closed ball of $\mathcal{E}^n$ for some n.

Our purpose is to prove, using the Mizar formalism, a number of properties of such locally Euclidean spaces and use them to demonstrate basic properties of a manifold. Let T be a locally Euclidean space. We prove that every interior point of T has a neighborhood homeomorphic to an open ball and that every boundary point of T has a neighborhood homeomorphic to a closed ball, where additionally this point is transformed into a point of the boundary of this ball. When T is n-dimensional, i.e. each point of T has a neighborhood that is homeomorphic to a closed ball of $\mathcal{E}^n$, we show that the interior of T is a locally Euclidean space without boundary of dimension n and the boundary of T is a locally Euclidean space without boundary of dimension $n-1$. Additionally, we show that every connected component of a compact locally Euclidean space is a locally Euclidean space of some dimension. We prove also that the Cartesian product of locally Euclidean spaces also forms a locally Euclidean space. We determine the interior and boundary of this product and show that its dimension is the sum of the dimensions of its factors. At the end, we present several consequences of these results for topological manifolds. This article is based on [14].

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The notation and terminology used in this paper have been introduced in the following articles: [30], [15], [19], [1], [10], [23], [24], [28], [11], [5], [12], [6], [7], [29], [3], [4], [8], [26], [33], [25], [32], [20], [34], [13], [21], and [9].

1. Preliminaries

From now on n, m denote natural numbers.

Now we state the proposition:

(1) Let us consider a non empty topological space M, a point q of M, a real number r, and a point p of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $r > 0$. Let us consider a neighbourhood U of q. Suppose $M\restriction U$ and $\mathbb{B}_r(p)$ are homeomorphic. Then there exists a neighbourhood W of q such that

(i) $W \subseteq \operatorname{Int} U$, and

(ii) $M\restriction W$ and $\mathrm{Tdisk}(p, r)$ are homeomorphic.

2. Locally Euclidean Spaces

In the sequel M, M_1, M_2 denote non empty topological spaces.

Let us consider M. We say that M is locally Euclidean if and only if

(Def. 1) Let us consider a point p of M. Then there exists a neighbourhood U of p and there exists n such that $M\restriction U$ and $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ are homeomorphic.

Let us consider n. We say that M is n-locally Euclidean if and only if

(Def. 2) Let us consider a point p of M. Then there exists a neighbourhood U of p such that $M\restriction U$ and $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ are homeomorphic.

Observe that $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ is n-locally Euclidean.

Note that there exists a non empty topological space which is n-locally Euclidean.

Observe that every non empty topological space which is n-locally Euclidean is also locally Euclidean.

3. Locally Euclidean Spaces With and Without a Boundary

Let M be a locally Euclidean non empty topological space. The functor $\operatorname{Int} M$ yielding a subset of M is defined by

(Def. 3) Let us consider a point p of M. Then $p \in it$ if and only if there exists a neighbourhood U of p and there exists n such that $M\restriction U$ and $\mathbb{B}_1(0_{\mathcal{E}^n_{\mathrm{T}}})$ are homeomorphic.

Observe that $\operatorname{Int} M$ is non empty and open.

The functor $\operatorname{Fr} M$ yielding a subset of M is defined by the term

(Def. 4) $(\mathrm{Int}\, M)^{\mathrm{c}}$.

Now we state the proposition:

(2) Boundary Points of Locally Euclidean Spaces:
Let us consider a locally Euclidean non empty topological space M and a point p of M. Then $p \in \mathrm{Fr}\, M$ if and only if there exists a neighbourhood U of p and there exists a natural number n and there exists a function h from $M \restriction U$ into $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ such that h is a homeomorphism and $h(p) \in \mathrm{Sphere}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$. Proof: If $p \in \mathrm{Fr}\, M$, then there exists a neighbourhood U of p and there exists a natural number n and there exists a function h from $M \restriction U$ into $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ such that h is a homeomorphism and $h(p) \in \mathrm{Sphere}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ by [34, (16)], [18, (25)], [6, (94)], [20, (18)]. □

4. Interior and Boundary of Locally Euclidean Spaces

Let M be a locally Euclidean non empty topological space. We say that M is without boundary if and only if

(Def. 5) $\mathrm{Int}\, M =$ the carrier of M.

Let us consider n. Let us observe that $\mathbb{B}_1(0_{\mathcal{E}^n_{\mathrm{T}}})$ is n-locally Euclidean and $\mathbb{B}_1(0_{\mathcal{E}^n_{\mathrm{T}}})$ is without boundary.

