FORMALIZED MATHEMATICS
Vol. 20, No. 4, Pages 281–290, 2012
DOI: 10.2478/v10037-012-0034-9

VERSITA
versita.com/fm/

Cayley-Dickson Construction[1]

Artur Korniłowicz
Institute of Informatics
University of Białystok
Sosnowa 64, 15-887 Białystok
Poland

Summary. Cayley-Dickson construction produces a sequence of normed algebras over real numbers. Its consequent applications result in complex numbers, quaternions, octonions, etc. In this paper we formalize the construction and prove its basic properties.

MML identifier: `CAYLDICK`, version: `8.0.01 5.3.1162`

The notation and terminology used here have been introduced in the following papers: [22], [12], [3], [1], [9], [8], [16], [13], [4], [5], [19], [15], [17], [14], [2], [6], [23], [20], [18], [21], [10], [11], and [7].

1. Preliminaries

We use the following convention: u, v, x, y, z, X, Y are sets and r, s are real numbers.

One can prove the following proposition

(1) For all real numbers a, b, c, d holds $(a+b)^{\mathbf{2}} + (c+d)^{\mathbf{2}} \leq (\sqrt{a^{\mathbf{2}} + c^{\mathbf{2}}} + \sqrt{b^{\mathbf{2}} + d^{\mathbf{2}}})^{\mathbf{2}}$.

Let X be a non trivial real normed space and let x be a non zero element of X. One can verify that $\|x\|$ is positive.

Let c be a non zero complex number. Note that $c^{\mathbf{2}}$ is non zero.

[1]This work has been supported by the Polish Ministry of Science and Higher Education project “Managing a Large Repository of Computer-verified Mathematical Knowledge” (N N519 385136).

Let x be a non empty set. Observe that $\langle x \rangle$ is non-empty.

Let us note that there exists a finite 0-sequence which is non-empty.

Let f, g be non-empty finite 0-sequences. Observe that $f \frown g$ is non-empty.

Let x, y be non empty sets. One can verify that $\langle x, y \rangle$ is non-empty.

The following propositions are true:

(2) If $\langle u \rangle = \langle x \rangle$, then $u = x$.

(3) If $\langle u, v \rangle = \langle x, y \rangle$, then $u = x$ and $v = y$.

(4) If $x \in X$, then $\langle x \rangle \in \prod \langle X \rangle$.

(5) If $z \in \prod \langle X \rangle$, then there exists x such that $x \in X$ and $z = \langle x \rangle$.

(6) If $x \in X$ and $y \in Y$, then $\langle x, y \rangle \in \prod \langle X, Y \rangle$.

(7) If $z \in \prod \langle X, Y \rangle$, then there exist x, y such that $x \in X$ and $y \in Y$ and $z = \langle x, y \rangle$.

Let D be a set. The functor $\operatorname{binop} D$ yielding a binary operation on D is defined by:

(Def. 1) $\operatorname{binop} D = D \times D \longmapsto$ the element of D.

Let D be a set. Observe that $\operatorname{binop} D$ is associative and commutative.

Let D be a set. One can verify that there exists a binary operation on D which is associative and commutative.

2. Conjunctive Normed Spaces

We introduce conjunctive normed algebra structures which are extensions of normed algebra structures and are systems

⟨ a carrier, a multiplication, an addition, an external multiplication, a one, a zero, a norm, a conjugate ⟩,

where the carrier is a set, the multiplication and the addition are binary operations on the carrier, the external multiplication is a function from $\mathbb{R} \times$ the carrier into the carrier, the one and the zero are elements of the carrier, the norm is a function from the carrier into $\mathbb{R}$, and the conjugate is a function from the carrier into the carrier.

Let us observe that there exists a conjunctive normed algebra structure which is non trivial and strict.

We use the following convention: N is a non empty conjunctive normed algebra structure and a, a_1, a_2, b, b_1, b_2 are elements of N.

Let N be a non empty conjunctive normed algebra structure and let a be an element of N. The functor $\overline{a}$ yields an element of N and is defined as follows:

(Def. 2) $\overline{a} = (\text{the conjugate of } N)(a)$.

