

# Isometric Differentiable Functions on Real Normed Space<sup>1</sup>

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**Summary.** In this article, we formalize isometric differentiable functions on real normed space [17], and their properties.

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The notation and terminology used in this paper have been introduced in the following articles: [3], [2], [8], [4], [5], [18], [10], [11], [19], [14], [16], [1], [6], [9], [15], [23], [24], [21], [22], [13], [25], and [7].

## 1. PRELIMINARIES

From now on  $S, T, W, Y$  denote real normed spaces,  $f, f_1, f_2$  denote partial functions from  $S$  to  $T$ ,  $Z$  denotes a subset of  $S$ , and  $i, n$  denote natural numbers.

Now we state the propositions:

- (1) Let us consider a set  $X$  and functions  $I, f$ . Then  $(f \upharpoonright X) \cdot I = (f \cdot I) \upharpoonright I^{-1}(X)$ .
- (2) Let us consider real normed spaces  $S, T$ , a linear operator  $L$  from  $S$  into  $T$ , and points  $x, y$  of  $S$ . Then  $L(x) - L(y) = L(x - y)$ .

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(3) Let us consider real normed spaces  $X, Y, W$ , a function  $I$  from  $X$  into  $Y$ , and partial functions  $f_1, f_2$  from  $Y$  to  $W$ . Then

(i)  $(f_1 + f_2) \cdot I = f_1 \cdot I + f_2 \cdot I$ , and

(ii)  $(f_1 - f_2) \cdot I = f_1 \cdot I - f_2 \cdot I$ .

PROOF: Set  $D_1 =$  the carrier of  $X$ . For every element  $s$  of  $D_1$ ,  $s \in \text{dom}((f_1 + f_2) \cdot I)$  iff  $s \in \text{dom}(f_1 \cdot I + f_2 \cdot I)$  by [4, (11)]. For every element  $z$  of  $D_1$  such that  $z \in \text{dom}((f_1 + f_2) \cdot I)$  holds  $((f_1 + f_2) \cdot I)(z) = (f_1 \cdot I + f_2 \cdot I)(z)$  by [4, (11), (12)]. For every element  $s$  of  $D_1$ ,  $s \in \text{dom}((f_1 - f_2) \cdot I)$  iff  $s \in \text{dom}(f_1 \cdot I - f_2 \cdot I)$  by [4, (11)]. For every element  $z$  of  $D_1$  such that  $z \in \text{dom}((f_1 - f_2) \cdot I)$  holds  $((f_1 - f_2) \cdot I)(z) = (f_1 \cdot I - f_2 \cdot I)(z)$  by [4, (11), (12)].  $\square$

(4) Let us consider real normed spaces  $X, Y, W$ , a function  $I$  from  $X$  into  $Y$ , a partial function  $f$  from  $Y$  to  $W$ , and a real number  $r$ . Then  $r \cdot (f \cdot I) = (r \cdot f) \cdot I$ . PROOF: Set  $D_1 =$  the carrier of  $X$ . For every element  $s$  of  $D_1$ ,  $s \in \text{dom}((r \cdot f) \cdot I)$  iff  $s \in \text{dom}(f \cdot I)$  by [4, (11)]. For every element  $s$  of  $D_1$ ,  $s \in \text{dom}(r \cdot (f \cdot I))$  iff  $I(s) \in \text{dom}(r \cdot f)$  by [4, (11)]. For every element  $z$  of  $D_1$  such that  $z \in \text{dom}(r \cdot (f \cdot I))$  holds  $(r \cdot (f \cdot I))(z) = ((r \cdot f) \cdot I)(z)$  by [4, (12)].  $\square$

(5) Let us consider a partial function  $f$  from  $T$  to  $W$ , a function  $g$  from  $S$  into  $T$ , and a point  $x$  of  $S$ . Suppose

(i)  $x \in \text{dom } g$ , and

(ii)  $g_x \in \text{dom } f$ , and

(iii)  $g$  is continuous in  $x$ , and

(iv)  $f$  is continuous in  $g_x$ .

Then  $f \cdot g$  is continuous in  $x$ . PROOF: Set  $h = f \cdot g$ . For every real number  $r$  such that  $0 < r$  there exists a real number  $s$  such that  $0 < s$  and for every point  $x_1$  of  $S$  such that  $x_1 \in \text{dom } h$  and  $\|x_1 - x\| < s$  holds  $\|h_{x_1} - h_x\| < r$  by [14, (7)], [12, (3), (4)].  $\square$

Let  $X, Y$  be real normed spaces and  $x$  be an element of  $X \times Y$ . The functor  $\text{reproj1}(x)$  yielding a function from  $X$  into  $X \times Y$  is defined by

(Def. 1) Let us consider an element  $r$  of  $X$ . Then  $it(r) = \langle r, x_2 \rangle$ .

