The $C^k$ Space\textsuperscript{1}

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Summary. In this article, we formalize continuous differentiability of real-valued functions on $n$-dimensional real normed linear spaces. Next, we give a definition of the $C^k$ space according to [23].

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The notation and terminology used in this paper have been introduced in the following articles: [1], [4], [10], [3], [5], [11], [17], [6], [7], [19], [18], [2], [8], [14], [12], [15], [13], [21], [22], [16], [20], and [9].

1. Definition of Continuously Differentiable Functions and Some Properties

Let $m$ be a non zero element of $\mathbb{N}$, $f$ be a partial function from $\mathbb{R}^m$ to $\mathbb{R}$, $k$ be an element of $\mathbb{N}$, and $Z$ be a set. We say that $f$ is continuously differentiable up to order of $k$ and $Z$ if and only if

(Def. 1) (i) $Z \subseteq \text{dom } f$, and
(ii) $f$ is partial differentiable up to order $k$ and $Z$, and
(iii) for every non empty finite sequence $I$ of elements of $\mathbb{N}$ such that $\text{len } I \leq k$ and $\text{rng } I \subseteq \text{Seg } m$ holds $f[I]Z$ is continuous on $Z$.

Now we state the propositions:

(1) Let us consider a non zero element $m$ of $\mathbb{N}$, a set $Z$, a non empty finite sequence $I$ of elements of $\mathbb{N}$, and a partial function $f$ from $\mathbb{R}^m$ to $\mathbb{R}$. Suppose $f$ is partially differentiable on $Z$ w.r.t. $I$. Then $\text{dom}(f[I]Z) = Z$.

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(2) Let us consider a non zero element \( m \) of \( \mathbb{N} \), an element \( k \) of \( \mathbb{N} \), a non empty subset \( X \) of \( \mathbb{R}^m \), and a partial function \( f \) from \( \mathbb{R}^m \) to \( \mathbb{R} \). Suppose
\[(i) \ X \text{ is open, and}
(ii) \ X \subseteq \text{dom } f.\]
Then \( f \) is continuously differentiable up to order of 1 and \( X \) if and only if \( f \) is differentiable on \( X \) and for every element \( x_0 \) of \( \mathbb{R}^m \) and for every real number \( r \) such that \( x_0 \in X \) and \( 0 < r \) there exists a real number \( s \) such that \( 0 < s \) and for every element \( x_1 \) of \( \mathbb{R}^m \) such that \( x_1 \in X \) and \( |x_1 - x_0| < s \) for every element \( v \) of \( \mathbb{R}^m \), \( |f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v| \).

(3) Let us consider a non zero element \( m \) of \( \mathbb{N} \), a non empty subset \( X \) of \( \mathbb{R}^m \), and a partial function \( f \) from \( \mathbb{R}^m \) to \( \mathbb{R} \). Suppose
\[(i) \ X \text{ is open, and}
(ii) \ X \subseteq \text{dom } f, \text{ and}
(iii) \ f \text{ is continuously differentiable up to order of } 1 \text{ and } X.\]
Then \( f \) is continuous on \( X \). The theorem is a consequence of (2).

(4) Let us consider a non zero element \( m \) of \( \mathbb{N} \), an element \( k \) of \( \mathbb{N} \), a non empty subset \( X \) of \( \mathbb{R}^m \), and partial functions \( f, g \) from \( \mathbb{R}^m \) to \( \mathbb{R} \). Suppose
\[(i) \ f \text{ is continuously differentiable up to order of } k \text{ and } X, \text{ and}
(ii) \ g \text{ is continuously differentiable up to order of } k \text{ and } X, \text{ and}
(iii) \ X \text{ is open.}\]
Then \( f + g \) is continuously differentiable up to order of \( k \) and \( X \). The theorem is a consequence of (1). PROOF: For every non empty finite sequence \( I \) of elements of \( \mathbb{N} \) such that \( \text{len } I \leq k \) and \( \text{rng } I \subseteq \text{Seg } m \) holds \( (f + g)|_I^X \) is continuous on \( X \). □

