

Quotient Module of \mathbb{Z} -module¹

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Summary. In this article we formalize a quotient module of \mathbb{Z} -module and a vector space constructed by the quotient module. We formally prove that for a \mathbb{Z} -module V and a prime number p , a quotient module V/pV has the structure of a vector space over \mathbb{F}_p . \mathbb{Z} -module is necessary for lattice problems, LLL (Lenstra, Lenstra and Lovász) base reduction algorithm and cryptographic systems with lattices [14]. Some theorems in this article are described by translating theorems in [20] and [19] into theorems of \mathbb{Z} -module.

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The terminology and notation used here have been introduced in the following articles: [4], [1], [16], [3], [21], [9], [5], [6], [18], [13], [15], [17], [2], [7], [11], [24], [25], [22], [20], [23], [12], [8], and [10].

1. QUOTIENT MODULE OF \mathbb{Z} -MODULE AND VECTOR SPACE

For simplicity, we follow the rules: x is a set, V is a \mathbb{Z} -module, u, v are vectors of V , F, G, H are finite sequences of elements of V , i is an element of \mathbb{N} , and f, g are sequences of V .

Let V be a \mathbb{Z} -module and let a be an integer number. The functor $a \cdot V$ yielding a non empty subset of V is defined by:

(Def. 1) $a \cdot V = \{a \cdot v : v \text{ ranges over elements of } V\}$.

Let V be a \mathbb{Z} -module and let a be an integer number. The functor $\text{Zero}(a, V)$ yielding an element of $a \cdot V$ is defined as follows:

(Def. 2) $\text{Zero}(a, V) = 0_V$.

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Let V be a \mathbb{Z} -module and let a be an integer number. The functor $\text{Add}(a, V)$ yielding a function from $(a \cdot V) \times (a \cdot V)$ into $a \cdot V$ is defined by:

(Def. 3) $\text{Add}(a, V) = (\text{the addition of } V) \upharpoonright ((a \cdot V) \times (a \cdot V))$.

Let V be a \mathbb{Z} -module and let a be an integer number. The functor $\text{Mult}(a, V)$ yielding a function from $\mathbb{Z} \times (a \cdot V)$ into $a \cdot V$ is defined by:

(Def. 4) $\text{Mult}(a, V) = (\text{the external multiplication of } V) \upharpoonright (\mathbb{Z} \times (a \cdot V))$.

Let V be a \mathbb{Z} -module and let a be an integer number. The functor $a \circ V$ yields a submodule of V and is defined as follows:

(Def. 5) $a \circ V = \langle a \cdot V, \text{Zero}(a, V), \text{Add}(a, V), \text{Mult}(a, V) \rangle$.

Let V be a \mathbb{Z} -module and let W be a submodule of V . The functor $\text{CosetSet}(V, W)$ yields a non empty family of subsets of V and is defined as follows:

(Def. 6) $\text{CosetSet}(V, W) = \{A : A \text{ ranges over cosets of } W\}$.

Let V be a \mathbb{Z} -module and let W be a submodule of V . The functor $\text{addCoset}(V, W)$ yields a binary operation on $\text{CosetSet}(V, W)$ and is defined as follows:

(Def. 7) For all elements A, B of $\text{CosetSet}(V, W)$ and for all vectors a, b of V such that $A = a + W$ and $B = b + W$ holds $(\text{addCoset}(V, W))(A, B) = a + b + W$.

Let V be a \mathbb{Z} -module and let W be a submodule of V . The functor $\text{zeroCoset}(V, W)$ yielding an element of $\text{CosetSet}(V, W)$ is defined by:

(Def. 8) $\text{zeroCoset}(V, W) = \text{the carrier of } W$.

Let V be a \mathbb{Z} -module and let W be a submodule of V . The functor $\text{lmultCoset}(V, W)$ yields a function from $\mathbb{Z} \times \text{CosetSet}(V, W)$ into $\text{CosetSet}(V, W)$ and is defined as follows:

(Def. 9) For every integer z and for every element A of $\text{CosetSet}(V, W)$ and for every vector a of V such that $A = a + W$ holds $(\text{lmultCoset}(V, W))(z, A) = z \cdot a + W$.

Let V be a \mathbb{Z} -module and let W be a submodule of V . The functor $\mathbb{Z}\text{-ModuleQuot}(V, W)$ yields a strict \mathbb{Z} -module and is defined by the conditions (Def. 10).

- (Def. 10)(i) The carrier of $\mathbb{Z}\text{-ModuleQuot}(V, W) = \text{CosetSet}(V, W)$,
(ii) the addition of $\mathbb{Z}\text{-ModuleQuot}(V, W) = \text{addCoset}(V, W)$,
(iii) $0_{\mathbb{Z}\text{-ModuleQuot}(V, W)} = \text{zeroCoset}(V, W)$, and
(iv) the external multiplication of $\mathbb{Z}\text{-ModuleQuot}(V, W) = \text{lmultCoset}(V, W)$.