The functor  $\text{reproj2}(x)$  yielding a function from  $Y$  into  $X \times Y$  is defined by

(Def. 2) Let us consider an element  $r$  of  $Y$ . Then  $it(r) = \langle x_1, r \rangle$ .

## 2. ISOMETRIES

Now we state the propositions:

(6) Let us consider a linear operator  $I$  from  $S$  into  $T$  and a point  $x$  of  $S$ . If  $I$  is isometric, then  $I$  is continuous in  $x$ .

- (7) Let us consider real normed spaces  $S$ ,  $T$  and a linear operator  $f$  from  $S$  into  $T$ . Then  $f$  is isometric if and only if for every element  $x$  of  $S$ ,  $\|f(x)\| = \|x\|$ . The theorem is a consequence of (2).
- (8) Let us consider a linear operator  $I$  from  $S$  into  $T$  and a subset  $Z$  of  $S$ . If  $I$  is isometric, then  $I$  is continuous on  $Z$ . The theorem is a consequence of (6).
- (9) Let us consider a linear operator  $I$  from  $S$  into  $T$ . Suppose  $I$  is one-to-one, onto, and isometric. Then there exists a linear operator  $J$  from  $T$  into  $S$  such that
- (i)  $J = I^{-1}$ , and
  - (ii)  $J$  is one-to-one, onto, and isometric.

The theorem is a consequence of (7). PROOF: Reconsider  $J = I^{-1}$  as a function from  $T$  into  $S$ . For every points  $v, w$  of  $T$ ,  $J(v+w) = J(v)+J(w)$  by [5, (113)], [4, (34)]. For every point  $v$  of  $T$  and for every real number  $r$ ,  $J(r \cdot v) = r \cdot J(v)$  by [5, (113)], [4, (34)]. For every point  $v$  of  $T$ ,  $\|J(v)\| = \|v\|$  by [5, (113)], [4, (34)].  $\square$

Let us consider a linear operator  $I$  from  $S$  into  $T$  and a sequence  $s_1$  of  $S$ . Now we state the propositions:

- (10) If  $I$  is isometric and  $s_1$  is convergent, then  $I \cdot s_1$  is convergent and  $\lim(I \cdot s_1) = I(\lim s_1)$ .
- (11) If  $I$  is one-to-one, onto, and isometric, then  $s_1$  is convergent iff  $I \cdot s_1$  is convergent.

Let us consider a linear operator  $I$  from  $S$  into  $T$  and a subset  $Z$  of  $S$ . Now we state the propositions:

- (12) If  $I$  is one-to-one, onto, and isometric, then  $Z$  is closed iff  $I^\circ Z$  is closed.
- (13) If  $I$  is one-to-one, onto, and isometric, then  $Z$  is open iff  $I^\circ Z$  is open.
- (14) If  $I$  is one-to-one, onto, and isometric, then  $Z$  is compact iff  $I^\circ Z$  is compact.

Now we state the propositions:

- (15) Let us consider a partial function  $f$  from  $T$  to  $W$  and a linear operator  $I$  from  $S$  into  $T$ . Suppose  $I$  is one-to-one, onto, and isometric. Let us consider a point  $x$  of  $S$ . Suppose  $I(x) \in \text{dom } f$ . Then  $f \cdot I$  is continuous in  $x$  if and only if  $f$  is continuous in  $I(x)$ . The theorem is a consequence of (9), (6), and (5).
- (16) Let us consider a partial function  $f$  from  $T$  to  $W$ , a linear operator  $I$  from  $S$  into  $T$ , and a set  $X$ . Suppose
- (i)  $X \subseteq \text{the carrier of } T$ , and
  - (ii)  $I$  is one-to-one, onto, and isometric.

Then  $f$  is continuous on  $X$  if and only if  $f \cdot I$  is continuous on  $I^{-1}(X)$ . The theorem is a consequence of (15) and (1). PROOF: For every point  $y$  of  $T$  such that  $y \in X$  holds  $f|_X$  is continuous in  $y$  by [5, (113)], [23, (57)].  
□

Let  $X, Y$  be real normed spaces. The functor  $\text{IsoCPNrSP}(X, Y)$  yielding a linear operator from  $X \times Y$  into  $\prod\langle X, Y \rangle$  is defined by

(Def. 3) Let us consider a point  $x$  of  $X$  and a point  $y$  of  $Y$ . Then  $it(x, y) = \langle x, y \rangle$ .