(5) Let us consider a non zero element \( m \) of \( \mathbb{N} \), an element \( k \) of \( \mathbb{N} \), a non empty subset \( X \) of \( \mathbb{R}^m \), a real number \( r \), and a partial function \( f \) from \( \mathbb{R}^m \) to \( \mathbb{R} \). Suppose
\[(i) \ f \text{ is continuously differentiable up to order of } k \text{ and } X, \text{ and}
(ii) \ X \text{ is open.}\]
Then \( r \cdot f \) is continuously differentiable up to order of \( k \) and \( X \). The theorem is a consequence of (1). PROOF: For every non empty finite sequence \( I \) of elements of \( \mathbb{N} \) such that \( \text{len } I \leq k \) and \( \text{rng } I \subseteq \text{Seg } m \) holds \( r \cdot f|_I^X \) is continuous on \( X \). □

(6) Let us consider a non zero element \( m \) of \( \mathbb{N} \), an element \( k \) of \( \mathbb{N} \), a non empty subset \( X \) of \( \mathbb{R}^m \), and partial functions \( f, g \) from \( \mathbb{R}^m \) to \( \mathbb{R} \). Suppose
\[(i) \ f \text{ is continuously differentiable up to order of } k \text{ and } X, \text{ and}
(ii) \ g \text{ is continuously differentiable up to order of } k \text{ and } X, \text{ and}
(iii) $X$ is open.

Then $f - g$ is continuously differentiable up to order of $k$ and $X$. The theorem is a consequence of (1). Proof: For every non empty finite sequence $I$ of elements of $\mathbb{N}$ such that $\text{len} \ I \leq k$ and $\text{rng} \ I \subseteq \text{Seg} \ m$ holds $(f - g)^{|I}X$ is continuous on $X$. □

Let us consider a non zero element $m$ of $\mathbb{N}$, a non empty subset $Z$ of $\mathbb{R}^m$, a partial function $f$ from $\mathbb{R}^m$ to $\mathbb{R}$, and non empty finite sequences $I, G$ of elements of $\mathbb{N}$. Now we state the propositions:

(7) $f^{|G - I}Z = (f^{|G}Z)^{|I}Z$.

(8) $f^{|G - I}Z$ is continuous on $Z$ if and only if $(f^{|G}Z)^{|I}Z$ is continuous on $Z$.

Now we state the propositions:

(9) Let us consider a non zero element $m$ of $\mathbb{N}$, a non empty subset $Z$ of $\mathbb{R}^m$, a partial function $f$ from $\mathbb{R}^m$ to $\mathbb{R}$, elements $i, j$ of $\mathbb{N}$, and a non empty finite sequence $I$ of elements of $\mathbb{N}$. Suppose

(i) $f$ is continuously differentiable up to order of $i + j$ and $Z$, and
(ii) $\text{rng} \ I \subseteq \text{Seg} \ m$, and
(iii) $\text{len} \ I = j$.

Then $f^{|I}Z$ is continuously differentiable up to order of $i$ and $Z$. The theorem is a consequence of (1) and (7).

(10) Let us consider a non zero element $m$ of $\mathbb{N}$, a non empty subset $Z$ of $\mathbb{R}^m$, a partial function $f$ from $\mathbb{R}^m$ to $\mathbb{R}$, and elements $i, j$ of $\mathbb{N}$. Suppose

(i) $f$ is continuously differentiable up to order of $i$ and $Z$, and
(ii) $j \leq i$.

Then $f$ is continuously differentiable up to order of $j$ and $Z$.

(11) Let us consider a non zero element $m$ of $\mathbb{N}$ and a non empty subset $Z$ of $\mathbb{R}^m$. Suppose $Z$ is open. Let us consider an element $k$ of $\mathbb{N}$ and partial functions $f, g$ from $\mathbb{R}^m$ to $\mathbb{R}$. Suppose

(i) $f$ is continuously differentiable up to order of $k$ and $Z$, and
(ii) $g$ is continuously differentiable up to order of $k$ and $Z$.

Then $f \cdot g$ is continuously differentiable up to order of $k$ and $Z$. The theorem is a consequence of (10), (1), (3), (9), and (7). Proof: Define $\mathcal{P}[\text{element of} \ \mathbb{N}] :=$ for every partial functions $f, g$ from $\mathbb{R}^m$ to $\mathbb{R}$ such that $f$ is continuously differentiable up to order of $k$ and $Z$ and $g$ is continuously differentiable up to order of $k$ and $Z$ holds $f \cdot g$ is continuously differentiable up to order of $k$ and $Z$. Set $Z0 = (0 \ \text{qua} \ \text{natural number})$. $\mathcal{P}[0]$. For every element $k$ of $\mathbb{N}$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. □

(12) Let us consider a non zero element $m$ of $\mathbb{N}$, a partial function $f$ from $\mathbb{R}^m$ to $\mathbb{R}$, a non empty subset $X$ of $\mathbb{R}^m$, and a real number $d$. Suppose
Let us consider an element $x$ of $\mathcal{R}$. If $x \in X$, then $f$ is differentiable in $x$ and $f'(x) = \mathcal{R} \mapsto 0$.