Let n be a non zero natural number. Let us observe that $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ has boundary.

Let us consider n. One can check that there exists an n-locally Euclidean non empty topological space which is without boundary.

Let n be a non zero natural number. One can verify that there exists an n-locally Euclidean non empty topological space which is compact and has boundary.

Let M be a without boundary locally Euclidean non empty topological space. Let us observe that $\mathrm{Fr}\, M$ is empty.

Let M be a locally Euclidean non empty topological space with boundary. Observe that $\mathrm{Fr}\, M$ is non empty.

Let n be a zero natural number. Let us observe that every n-locally Euclidean non empty topological space is without boundary.

Now we state the propositions:

(3) M is a without boundary locally Euclidean non empty topological space if and only if for every point p of M, there exists a neighbourhood U of p and there exists n such that $M \restriction U$ and $\mathbb{B}_1(0_{\mathcal{E}^n_{\mathrm{T}}})$ are homeomorphic.

(4) Let us consider a locally Euclidean non empty topological space M with boundary, a point p of M, and n. Suppose there exists a neighbourhood U of p such that $M \restriction U$ and $\mathrm{Tdisk}(0_{\mathcal{E}^{n+1}_{\mathrm{T}}}, 1)$ are homeomorphic. Let us consider a point p_1 of $M \restriction \mathrm{Fr}\, M$. Suppose $p = p_1$. Then there exists a neighbourhood

U of p_1 such that $(M \restriction \mathrm{Fr}\, M) \restriction U$ and $\mathbb{B}_1(0_{\mathcal{E}^n_{\mathrm{T}}})$ are homeomorphic. Proof: Set $n_1 = n + 1$. Set $T_1 = \mathcal{E}^{n_1}_{\mathrm{T}}$. Consider W being a neighbourhood of p such that $M \restriction W$ and $\mathrm{Tdisk}(0_{T_1}, 1)$ are homeomorphic. Set $T_2 = \mathcal{E}^n_{\mathrm{T}}$. Set $S = \mathrm{Sphere}(0_{T_1}, 1)$. Set $F = \mathrm{Fr}\, M$. Set $M_4 = M \restriction F$. Consider U being a neighbourhood of p, m being a natural number, h being a function from $M \restriction U$ into $\mathrm{Tdisk}(0_{\mathcal{E}^m_{\mathrm{T}}}, 1)$ such that h is a homeomorphism and $h(p) \in \mathrm{Sphere}(0_{\mathcal{E}^m_{\mathrm{T}}}, 1)$. Reconsider $I_3 = \mathrm{Int}\, U$ as a subset of $M \restriction U$. Set $M_6 = M \restriction U$. Reconsider $F_1 = F \cap \mathrm{Int}\, U$ as a non empty subset of M_6. Consider W being a subset of T_1 such that $W \in$ the topology of T_1 and $h^\circ I_3 = W \cap \Omega_{\mathrm{Tdisk}(0_{T_1}, 1)}$. Reconsider $h_{14} = h(p)$ as a point of T_1. Reconsider $H_3 = h_{14}$ as a point of $\mathcal{E}^{n_1}$. Consider s being a real number such that $s > 0$ and $\mathrm{Ball}(H_3, s) \subseteq W$. Set $m = \min(\frac{s}{2}, \frac{1}{2})$. Set $V_0 = S \cap \mathrm{Ball}(h_{14}, m)$. Set $h_9 = h^{-1}(V_0)$. $h_9 \subseteq F$ by [20, (9)], (2). Reconsider $h_8 = h^\circ F_1$ as a subset of T_1. $V_0 \subseteq h_8$. $h_8 \cap \mathrm{Ball}(h_{14}, m) \subseteq V_0$ by [11, (67)], [34, (23)], [33, (123)], [31, (5)]. □

Let M be a locally Euclidean non empty topological space. Note that $M \restriction \mathrm{Int}\, M$ is locally Euclidean and $M \restriction \mathrm{Int}\, M$ is without boundary.

Let M be a locally Euclidean non empty topological space with boundary. Note that $M \restriction \mathrm{Fr}\, M$ is locally Euclidean and $M \restriction \mathrm{Fr}\, M$ is without boundary.

5. Cartesian Product of Locally Euclidean Spaces

Let N, M be locally Euclidean non empty topological spaces. Note that $N \times M$ is locally Euclidean.