Now we state the proposition:

(17) Let us consider real normed spaces  $X, Y$ . Then  $0_{\prod\langle X, Y \rangle} = (\text{IsoCPNrSP}(X, Y))(0_{X \times Y})$ .

Let  $X, Y$  be real normed spaces. Observe that  $\text{IsoCPNrSP}(X, Y)$  is one-to-one onto and isometric.

Let us note that there exists a linear operator from  $X \times Y$  into  $\prod\langle X, Y \rangle$  which is one-to-one, onto, and isometric.

Let  $f$  be a one-to-one onto isometric linear operator from  $X \times Y$  into  $\prod\langle X, Y \rangle$ . Let us note that the functor  $f^{-1}$  yields a linear operator from  $\prod\langle X, Y \rangle$  into  $X \times Y$ . One can verify that  $f^{-1}$  is one-to-one onto and isometric as a linear operator from  $\prod\langle X, Y \rangle$  into  $X \times Y$ .

Observe that there exists a linear operator from  $\prod\langle X, Y \rangle$  into  $X \times Y$  which is one-to-one, onto, and isometric.

Now we state the propositions:

(18) Let us consider real normed spaces  $X, Y$ , a point  $x$  of  $X$ , and a point  $y$  of  $Y$ . Then  $(\text{IsoCPNrSP}(X, Y))^{-1}(\langle x, y \rangle) = \langle x, y \rangle$ . PROOF: Set  $I = \text{IsoCPNrSP}(X, Y)$ . Set  $J = I^{-1}$ . For every point  $x$  of  $X$  and for every point  $y$  of  $Y$ ,  $J(\langle x, y \rangle) = \langle x, y \rangle$  by [4, (34)]. □

(19) Let us consider real normed spaces  $X, Y$ . Then  $(\text{IsoCPNrSP}(X, Y))^{-1}(0_{\prod\langle X, Y \rangle}) = 0_{X \times Y}$ . The theorem is a consequence of (17).

(20) Let us consider real normed spaces  $X, Y$  and a subset  $Z$  of  $X \times Y$ . Then  $\text{IsoCPNrSP}(X, Y)$  is continuous on  $Z$ .

(21) Let us consider real normed spaces  $X, Y$  and a subset  $Z$  of  $\prod\langle X, Y \rangle$ . Then  $(\text{IsoCPNrSP}(X, Y))^{-1}$  is continuous on  $Z$ .

(22) Let us consider real normed spaces  $S, T, W$ , a point  $f$  of the real norm space of bounded linear operators from  $S$  into  $W$ , a point  $g$  of the real norm space of bounded linear operators from  $T$  into  $W$ , and a linear operator  $I$  from  $S$  into  $T$ . Suppose

- (i)  $I$  is one-to-one, onto, and isometric, and
- (ii)  $f = g \cdot I$ .

Then  $\|f\| = \|g\|$ . The theorem is a consequence of (9) and (7). PROOF: Consider  $J$  being a linear operator from  $T$  into  $S$  such that  $J = I^{-1}$  and

$J$  is one-to-one, onto, and isometric. Reconsider  $g_0 = g$  as a Lipschitzian linear operator from  $T$  into  $W$ . Reconsider  $g_3 = g \cdot I$  as a Lipschitzian linear operator from  $S$  into  $W$ . For every element  $x$ ,  $x \in \{\|g_0(t)\|\}$ , where  $t$  is a vector of  $T : \|t\| \leq 1\}$  iff  $x \in \{\|g_3(w)\|\}$ , where  $w$  is a vector of  $S : \|w\| \leq 1\}$  by [4, (13), (35)].  $\square$

Let us consider  $S$  and  $T$ . One can verify that every linear operator from  $S$  into  $T$  which is isometric is also Lipschitzian.