The following propositions are true:

- (1) Let p be an integer, V be a \mathbb{Z} -module, W be a submodule of V , and x be a vector of $\mathbb{Z}\text{-ModuleQuot}(V, W)$. If $W = p \circ V$, then $p \cdot x = 0_{\mathbb{Z}\text{-ModuleQuot}(V, W)}$.

- (2) Let p, i be integers, V be a \mathbb{Z} -module, W be a submodule of V , and x be a vector of $\mathbb{Z}\text{-ModuleQuot}(V, W)$. If $p \neq 0$ and $W = p \circ V$, then $i \cdot x = (i \bmod p) \cdot x$.
- (3) Let p, q be integers, V be a \mathbb{Z} -module, W be a submodule of V , and v be a vector of V . Suppose $W = p \circ V$ and $p > 1$ and $q > 1$ and p and q are relative prime. If $q \cdot v = 0_V$, then $v + W = 0_{\mathbb{Z}\text{-ModuleQuot}(V, W)}$.

Let p be a prime number and let V be a \mathbb{Z} -module. The functor $\text{MultModpV}(V, p)$ yields a function from (the carrier of $\text{GF}(p)$) \times (the carrier of $\mathbb{Z}\text{-ModuleQuot}(V, p \circ V)$) into the carrier of $\mathbb{Z}\text{-ModuleQuot}(V, p \circ V)$ and is defined by the condition (Def. 11).

- (Def. 11) Let a be an element of $\text{GF}(p)$, i be an integer, and x be an element of $\mathbb{Z}\text{-ModuleQuot}(V, p \circ V)$. If $a = i \bmod p$, then $(\text{MultModpV}(V, p))(a, x) = (i \bmod p) \cdot x$.

Let p be a prime number and let V be a \mathbb{Z} -module. The functor $\mathbb{Z}\text{-MQVectSp}(V, p)$ yielding a non empty strict vector space structure over $\text{GF}(p)$ is defined by:

- (Def. 12) $\mathbb{Z}\text{-MQVectSp}(V, p) = \langle \text{the carrier of } \mathbb{Z}\text{-ModuleQuot}(V, p \circ V), \text{ the addition of } \mathbb{Z}\text{-ModuleQuot}(V, p \circ V), \text{ the zero of } \mathbb{Z}\text{-ModuleQuot}(V, p \circ V), \text{MultModpV}(V, p) \rangle$.

Let p be a prime number and let V be a \mathbb{Z} -module. Observe that $\mathbb{Z}\text{-MQVectSp}(V, p)$ is scalar distributive, vector distributive, scalar associative, scalar unital, add-associative, right zeroed, right complementable, and Abelian.

Let p be a prime number, let V be a \mathbb{Z} -module, and let v be a vector of V . The functor $\mathbb{Z}\text{-MtoMQV}(V, p, v)$ yields a vector of $\mathbb{Z}\text{-MQVectSp}(V, p)$ and is defined as follows:

- (Def. 13) $\mathbb{Z}\text{-MtoMQV}(V, p, v) = v + p \circ V$.

Let X be a \mathbb{Z} -module. The functor $\text{MultINT} * X$ yielding a function from (the carrier of $\mathbb{Z}^{\mathbb{R}}$) \times (the carrier of X) into the carrier of X is defined by:

- (Def. 14) $\text{MultINT} * X = \text{the external multiplication of } X$.

Let X be a \mathbb{Z} -module. The functor $\text{PreNorms } X$ yielding a non empty strict vector space structure over $\mathbb{Z}^{\mathbb{R}}$ is defined by:

- (Def. 15) $\text{PreNorms } X = \langle \text{the carrier of } X, \text{ the addition of } X, \text{ the zero of } X, \text{MultINT} * X \rangle$.

Let X be a \mathbb{Z} -module. Observe that $\text{PreNorms } X$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

Let X be a left module over $\mathbb{Z}^{\mathbb{R}}$. The functor $\text{MultINT} * X$ yielding a function from $\mathbb{Z} \times \text{the carrier of } X$ into the carrier of X is defined as follows:

- (Def. 16) $\text{MultINT} * X = \text{the left multiplication of } X$.

Let X be a left module over $\mathbb{Z}^{\mathbb{R}}$. The functor $\text{PreNorms } X$ yields a non empty strict \mathbb{Z} -module structure and is defined as follows:

(Def. 17) $\text{PreNorms } X = \langle \text{the carrier of } X, \text{ the zero of } X, \text{ the addition of } X, \text{ MultINT}^* X \rangle$.