Theorem 13 Let us consider a non zero element $m$ of $\mathbb{N}$, a partial function $f$ from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset $X$ of $\mathcal{R}^m$, and a real number $d$. Suppose

(i) $X$ is open, and
(ii) $f = X \mapsto d$.

Let us consider an element $x_0$ of $\mathcal{R}^m$ and a real number $r$. Suppose

(iii) $x_0 \in X$, and
(iv) $0 < r$.

Then there exists a real number $s$ such that

(v) $0 < s$, and
(vi) for every element $x_1$ of $\mathcal{R}^m$ such that $x_1 \in X$ and $|x_1 - x_0| < s$ for every element $v$ of $\mathcal{R}^m$, $|f'(x_1)(v) - f'(x_0)(v)| \leq r \cdot |v|$.

The theorem is a consequence of (12).

Theorem 14 Let us consider a non zero element $m$ of $\mathbb{N}$, a partial function $f$ from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset $X$ of $\mathcal{R}^m$, and a real number $d$. Suppose

(i) $X$ is open, and
(ii) $f = X \mapsto d$.

Then

(iii) $f$ is differentiable on $X$, and
(iv) $\text{dom} f'\big|_X = X$, and
(v) for every element $x$ of $\mathcal{R}^m$ such that $x \in X$ holds $(f'\big|_X)_x = \mathcal{R} \mapsto 0$.

The theorem is a consequence of (12).

Theorem 15 Let us consider a non zero element $m$ of $\mathbb{N}$, a partial function $f$ from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset $X$ of $\mathcal{R}^m$, a real number $d$, and an element $i$ of $\mathbb{N}$. Suppose

(i) $X$ is open, and
(ii) $f = X \mapsto d$, and
(iii) $1 \leq i \leq m$.

Then

(iv) $f$ is partially differentiable on $X$ w.r.t. $i$, and
(v) $f'\big|_X$ is continuous on $X$.

The theorem is a consequence of (14) and (13).
(16) Let us consider a non zero element \( m \) of \( \mathbb{N} \), an element \( i \) of \( \mathbb{N} \), a partial function \( f \) from \( \mathbb{R}^m \) to \( \mathbb{R} \), a non empty subset \( X \) of \( \mathbb{R}^m \), and a real number \( d \). Suppose

(i) \( X \) is open, and

(ii) \( f = X \mapsto d \), and

(iii) \( 1 \leq i \leq m \).

Then \( f|_X^i = X \mapsto 0 \). The theorem is a consequence of (15) and (12).

Let us consider a non zero element \( m \) of \( \mathbb{N} \), a non empty finite sequence \( I \) of elements of \( \mathbb{N} \), a non empty subset \( X \) of \( \mathbb{R}^m \), a partial function \( f \) from \( \mathbb{R}^m \) to \( \mathbb{R} \), and a real number \( d \). Now we state the propositions:

(17) Suppose \( X \) is open and \( f = X \mapsto d \) and \( \text{rng} \, I \subseteq \text{Seg} \, m \). Then

(i) \( \text{PartDiffSeq}(f, X, I)(0) = X \mapsto d \), and

(ii) for every element \( i \) of \( \mathbb{N} \) such that \( 1 \leq i \leq \text{len} \, I \) holds

\( \text{PartDiffSeq}(f, X, I)(i) = X \mapsto 0 \).

(18) Suppose \( X \) is open and \( f = X \mapsto d \) and \( \text{rng} \, I \subseteq \text{Seg} \, m \). Then

(i) \( f \) is partially differentiable on \( X \) w.r.t. \( I \), and

(ii) \( f|_X^I \) is continuous on \( X \).

Now we state the proposition:

(19) Let us consider a non zero element \( m \) of \( \mathbb{N} \), an element \( k \) of \( \mathbb{N} \), a non empty subset \( X \) of \( \mathbb{R}^m \), a partial function \( f \) from \( \mathbb{R}^m \) to \( \mathbb{R} \), and a real number \( d \). Suppose

(i) \( X \) is open, and

(ii) \( f = X \mapsto d \).