### 3. ISOMETRIC DIFFERENTIABLE FUNCTIONS ON REAL NORMED SPACE

Let us consider a real norm space sequence  $G$ , a real normed space  $F$ , a set  $i$ , partial functions  $f, g$  from  $\prod G$  to  $F$ , and a subset  $X$  of  $\prod G$ . Now we state the propositions:

- (23) Suppose  $X$  is open and  $i \in \text{dom } G$  and  $f$  is partially differentiable on  $X$  w.r.t.  $i$  and  $g$  is partially differentiable on  $X$  w.r.t.  $i$ . Then
- (i)  $f + g$  is partially differentiable on  $X$  w.r.t.  $i$ , and
  - (ii)  $(f + g)|^i X = (f|^i X) + (g|^i X)$ .
- (24) Suppose  $X$  is open and  $i \in \text{dom } G$  and  $f$  is partially differentiable on  $X$  w.r.t.  $i$  and  $g$  is partially differentiable on  $X$  w.r.t.  $i$ . Then
- (i)  $f - g$  is partially differentiable on  $X$  w.r.t.  $i$ , and
  - (ii)  $(f - g)|^i X = (f|^i X) - (g|^i X)$ .

Now we state the propositions:

- (25) Let us consider a real norm space sequence  $G$ , a real normed space  $F$ , a set  $i$ , a partial function  $f$  from  $\prod G$  to  $F$ , a real number  $r$ , and a subset  $X$  of  $\prod G$ . Suppose
- (i)  $X$  is open, and
  - (ii)  $i \in \text{dom } G$ , and
  - (iii)  $f$  is partially differentiable on  $X$  w.r.t.  $i$ .

Then

- (iv)  $r \cdot f$  is partially differentiable on  $X$  w.r.t.  $i$ , and
- (v)  $r \cdot f|^i X = r \cdot (f|^i X)$ .

PROOF: Set  $h = r \cdot f$ . For every point  $x$  of  $\prod G$  such that  $x \in X$  holds  $h$  is partially differentiable in  $x$  w.r.t.  $i$  and  $\text{partdiff}(h, x, i) = r \cdot \text{partdiff}(f, x, i)$  by [19, (24), (30)]. Set  $f_3 = f|^i X$ . For every point  $x$  of  $\prod G$  such that  $x \in X$  holds  $(r \cdot f_3)_x = \text{partdiff}(h, x, i)$ .  $\square$

- (26) Let us consider real normed spaces  $S, T$ , a Lipschitzian linear operator  $L$  from  $S$  into  $T$ , and a point  $x_0$  of  $S$ . Then

- (i)  $L$  is differentiable in  $x_0$ , and
- (ii)  $L'(x_0) = L$ .

The theorem is a consequence of (2). PROOF: Reconsider  $L_0 = L$  as a point of the real norm space of bounded linear operators from  $S$  into  $T$ . Reconsider  $R = (\text{the carrier of } S) \mapsto 0_T$  as a partial function from  $S$  to  $T$ . Set  $N =$  the neighbourhood of  $x_0$ . For every point  $x$  of  $S$  such that  $x \in N$  holds  $L_{0x} - L_{0x_0} = L(x - x_0) + R_{x-x_0}$  by [20, (7)], [21, (4)].  $\square$

- (27) Let us consider a partial function  $f$  from  $T$  to  $W$ , a Lipschitzian linear operator  $I$  from  $S$  into  $T$ , and a point  $I_0$  of the real norm space of bounded linear operators from  $S$  into  $T$ . Suppose  $I_0 = I$ . Let us consider a point  $x$  of  $S$ . Suppose  $f$  is differentiable in  $I(x)$ . Then

- (i)  $f \cdot I$  is differentiable in  $x$ , and
- (ii)  $(f \cdot I)'(x) = f'(I(x)) \cdot I_0$ .

The theorem is a consequence of (26).

- (28) Let us consider a partial function  $f$  from  $T$  to  $W$  and a linear operator  $I$  from  $S$  into  $T$ . Suppose

- (i)  $I$  is one-to-one and onto, and
- (ii)  $I$  is isometric.

Let us consider a point  $x$  of  $S$ . Then  $f \cdot I$  is differentiable in  $x$  if and only if  $f$  is differentiable in  $I(x)$ . The theorem is a consequence of (9), (26), and (27).

- (29) Let us consider a partial function  $f$  from  $T$  to  $W$ , a linear operator  $I$  from  $S$  into  $T$ , and a set  $X$ . Suppose

- (i)  $X \subseteq$  the carrier of  $T$ , and
- (ii)  $I$  is one-to-one and onto, and
- (iii)  $I$  is isometric.