Let N be a non empty conjunctive normed algebra structure and let a be an element of N. We say that a is properly conjugated if and only if:

(Def. 3)(i) $\overline{a} \cdot a = \|a\|^{\mathbf{2}} \cdot 1_N$ if a is non zero,

(ii) $\overline{a}$ is zero, otherwise.

Let N be a non empty conjunctive normed algebra structure. We say that N is properly conjugated if and only if:

(Def. 4) Every element of N is properly conjugated.

We say that N is additively conjugative if and only if:

(Def. 5) For all elements a, b of N holds $\overline{a+b} = \overline{a} + \overline{b}$.

We say that N is norm-wise conjugative if and only if:

(Def. 6) For every element a of N holds $\|\overline{a}\| = \|a\|$.

We say that N is scalar-wise conjugative if and only if:

(Def. 7) For every real number r and for every element a of N holds $r \cdot \overline{a} = \overline{r \cdot a}$.

Let D be a real-membered set, let a, m be binary operations on D, let M be a function from $\mathbb{R} \times D$ into D, let O, Z be elements of D, let n be a function from D into $\mathbb{R}$, and let c be a function from D into D. Observe that $\langle D, m, a, M, O, Z, n, c \rangle$ is real-membered.

Let D be a set, let a be an associative binary operation on D, let m be a binary operation on D, let M be a function from $\mathbb{R} \times D$ into D, let O, Z be elements of D, let n be a function from D into $\mathbb{R}$, and let c be a function from D into D. Observe that $\langle D, m, a, M, O, Z, n, c \rangle$ is add-associative.

Let D be a set, let a be a commutative binary operation on D, let m be a binary operation on D, let M be a function from $\mathbb{R} \times D$ into D, let O, Z be elements of D, let n be a function from D into $\mathbb{R}$, and let c be a function from D into D. Observe that $\langle D, m, a, M, O, Z, n, c \rangle$ is Abelian.

Let D be a set, let a be a binary operation on D, let m be an associative binary operation on D, let M be a function from $\mathbb{R} \times D$ into D, let O, Z be elements of D, let n be a function from D into $\mathbb{R}$, and let c be a function from D into D. One can verify that $\langle D, m, a, M, O, Z, n, c \rangle$ is associative.

Let D be a set, let a be a binary operation on D, let m be a commutative binary operation on D, let M be a function from $\mathbb{R} \times D$ into D, let O, Z be elements of D, let n be a function from D into $\mathbb{R}$, and let c be a function from D into D. One can check that $\langle D, m, a, M, O, Z, n, c \rangle$ is commutative.

The strict conjunctive normed algebra structure N-Real is defined by:

(Def. 8) N-Real $= \langle \mathbb{R}, \cdot_{\mathbb{R}}, +_{\mathbb{R}}, \cdot_{\mathbb{R}}, 1(\in \mathbb{R}), 0(\in \mathbb{R}), |\square|_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}} \rangle$.

Let us observe that N-Real is non degenerated, real-membered, add-associative, Abelian, associative, and commutative. Let a, b be elements of N-Real and r, s be real numbers. We identify $r+s$ with $a+b$ where $a = r$ and $b = s$. We identify $r \cdot s$ with $a \cdot b$ where $a = r$ and $b = s$.

One can check the following observations:

* every Abelian non empty additive magma which is right add-cancelable is also left add-cancelable,

* every Abelian non empty additive magma which is left add-cancelable is also right add-cancelable,
* every Abelian non empty additive loop structure which is left complementable is also right complementable,
* every Abelian commutative non empty double loop structure which is left distributive is also right distributive,
* every Abelian commutative non empty double loop structure which is right distributive is also left distributive,
* every commutative non empty multiplicative loop with zero structure which is almost left invertible is also almost right invertible,
* every commutative non empty multiplicative loop with zero structure which is almost right invertible is also almost left invertible,
* every commutative non empty multiplicative loop with zero structure which is almost right cancelable is also almost left cancelable,
* every commutative non empty multiplicative loop with zero structure which is almost left cancelable is also almost right cancelable,
* every commutative non empty multiplicative magma which is right mult-cancelable is also left mult-cancelable, and
* every commutative non empty multiplicative magma which is left mult-cancelable is also right mult-cancelable.

One can verify that N-Real is right complementable and right add-cancelable.

We identify $-r$ with $-a$ where $a = r$.