Let us consider locally Euclidean non empty topological spaces N, M. Now we state the propositions:

(5) $\mathrm{Int}(N \times M) = \mathrm{Int}\, N \times \mathrm{Int}\, M$. Proof: Set $N_1 = N \times M$. Set $I_2 = \mathrm{Int}\, N$. Set $I_1 = \mathrm{Int}\, M$. $\mathrm{Int}\, N_1 \subseteq I_2 \times I_1$ by [9, (87)], (2), [20, (19)], [27, (19), (15)]. □

(6) $\mathrm{Fr}(N \times M) = \Omega_N \times \mathrm{Fr}\, M \cup \mathrm{Fr}\, N \times \Omega_M$. The theorem is a consequence of (5).

Let N, M be without boundary locally Euclidean non empty topological spaces. Let us observe that $N \times M$ is without boundary.

Let N be a locally Euclidean non empty topological space and M be a locally Euclidean non empty topological space with boundary. Note that $N \times M$ has boundary and $M \times N$ has boundary.

6. Fixed Dimension Locally Euclidean Spaces

Let us consider n. Let M be an n-locally Euclidean non empty topological space. Observe that the functor $\operatorname{Int} M$ yields a subset of M and is defined by

(Def. 6) Let us consider a point p of M. Then $p \in it$ if and only if there exists a neighbourhood U of p such that $M \restriction U$ and $\mathbb{B}_1(0_{\mathcal{E}^n_{\rm T}})$ are homeomorphic.

Let us note that the functor $\operatorname{Fr} M$ yields a subset of M and is defined by

(Def. 7) Let us consider a point p of M. Then $p \in it$ if and only if there exists a neighbourhood U of p and there exists a function h from $M \restriction U$ into $\operatorname{Tdisk}(0_{\mathcal{E}^n_{\rm T}}, 1)$ such that h is a homeomorphism and $h(p) \in \operatorname{Sphere}(0_{\mathcal{E}^n_{\rm T}}, 1)$.

Now we state the propositions:

(7) If M_1 is locally Euclidean and M_1 and M_2 are homeomorphic, then M_2 is locally Euclidean.

(8) If M_1 is n-locally Euclidean and M_2 is locally Euclidean and M_1 and M_2 are homeomorphic, then M_2 is n-locally Euclidean.

Now we state the propositions:

(9) Topological Invariance of Dimension of Locally Euclidean Spaces:
If M is n-locally Euclidean and m-locally Euclidean, then $n = m$.

(10) M is a without boundary n-locally Euclidean non empty topological space if and only if for every point p of M, there exists a neighbourhood U of p such that $M \restriction U$ and $\mathbb{B}_1(0_{\mathcal{E}^n_{\rm T}})$ are homeomorphic. Proof: M is n-locally Euclidean by [20, (16)], [16, (9)], [17, (21)], [34, (16)]. M is without boundary. □

Let n, m be elements of $\mathbb{N}$, N be an n-locally Euclidean non empty topological space, and M be an m-locally Euclidean non empty topological space.

Dimension of the Cartesian Product of Locally Euclidean Spaces: $N \times M$ is $(n+m)$-locally Euclidean.

Let us consider n. Let M be an n-locally Euclidean non empty topological space.

Dimension of the Interior of Locally Euclidean Spaces: $M \restriction \operatorname{Int} M$ is n-locally Euclidean as a non empty topological space.

Let n be a non zero natural number and M be an n-locally Euclidean non empty topological space with boundary.

Dimension of the Boundary of Locally Euclidean Spaces: $M \restriction \operatorname{Fr} M$ is $(n -' 1)$-locally Euclidean as a non empty topological space.

7. Connected Components of Locally Euclidean Spaces

Now we state the proposition:

(11) Let us consider a compact locally Euclidean non empty topological space M and a subset C of M. Suppose C is a component. Then

(i) C is open, and

(ii) there exists n such that $M\restriction C$ is an n-locally Euclidean non empty topological space.