Then  $f$  is differentiable on  $X$  if and only if  $f \cdot I$  is differentiable on  $I^{-1}(X)$ . The theorem is a consequence of (28) and (1). PROOF: For every point  $y$  of  $T$  such that  $y \in X$  holds  $f \upharpoonright X$  is differentiable in  $y$  by [5, (113)].  $\square$

- (30) Let us consider real normed spaces  $X, Y$ , a partial function  $f$  from  $\prod\langle X, Y \rangle$  to  $W$ , and a subset  $D$  of  $\prod\langle X, Y \rangle$ . Suppose  $f$  is differentiable on  $D$ . Let us consider a point  $z$  of  $\prod\langle X, Y \rangle$ . Suppose  $z \in \text{dom } f'_{\upharpoonright D}$ . Then  $f'_{\upharpoonright D}(z) = ((f \cdot \text{IsoCPNrSP}(X, Y))'_{\upharpoonright (\text{IsoCPNrSP}(X, Y))^{-1}(D)})_{(\text{IsoCPNrSP}(X, Y))^{-1}(z)} \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$ . The theorem is a consequence of (17), (29), and (27). PROOF: Set  $I = \text{IsoCPNrSP}(X, Y)$ . Set  $J = (\text{IsoCPNrSP}(X, Y))^{-1}$ . Set  $g = f \cdot I$ . Set  $E = I^{-1}(D)$ . For every point  $z$  of  $\prod\langle X, Y \rangle$  such that  $z \in \text{dom } f'_{\upharpoonright D}$  holds  $f'_{\upharpoonright D}(z) = (g'_{\upharpoonright E})_{J(z)} \cdot I^{-1}$  by [10, (31)], [5, (113)], [23, (36)].  $\square$

- (31) Let us consider real normed spaces  $X, Y$ , a partial function  $f$  from  $X \times Y$  to  $W$ , and a subset  $D$  of  $X \times Y$ . Suppose  $f$  is differentiable on  $D$ . Let us consider a point  $z$  of  $X \times Y$ . Suppose  $z \in \text{dom } f'_{|D}$ . Then  $f'_{|D}(z) = ((f \cdot (\text{IsoCPNrSP}(X, Y))^{-1})'_{|((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(D)})_{(\text{IsoCPNrSP}(X, Y))(z)} \cdot ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}$ . The theorem is a consequence of (18), (19), (17), (29), and (27). PROOF: Set  $I = (\text{IsoCPNrSP}(X, Y))^{-1}$ . Set  $J = \text{IsoCPNrSP}(X, Y)$ . Set  $g = f \cdot I$ . Set  $E = I^{-1}(D)$ . For every point  $z$  of  $X \times Y$  such that  $z \in \text{dom } f'_{|D}$  holds  $f'_{|D}(z) = (g'_{|E})_{J(z)} \cdot I^{-1}$  by [10, (31)], [5, (113)], [23, (36)].  $\square$
- (32) Let us consider real normed spaces  $X, Y$  and a point  $z$  of  $X \times Y$ . Then
- (i)  $\text{reproj1}(z) = (\text{IsoCPNrSP}(X, Y))^{-1} \cdot \text{reproj}(1(\in \text{dom}\langle X, Y \rangle), (\text{IsoCPNrSP}(X, Y))(z))$ , and
  - (ii)  $\text{reproj2}(z) = (\text{IsoCPNrSP}(X, Y))^{-1} \cdot \text{reproj}(2(\in \text{dom}\langle X, Y \rangle), (\text{IsoCPNrSP}(X, Y))(z))$ .

The theorem is a consequence of (18).

Let  $X, Y$  be real normed spaces and  $z$  be a point of  $X \times Y$ . Let us note that the functor  $z_1$  yields a point of  $X$ . One can verify that the functor  $z_2$  yields a point of  $Y$ . Let  $X, Y, W$  be real normed spaces. Let  $f$  be a partial function from  $X \times Y$  to  $W$ . We say that  $f$  is partially differentiable in  $z$  w.r.t. 1 if and only if

(Def. 4)  $f \cdot \text{reproj1}(z)$  is differentiable in  $z_1$ .

We say that  $f$  is partially differentiable in  $z$  w.r.t. 2 if and only if

(Def. 5)  $f \cdot \text{reproj2}(z)$  is differentiable in  $z_2$ .