We identify $r - s$ with $a - b$ where $a = r$ and $b = s$.

We identify $r \cdot s$ with $r \cdot a$ where $a = s$.

We identify $|a|$ with $\|a\|$.

The following proposition is true

(8) For every element a of N-Real holds $a \cdot a = \|a\|^{\mathbf{2}}$.

Let us observe that $\overline{a}$ reduces to a.

One can verify that N-Real is reflexive, discernible, well unital, real normed space-like, right zeroed, right distributive, vector associative, vector distributive, scalar distributive, scalar associative, scalar unital, Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, almost left invertible, almost left cancelable, properly conjugated, additively conjugative, norm-wise conjugative, and scalar-wise conjugative.

One can verify that there exists a non empty conjunctive normed algebra structure which is strict, non degenerated, real-membered, reflexive, discernible, zeroed, complementable, add-associative, Abelian, associative, commutative, distributive, well unital, add-cancelable, vector associative, vector distributive, scalar distributive, scalar associative, scalar unital, Banach Algebra-like1, Banach Algebra-like2, Banach Algebra-like3, properly conjugated, additively conjugative, norm-wise conjugative, scalar-wise conjugative, almost left invertible, almost left cancelable, and real normed space-like.

One can check that $0_{\text{N-Real}}$ is non left invertible and non right invertible.

We identify r^{-1} with a^{-1} where $a = r$.

Let X be a discernible non trivial conjunctive normed algebra structure and let x be a non zero element of X. One can check that $\|x\|$ is non zero.

Let us mention that every non zero element of N-Real is non empty.

Let us observe that every non zero element of N-Real is mult-cancelable.

Let N be a properly conjugated non empty conjunctive normed algebra structure. Observe that every element of N is properly conjugated.

Let N be a properly conjugated non empty conjunctive normed algebra structure and let a be a zero element of N. Observe that $\overline{a}$ is zero.

Let us observe that $\overline{0_N}$ reduces to 0_N.

Let N be a properly conjugated discernible add-associative right zeroed right complementable left distributive scalar distributive scalar associative scalar unital vector distributive non degenerated conjunctive normed algebra structure and let a be a non zero element of N. Note that $\overline{a}$ is non zero.

The following propositions are true:

(9) Suppose that N is add-associative, right zeroed, right complementable, properly conjugated, reflexive, scalar distributive, scalar unital, vector distributive, and left distributive. Let given a. Then $\overline{a} \cdot a = \|a\|^{\mathbf{2}} \cdot 1_N$.

Let N be left unital Banach Algebra-like2 almost right cancelable properly conjugated scalar unital nonempty conjunctive normed algebra structure. Let us observe that $\overline{\overline{a}}$ reduces to a.

Let N be right unital Banach Algebra-like2 almost right cancelable properly conjugated scalar unital nonempty conjunctive normed algebra structure. Let us observe that $\overline{1_N}$ reduces to 1_N.

(10) Suppose that N is properly conjugated, reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, right complementable, associative, distributive, well unital, non degenerated, and almost left invertible. Then $\overline{-a} = -\overline{a}$.

(11) Suppose that N is properly conjugated, reflexive, discernible, real normed space-like, vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, right complementable, associative, distributive, well unital, non degenerated, almost left invertible, and additively conjugative. Then $\overline{a - b} = \overline{a} - \overline{b}$.

3. Cayley-Dickson Construction

Let N be a non empty conjunctive normed algebra structure. The functor Cayley-Dickson N yielding a strict conjunctive normed algebra structure is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of Cayley-Dickson $N = \prod\langle$the carrier of N, the carrier of $N\rangle$,

(ii) the zero of Cayley-Dickson $N = \langle 0_N, 0_N \rangle$,

(iii) the one of Cayley-Dickson $N = \langle 1_N, 0_N \rangle$,

(iv) for all elements a_1, a_2, b_1, b_2 of N holds (the addition of Cayley-Dickson N)$(\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle) = \langle a_1 + a_2, b_1 + b_2 \rangle$ and (the multiplication of Cayley-Dickson N)$(\langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle) = \langle a_1 \cdot a_2 - \overline{b_2} \cdot b_1, b_2 \cdot a_1 + b_1 \cdot \overline{a_2} \rangle$,

(v) for every real number r and for all elements a, b of N holds (the external multiplication of Cayley-Dickson N)$(r, \langle a, b \rangle) = \langle r \cdot a, r \cdot b \rangle$, and

(vi) for all elements a, b of N holds (the norm of Cayley-Dickson N)$(\langle a, b \rangle) = \sqrt{\|a\|^{\mathbf{2}} + \|b\|^{\mathbf{2}}}$ and (the conjugate of Cayley-Dickson N)$(\langle a, b \rangle) = \langle \overline{a}, -b \rangle$.