Proof: Define $\mathcal{P}[\text{point of } M, \text{subset of } M] \equiv \$_2$ is a neighbourhood of $\$_1$ and there exists n such that $M\restriction\$_2$ and $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ are homeomorphic. Consider p being an object such that $p \in C$. For every point x of M, there exists an element y of 2^α such that $\mathcal{P}[x, y]$, where α is the carrier of M. Consider W being a function from M into $2^{(\text{the carrier of } M)}$ such that for every point x of M, $\mathcal{P}[x, W(x)]$ from [7, Sch. 3]. Reconsider $M_3 = M\restriction C$ as a non empty connected topological space. Define $\mathcal{D}[\text{object}, \text{object}] \equiv \$_2 \in C$ and for every subset A of M such that $A = W(\$_2)$ holds $\mathrm{Int}\, A = \$_1$. Set $I_5 = \{\mathrm{Int}\, U$, where U is a subset of $M : U \in \mathrm{rng}(W\restriction C)\}$. $I_5 \subseteq 2^\alpha$, where α is the carrier of M. Reconsider $R = I_5 \cup \{C^{\mathrm{c}}\}$ as a family of subsets of M. For every subset A of M such that $A \in R$ holds A is open by [9, (136)]. For every subset A of M such that $A \in \mathrm{rng}\, W$ holds A is connected and $\mathrm{Int}\, A$ is not empty by [33, (113)], [23, (14)]. The carrier of $M \subseteq \bigcup R$ by [33, (57)], [6, (47)], [9, (136)]. Consider R_1 being a family of subsets of M such that $R_1 \subseteq R$ and R_1 is a cover of M and R_1 is finite. Set $R_2 = R_1 \setminus \{C^{\mathrm{c}}\}$. Consider x_1 being a set such that $p \in x_1$ and $x_1 \in R_2$. For every set x, $x \in C$ iff there exists a subset Q of M such that Q is open and $Q \subseteq C$ and $x \in Q$ by [34, (16)], [22, (16)]. $\bigcup R_2 \subseteq C$ by [9, (56), (136)], [34, (16)], [6, (47)]. For every object x such that $x \in R_2$ there exists an object y such that $\mathcal{D}[x, y]$ by [9, (56), (136)], [6, (47)]. Consider c being a function such that $\mathrm{dom}\, c = R_2$ and for every object x such that $x \in R_2$ holds $\mathcal{D}[x, c(x)]$ from [2, Sch. 1]. Reconsider $c_3 = c(x_1)$ as a point of M. Consider n such that $M\restriction W(c_3)$ and $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ are homeomorphic. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \overline{\overline{R_2}}$, then there exists a family R_3 of subsets of M such that $\overline{\overline{R_3}} = \$_1$ and $R_3 \subseteq R_2$ and $\bigcup(W^\circ(c^\circ R_3))$ is a connected subset of M and for every subsets A, B of M such that $A \in R_3$ and $B = W(c(A))$ holds $M\restriction B$ and $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ are homeomorphic. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (13), (44)], [1, (68)], [9, (56), (136), (74)]. $\mathcal{P}[0]$ by [9, (2)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. For every point p of M_3, there exists a neighbourhood U of p such that $M_3\restriction U$ and $\mathrm{Tdisk}(0_{\mathcal{E}^n_{\mathrm{T}}}, 1)$ are homeomorphic by [34, (16)], [22, (16), (28)], [34, (22)]. □

Let us consider a compact locally Euclidean non empty topological space M. Now we state the propositions:

(12) There exists a partition P of the carrier of M such that for every subset A of M such that $A \in P$ holds A is open and a component and there exists n such that $M \restriction A$ is an n-locally Euclidean non empty topological space. Proof: Set $P = \{$the component of p, where p is a point of M : not contradiction$\}$. $P \subseteq 2^{\alpha}$, where α is the carrier of M. The carrier of $M \subseteq \bigcup P$ by [23, (38)]. For every subset A of M such that $A \in P$ holds $A \neq \emptyset$ and for every subset B of M such that $B \in P$ holds $A = B$ or A misses B by [23, (42)]. $\square$

(13) If M is connected, then there exists n such that M is n-locally Euclidean. The theorem is a consequence of (11) and (8).

8. Topological Manifold

Let us consider n. Observe that there exists a non empty topological space which is second-countable, Hausdorff, and n-locally Euclidean.

A topological manifold is a second-countable Hausdorff locally Euclidean non empty topological space. Let us consider n. Let M be a topological manifold. We introduce M is n-dimensional as a synonym of M is n-locally Euclidean.

Note that there exists a topological manifold which is n-dimensional and without boundary.

Let n be a non zero natural number. Note that there exists a topological manifold which is n-dimensional and compact and has boundary.

Let M be a topological manifold. Let us observe that every non empty subspace of M is second-countable and Hausdorff.

Let M_1, M_2 be topological manifolds. Observe that $M_1 \times M_2$ is second-countable and Hausdorff.

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