Now we state the propositions:

- (33) Let us consider real normed spaces  $X, Y$  and a point  $z$  of  $X \times Y$ . Then
- (i)  $z_1 = \text{the projection onto } 1(\in \text{dom}\langle X, Y \rangle)((\text{IsoCPNrSP}(X, Y))(z))$ , and
  - (ii)  $z_2 = \text{the projection onto } 2(\in \text{dom}\langle X, Y \rangle)((\text{IsoCPNrSP}(X, Y))(z))$ .
- (34) Let us consider real normed spaces  $X, Y, W$ , a point  $z$  of  $X \times Y$ , and a partial function  $f$  from  $X \times Y$  to  $W$ . Then
- (i)  $f$  is partially differentiable in  $z$  w.r.t. 1 iff  $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$  is partially differentiable in  $(\text{IsoCPNrSP}(X, Y))(z)$  w.r.t. 1, and
  - (ii)  $f$  is partially differentiable in  $z$  w.r.t. 2 iff  $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$  is partially differentiable in  $(\text{IsoCPNrSP}(X, Y))(z)$  w.r.t. 2.

The theorem is a consequence of (32) and (33).

Let  $X, Y, W$  be real normed spaces,  $z$  be a point of  $X \times Y$ , and  $f$  be a partial function from  $X \times Y$  to  $W$ . The functor  $\text{partdiff}(f, z)$  w.r.t. 1 yielding a point of the real norm space of bounded linear operators from  $X$  into  $W$  is defined by the term

(Def. 6)  $(f \cdot \text{reproj1}(z))'(z_1)$ .

The functor  $\text{partdiff}(f, z)$  w.r.t. 2 yielding a point of the real norm space of bounded linear operators from  $Y$  into  $W$  is defined by the term

(Def. 7)  $(f \cdot \text{reproj2}(z))'(z_2)$ .

Now we state the proposition:

(35) Let us consider real normed spaces  $X, Y, W$ , a point  $z$  of  $X \times Y$ , and a partial function  $f$  from  $X \times Y$  to  $W$ . Then

- (i)  $\text{partdiff}(f, z)$  w.r.t. 1 =  $\text{partdiff}(f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}, (\text{IsoCPNrSP}(X, Y))(z), 1)$ , and
- (ii)  $\text{partdiff}(f, z)$  w.r.t. 2 =  $\text{partdiff}(f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}, (\text{IsoCPNrSP}(X, Y))(z), 2)$ .

The theorem is a consequence of (32) and (33).

Let us consider real normed spaces  $X, Y, W$ , a point  $z$  of  $X \times Y$ , and partial functions  $f_1, f_2$  from  $X \times Y$  to  $W$ . Now we state the propositions:

(36) Suppose  $f_1$  is partially differentiable in  $z$  w.r.t. 1 and  $f_2$  is partially differentiable in  $z$  w.r.t. 1. Then

- (i)  $f_1 + f_2$  is partially differentiable in  $z$  w.r.t. 1, and
- (ii)  $\text{partdiff}(f_1 + f_2, z)$  w.r.t. 1 =  $\text{partdiff}(f_1, z)$  w.r.t. 1 +  $\text{partdiff}(f_2, z)$  w.r.t. 1, and
- (iii)  $f_1 - f_2$  is partially differentiable in  $z$  w.r.t. 1, and
- (iv)  $\text{partdiff}(f_1 - f_2, z)$  w.r.t. 1 =  $\text{partdiff}(f_1, z)$  w.r.t. 1 -  $\text{partdiff}(f_2, z)$  w.r.t. 1.

(37) Suppose  $f_1$  is partially differentiable in  $z$  w.r.t. 2 and  $f_2$  is partially differentiable in  $z$  w.r.t. 2. Then

- (i)  $f_1 + f_2$  is partially differentiable in  $z$  w.r.t. 2, and
- (ii)  $\text{partdiff}(f_1 + f_2, z)$  w.r.t. 2 =  $\text{partdiff}(f_1, z)$  w.r.t. 2 +  $\text{partdiff}(f_2, z)$  w.r.t. 2, and
- (iii)  $f_1 - f_2$  is partially differentiable in  $z$  w.r.t. 2, and
- (iv)  $\text{partdiff}(f_1 - f_2, z)$  w.r.t. 2 =  $\text{partdiff}(f_1, z)$  w.r.t. 2 -  $\text{partdiff}(f_2, z)$  w.r.t. 2.

Let us consider real normed spaces  $X, Y, W$ , a point  $z$  of  $X \times Y$ , a real number  $r$ , and a partial function  $f$  from  $X \times Y$  to  $W$ . Now we state the propositions:

(38) Suppose  $f$  is partially differentiable in  $z$  w.r.t. 1. Then

- (i)  $r \cdot f$  is partially differentiable in  $z$  w.r.t. 1, and
- (ii)  $\text{partdiff}(r \cdot f, z)$  w.r.t. 1 =  $r \cdot \text{partdiff}(f, z)$  w.r.t. 1.