In the sequel c, c_1, c_2 are elements of Cayley-Dickson N.

Let N be a non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is non empty.

We now state two propositions:

(12) There exist elements a, b of N such that $c = \langle a, b \rangle$.

(13) For every element c of Cayley-Dickson Cayley-Dickson N there exist a_1, b_1, a_2, b_2 such that $c = \langle \langle a_1, b_1 \rangle, \langle a_2, b_2 \rangle \rangle$.

Let us consider N, a, b. Then $\langle a, b \rangle$ is an element of Cayley-Dickson N.

Let us consider N and let a, b be zero elements of N. Observe that $\langle a, b \rangle$ is zero.

Let N be a non degenerated non empty conjunctive normed algebra structure, let a be a non zero element of N, and let b be an element of N. One can check that $\langle a, b \rangle$ is non zero.

Let N be a reflexive non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is reflexive.

Let N be a discernible non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is discernible.

We now state a number of propositions:

(14) If a is left complementable and b is left complementable, then $\langle a, b \rangle$ is left complementable.

(15) If $\langle a, b \rangle$ is left complementable, then a is left complementable and b is left complementable.

(16) If a is right complementable and b is right complementable, then $\langle a, b \rangle$ is right complementable.

(17) If $\langle a, b\rangle$ is right complementable, then a is right complementable and b is right complementable.

(18) If a is complementable and b is complementable, then $\langle a, b\rangle$ is complementable.

(19) If $\langle a, b\rangle$ is complementable, then a is complementable and b is complementable.

(20) If a is left add-cancelable and b is left add-cancelable, then $\langle a, b\rangle$ is left add-cancelable.

(21) If $\langle a, b\rangle$ is left add-cancelable, then a is left add-cancelable and b is left add-cancelable.

(22) If a is right add-cancelable and b is right add-cancelable, then $\langle a, b\rangle$ is right add-cancelable.

(23) If $\langle a, b\rangle$ is right add-cancelable, then a is right add-cancelable and b is right add-cancelable.

(24) If a is add-cancelable and b is add-cancelable, then $\langle a, b\rangle$ is add-cancelable.

(25) If $\langle a, b\rangle$ is add-cancelable, then a is add-cancelable and b is add-cancelable.

(26) If $\langle a, b\rangle$ is left complementable and right add-cancelable, then $-\langle a, b\rangle = \langle -a, -b\rangle$.

Let N be an add-associative non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is add-associative.

Let N be a right zeroed non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is right zeroed.

Let N be a left zeroed non empty conjunctive normed algebra structure. One can verify that Cayley-Dickson N is left zeroed.

Let N be a right complementable non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is right complementable.

Let N be a left complementable non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is left complementable.

Let N be an Abelian non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is Abelian.

One can prove the following propositions:

(27) If N is add-associative, right zeroed, and right complementable, then $-\langle a, b\rangle = \langle -a, -b\rangle$.

(28) If N is add-associative, right zeroed, and right complementable, then $\langle a_1, b_1\rangle - \langle a_2, b_2\rangle = \langle a_1 - a_2, b_1 - b_2\rangle$.

Let N be a well unital add-associative right zeroed right complementable distributive Banach Algebra-like2 properly conjugated scalar unital almost right cancelable non empty conjunctive normed algebra structure. Observe that

Cayley-Dickson N is well unital.

Let N be a non degenerated non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is non degenerated.

Let N be an additively conjugative add-associative right zeroed right complementable Abelian non empty conjunctive normed algebra structure. One can verify that Cayley-Dickson N is additively conjugative.

Let N be a norm-wise conjugative reflexive discernible real normed space-like vector distributive scalar distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is norm-wise conjugative.