Then \( f \) is continuously differentiable up to order of \( k \) and \( X \). The theorem is a consequence of (18).

Let \( m \) be a non zero element of \( \mathbb{N} \). Observe that there exists a non empty subset of \( \mathbb{R}^m \) which is open.

2. Definition of the \( \mathbb{C}^k \) Space

Let \( m \) be a non zero element of \( \mathbb{N} \), \( k \) be an element of \( \mathbb{N} \), and \( X \) be a non empty open subset of \( \mathbb{R}^m \). The functor the \( \mathbb{C}^k \) functions of \( k \) and \( X \) yielding a non empty subset of \( \mathbb{R} \text{Algebra} \) \( X \) is defined by the term

\[ \{ f \text{ where } f \text{ is a partial function from } \mathbb{R}^m \text{ to } \mathbb{R} : f \text{ is continuously differentiable up to order of } k \text{ and } X \text{ and } \text{dom } f = X \}. \]
Let us note that the $C^k$ functions of $k$ and $X$ is additively linearly closed and multiplicatively closed.

The functor the $\mathbb{R}$ algebra of $C^k$ functions of $k$ and $X$ yielding a subalgebra of $\text{RAlgebra } X$ is defined by the term

$$\langle \text{the } C^k \text{ functions of } k \text{ and } X, \text{ mult(} \text{the } C^k \text{ functions of } k \text{ and } X, \text{ RAlgebra } X), \text{ Add(} \text{the } C^k \text{ functions of } k \text{ and } X, \text{ RAlgebra } X), \text{ One(} \text{the } C^k \text{ functions of } k \text{ and } X, \text{ RAlgebra } X), \text{ Zero(} \text{the } C^k \text{ functions of } k \text{ and } X, \text{ RAlgebra } X) \rangle.$$

Let us note that the $\mathbb{R}$ algebra of $C^k$ functions of $k$ and $X$ is Abelian additive associative right zeroed right complementable vector distributive scalar distributive scalar associative scalar unital commutative associative right unital right distributive and vector associative.

Now we state the propositions:

(20) Let us consider a non zero element $m$ of $\mathbb{N}$, an element $k$ of $\mathbb{N}$, a non empty open subset $X$ of $\mathbb{R}^m$, vectors $F$, $G$, $H$ of the $\mathbb{R}$ algebra of $C^k$ functions of $k$ and $X$, and partial functions $f$, $g$, $h$ from $\mathbb{R}^m$ to $\mathbb{R}$. Suppose

(i) $f = F$, and
(ii) $g = G$, and
(iii) $h = H$.

Then $H = F + G$ if and only if for every element $x$ of $X$, $h(x) = f(x) + g(x)$.

(21) Let us consider a non zero element $m$ of $\mathbb{N}$, an element $k$ of $\mathbb{N}$, a non empty open subset $X$ of $\mathbb{R}^m$, vectors $F$, $G$, $H$ of the $\mathbb{R}$ algebra of $C^k$ functions of $k$ and $X$, partial functions $f$, $g$, $h$ from $\mathbb{R}^m$ to $\mathbb{R}$, and a real number $a$. Suppose

(i) $f = F$, and
(ii) $g = G$.

Then $G = a \cdot F$ if and only if for every element $x$ of $X$, $g(x) = a \cdot f(x)$.

(22) Let us consider a non zero element $m$ of $\mathbb{N}$, an element $k$ of $\mathbb{N}$, a non empty open subset $X$ of $\mathbb{R}^m$, vectors $F$, $G$, $H$ of the $\mathbb{R}$ algebra of $C^k$ functions of $k$ and $X$, and partial functions $f$, $g$, $h$ from $\mathbb{R}^m$ to $\mathbb{R}$. Suppose

(i) $f = F$, and
(ii) $g = G$, and
(iii) $h = H$.

Then $H = F \cdot G$ if and only if for every element $x$ of $X$, $h(x) = f(x) \cdot g(x)$.

Let us consider a non zero element $m$ of $\mathbb{N}$, an element $k$ of $\mathbb{N}$, and a non empty open subset $X$ of $\mathbb{R}^m$. Now we state the propositions:

(23) $0_\alpha = X \mapsto 0$, where $\alpha$ is the $\mathbb{R}$ algebra of $C^k$ functions of $k$ and $X$.
(24) $1_\alpha = X \mapsto 1$, where $\alpha$ is the $\mathbb{R}$ algebra of $C^k$ functions of $k$ and $X$. 
The $C^k$ space

References


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