(39) Suppose  $f$  is partially differentiable in  $z$  w.r.t. 2. Then



- (i)  $r \cdot f$  is partially differentiable in  $z$  w.r.t. 2, and
- (ii)  $\text{partdiff}(r \cdot f, z)$  w.r.t. 2 =  $r \cdot \text{partdiff}(f, z)$  w.r.t. 2.

Let  $X, Y, W$  be real normed spaces,  $Z$  be a set, and  $f$  be a partial function from  $X \times Y$  to  $W$ . We say that  $f$  is partially differentiable on  $Z$  w.r.t. 1 if and only if

- (Def. 8) (i)  $Z \subseteq \text{dom } f$ , and
- (ii) for every point  $z$  of  $X \times Y$  such that  $z \in Z$  holds  $f \upharpoonright Z$  is partially differentiable in  $z$  w.r.t. 1.

We say that  $f$  is partially differentiable on  $Z$  w.r.t. 2 if and only if

- (Def. 9) (i)  $Z \subseteq \text{dom } f$ , and
- (ii) for every point  $z$  of  $X \times Y$  such that  $z \in Z$  holds  $f \upharpoonright Z$  is partially differentiable in  $z$  w.r.t. 2.

Now we state the proposition:

- (40) Let us consider real normed spaces  $X, Y, W$ , a subset  $Z$  of  $X \times Y$ , and a partial function  $f$  from  $X \times Y$  to  $W$ . Then
- (i)  $f$  is partially differentiable on  $Z$  w.r.t. 1 iff  $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$  is partially differentiable on  $((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$  w.r.t. 1, and
  - (ii)  $f$  is partially differentiable on  $Z$  w.r.t. 2 iff  $f \cdot (\text{IsoCPNrSP}(X, Y))^{-1}$  is partially differentiable on  $((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$  w.r.t. 2.

The theorem is a consequence of (18), (19), (17), (34), and (1). PROOF: Set  $I = (\text{IsoCPNrSP}(X, Y))^{-1}$ . Set  $g = f \cdot I$ . Set  $E = I^{-1}(Z)$ .  $f$  is partially differentiable on  $Z$  w.r.t. 1 iff  $g$  is partially differentiable on  $E$  w.r.t. 1 by [5, (113)], [4, (34)], [5, (38)].  $f$  is partially differentiable on  $Z$  w.r.t. 2 iff  $g$  is partially differentiable on  $E$  w.r.t. 2 by [5, (113)], [4, (34)], [5, (38)].  $\square$

Let  $X, Y, W$  be real normed spaces,  $Z$  be a set, and  $f$  be a partial function from  $X \times Y$  to  $W$ . Assume  $f$  is partially differentiable on  $Z$  w.r.t. 1. The functor  $f \upharpoonright^1 Z$  yielding a partial function from  $X \times Y$  to the real norm space of bounded linear operators from  $X$  into  $W$  is defined by

- (Def. 10) (i)  $\text{dom } it = Z$ , and
- (ii) for every point  $z$  of  $X \times Y$  such that  $z \in Z$  holds  $it_z = \text{partdiff}(f, z)$  w.r.t. 1.

Assume  $f$  is partially differentiable on  $Z$  w.r.t. 2. The functor  $f \upharpoonright^2 Z$  yielding a partial function from  $X \times Y$  to the real norm space of bounded linear operators from  $Y$  into  $W$  is defined by

- (Def. 11) (i)  $\text{dom } it = Z$ , and
- (ii) for every point  $z$  of  $X \times Y$  such that  $z \in Z$  holds  $it_z = \text{partdiff}(f, z)$  w.r.t. 2.

Let us consider real normed spaces  $X, Y, W$ , a subset  $Z$  of  $X \times Y$ , and a partial function  $f$  from  $X \times Y$  to  $W$ . Now we state the propositions:

- (41) Suppose  $f$  is partially differentiable on  $Z$  w.r.t. 1. Then  $f \upharpoonright^1 Z = (f \cdot (\text{IsoCPNrSP}(X, Y))^{-1} \upharpoonright^1 ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)) \cdot \text{IsoCPNrSP}(X, Y)$ .
- (42) Suppose  $f$  is partially differentiable on  $Z$  w.r.t. 2. Then  $f \upharpoonright^2 Z = (f \cdot (\text{IsoCPNrSP}(X, Y))^{-1} \upharpoonright^2 ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)) \cdot \text{IsoCPNrSP}(X, Y)$ .
- (43) Suppose  $Z$  is open. Then  $f$  is partially differentiable on  $Z$  w.r.t. 1 if and only if  $Z \subseteq \text{dom } f$  and for every point  $x$  of  $X \times Y$  such that  $x \in Z$  holds  $f$  is partially differentiable in  $x$  w.r.t. 1.
- (44) Suppose  $Z$  is open. Then  $f$  is partially differentiable on  $Z$  w.r.t. 2 if and only if  $Z \subseteq \text{dom } f$  and for every point  $x$  of  $X \times Y$  such that  $x \in Z$  holds  $f$  is partially differentiable in  $x$  w.r.t. 2.