Let N be a scalar-wise conjugative add-associative right zeroed right complementable Abelian scalar distributive scalar associative scalar unital vector distributive non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is scalar-wise conjugative.

Let N be a distributive add-associative right zeroed right complementable Abelian non empty conjunctive normed algebra structure.

Note that Cayley-Dickson N is left distributive.

Let N be a distributive add-associative right zeroed right complementable additively conjugative Abelian non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is right distributive.

Let N be a reflexive discernible real normed space-like vector distributive scalar distributive scalar associative scalar unital Abelian add-associative right zeroed right complementable non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is real normed space-like.

Let N be a vector distributive non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is vector distributive.

Let N be a vector associative Banach Algebra-like3 add-associative right zeroed right complementable Abelian scalar distributive scalar associative scalar unital vector distributive non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is vector associative.

Let N be a scalar distributive non empty conjunctive normed algebra structure. One can verify that Cayley-Dickson N is scalar distributive.

Let N be a scalar associative non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is scalar associative.

Let N be a scalar unital non empty conjunctive normed algebra structure. One can check that Cayley-Dickson N is scalar unital.

Let N be a reflexive Banach Algebra-like2 non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is Banach Algebra-like2.

Let N be a Banach Algebra-like3 add-associative right zeroed right complementable Abelian scalar distributive scalar associative scalar unital vector distributive vector associative scalar-wise conjugative non empty conjunctive normed algebra structure. Observe that Cayley-Dickson N is Banach Algebra-like3.

Next we state the proposition

(29) Let N be an almost left invertible associative add-associative right zeroed right complementable well unital distributive Abelian scalar distributive scalar associative scalar unital vector distributive vector associative reflexive discernible real normed space-like almost right cancelable properly conjugated additively conjugative Banach Algebra-like2 Banach Algebra-like3 non degenerated conjunctive normed algebra structure and a, b be elements of N. Suppose a is non zero or b is non zero but $\langle a, b\rangle$ is right mult-cancelable and left invertible. Then $\langle a, b\rangle^{-1} = \langle \frac{1}{\|a\|^2+\|b\|^2} \cdot \overline{a}, \frac{1}{\|a\|^2+\|b\|^2} \cdot -b\rangle$.

Let N be an add-associative right zeroed right complementable distributive scalar distributive scalar unital vector distributive discernible reflexive properly conjugated non empty conjunctive normed algebra structure. Note that Cayley-Dickson N is properly conjugated.

Let us mention that Cayley-Dickson N-Real is associative and commutative.

The following propositions are true:

(30) $\langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle \cdot \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle = \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle\rangle$.

(31) $\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle \cdot \langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle = \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, -1_{\text{N-Real}}\rangle\rangle$.

One can verify that Cayley-Dickson Cayley-Dickson N-Real is associative and non commutative.

We now state four propositions:

(32) $\langle\langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle = \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle$.

(33) $\langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot \langle\langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle = \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, -1_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle$.

(34) $\langle\langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle = \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle -1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle$.

(35) $\langle\langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot (\langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle \cdot \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 1_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle) = \langle\langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle, \langle\langle 0_{\text{N-Real}}, 0_{\text{N-Real}}\rangle, \langle 1_{\text{N-Real}}, 0_{\text{N-Real}}\rangle\rangle\rangle$.

One can check that Cayley-Dickson Cayley-Dickson Cayley-Dickson N-Real is non associative and non commutative.

References

[1] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[10] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[11] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[12] Michał Muzalewski and Wojciech Skaba. From loops to abelian multiplicative groups with zero. *Formalized Mathematics*, 1(**5**):833–840, 1990.
[13] Henryk Oryszczyszyn and Krzysztof Prażmowski. Real functions spaces. *Formalized Mathematics*, 1(**3**):555–561, 1990.
[14] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[15] Yasunari Shidama. The Banach algebra of bounded linear operators. *Formalized Mathematics*, 12(**2**):103–108, 2004.
[16] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[17] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(**4**):341–347, 2003.
[18] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[19] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[20] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[22] Tetsuya Tsunetou, Grzegorz Bancerek, and Yatsuka Nakamura. Zero-based finite sequences. *Formalized Mathematics*, 9(**4**):825–829, 2001.
[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received August 6, 2012