Let us consider real normed spaces  $X, Y, W$ , a subset  $Z$  of  $X \times Y$ , and partial functions  $f, g$  from  $X \times Y$  to  $W$ . Now we state the propositions:

- (45) Suppose  $Z$  is open and  $f$  is partially differentiable on  $Z$  w.r.t. 1 and  $g$  is partially differentiable on  $Z$  w.r.t. 1. Then
  - (i)  $f + g$  is partially differentiable on  $Z$  w.r.t. 1, and
  - (ii)  $(f + g) \upharpoonright^1 Z = (f \upharpoonright^1 Z) + (g \upharpoonright^1 Z)$ .
- (46) Suppose  $Z$  is open and  $f$  is partially differentiable on  $Z$  w.r.t. 1 and  $g$  is partially differentiable on  $Z$  w.r.t. 1. Then
  - (i)  $f - g$  is partially differentiable on  $Z$  w.r.t. 1, and
  - (ii)  $(f - g) \upharpoonright^1 Z = (f \upharpoonright^1 Z) - (g \upharpoonright^1 Z)$ .
- (47) Suppose  $Z$  is open and  $f$  is partially differentiable on  $Z$  w.r.t. 2 and  $g$  is partially differentiable on  $Z$  w.r.t. 2. Then
  - (i)  $f + g$  is partially differentiable on  $Z$  w.r.t. 2, and
  - (ii)  $(f + g) \upharpoonright^2 Z = (f \upharpoonright^2 Z) + (g \upharpoonright^2 Z)$ .
- (48) Suppose  $Z$  is open and  $f$  is partially differentiable on  $Z$  w.r.t. 2 and  $g$  is partially differentiable on  $Z$  w.r.t. 2. Then
  - (i)  $f - g$  is partially differentiable on  $Z$  w.r.t. 2, and
  - (ii)  $(f - g) \upharpoonright^2 Z = (f \upharpoonright^2 Z) - (g \upharpoonright^2 Z)$ .

Let us consider real normed spaces  $X, Y, W$ , a subset  $Z$  of  $X \times Y$ , a real number  $r$ , and a partial function  $f$  from  $X \times Y$  to  $W$ . Now we state the propositions:

- (49) Suppose  $Z$  is open and  $f$  is partially differentiable on  $Z$  w.r.t. 1. Then
  - (i)  $r \cdot f$  is partially differentiable on  $Z$  w.r.t. 1, and
  - (ii)  $r \cdot f \upharpoonright^1 Z = r \cdot (f \upharpoonright^1 Z)$ .
- (50) Suppose  $Z$  is open and  $f$  is partially differentiable on  $Z$  w.r.t. 2. Then

- (i)  $r \cdot f$  is partially differentiable on  $Z$  w.r.t. 2, and
- (ii)  $r \cdot f \upharpoonright^2 Z = r \cdot (f \upharpoonright^2 Z)$ .

Let us consider real normed spaces  $X, Y, W$ , a subset  $Z$  of  $X \times Y$ , and a partial function  $f$  from  $X \times Y$  to  $W$ . Now we state the propositions:

- (51) Suppose  $f$  is differentiable on  $Z$ . Then  $f'_{\upharpoonright Z}$  is continuous on  $Z$  if and only if  $(f \cdot (\text{IsoCPNrSP}(X, Y))^{-1})'_{\upharpoonright ((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)}$  is continuous on  $((\text{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$ .
- (52) Suppose  $Z$  is open. Then  $f$  is partially differentiable on  $Z$  w.r.t. 1 and  $f$  is partially differentiable on  $Z$  w.r.t. 2 and  $f \upharpoonright^1 Z$  is continuous on  $Z$  and  $f \upharpoonright^2 Z$  is continuous on  $Z$  if and only if  $f$  is differentiable on  $Z$  and  $f'_{\upharpoonright Z}$  is continuous on  $Z$ .

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