Let X be a left module over $\mathbb{Z}^{\mathbb{R}}$. Note that $\text{PreNorms } X$ is Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, and scalar unital.

We now state four propositions:

- (4) Let X be a \mathbb{Z} -module, v, w be elements of X , and v_1, w_1 be elements of $\text{PreNorms } X$. If $v = v_1$ and $w = w_1$, then $v + w = v_1 + w_1$ and $v - w = v_1 - w_1$.
- (5) Let X be a \mathbb{Z} -module, v be an element of X , v_1 be an element of $\text{PreNorms } X$, a be an integer, and a_1 be an element of $\mathbb{Z}^{\mathbb{R}}$. If $v = v_1$ and $a = a_1$, then $a \cdot v = a_1 \cdot v_1$.
- (6) Let X be a left module over $\mathbb{Z}^{\mathbb{R}}$, v, w be elements of X , and v_1, w_1 be elements of $\text{PreNorms } X$. If $v = v_1$ and $w = w_1$, then $v + w = v_1 + w_1$ and $v - w = v_1 - w_1$.
- (7) Let X be a left module over $\mathbb{Z}^{\mathbb{R}}$, v be an element of X , v_1 be an element of $\text{PreNorms } X$, a be an element of $\mathbb{Z}^{\mathbb{R}}$, and a_1 be an integer. If $v = v_1$ and $a = a_1$, then $a \cdot v = a_1 \cdot v_1$.

2. LINEAR COMBINATION OF \mathbb{Z} -MODULE

Let V be a non empty zero structure. An element of $\mathbb{Z}^{\text{the carrier of } V}$ is said to be a \mathbb{Z} -linear combination of V if:

(Def. 18) There exists a finite subset T of V such that for every element v of V such that $v \notin T$ holds $\text{it}(v) = 0$.

In the sequel K, L, L_1, L_2, L_3 denote \mathbb{Z} -linear combinations of V .

Let V be a non empty additive loop structure and let L be a \mathbb{Z} -linear combination of V . The support of L yielding a finite subset of V is defined by:

(Def. 19) The support of $L = \{v \in V : L(v) \neq 0\}$.

Next we state the proposition

- (8) Let V be a non empty additive loop structure, L be a \mathbb{Z} -linear combination of V , and v be an element of V . Then $L(v) = 0$ if and only if $v \notin$ the support of L .

Let V be a non empty additive loop structure. The functor $\mathbb{Z}\text{-ZeroLC } V$ yields a \mathbb{Z} -linear combination of V and is defined by:

(Def. 20) The support of $\mathbb{Z}\text{-ZeroLC } V = \emptyset$.

One can prove the following proposition

- (9) For every non empty additive loop structure V and for every element v of V holds $(\mathbb{Z}\text{-ZeroLCV})(v) = 0$.

Let V be a non empty additive loop structure and let A be a subset of V . A \mathbb{Z} -linear combination of V is said to be a \mathbb{Z} -linear combination of A if:

(Def. 21) The support of it $\subseteq A$.

For simplicity, we adopt the following convention: a, b are integers, $G, H_1, H_2, F, F_1, F_2, F_3$ are finite sequences of elements of V , A, B are subsets of V , $v_1, v_2, v_3, u_1, u_2, u_3$ are vectors of V , f is a function from the carrier of V into \mathbb{Z} , i is an element of \mathbb{N} , and l, l_1, l_2 are \mathbb{Z} -linear combinations of A .

One can prove the following propositions:

- (10) If $A \subseteq B$, then l is a \mathbb{Z} -linear combination of B .
 (11) $\mathbb{Z}\text{-ZeroLCV}$ is a \mathbb{Z} -linear combination of A .
 (12) For every \mathbb{Z} -linear combination l of $\emptyset_{\text{the carrier of } V}$ holds $l = \mathbb{Z}\text{-ZeroLCV}$.

Let us consider V, F, f . The functor $f \cdot F$ yields a finite sequence of elements of V and is defined by:

(Def. 22) $\text{len}(f \cdot F) = \text{len } F$ and for every i such that $i \in \text{dom}(f \cdot F)$ holds $(f \cdot F)(i) = f(F_i) \cdot F_i$.

Next we state several propositions:

- (13) If $i \in \text{dom } F$ and $v = F(i)$, then $(f \cdot F)(i) = f(v) \cdot v$.
 (14) $f \cdot \varepsilon_{(\text{the carrier of } V)} = \varepsilon_{(\text{the carrier of } V)}$.
 (15) $f \cdot \langle v \rangle = \langle f(v) \cdot v \rangle$.
 (16) $f \cdot \langle v_1, v_2 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2 \rangle$.
 (17) $f \cdot \langle v_1, v_2, v_3 \rangle = \langle f(v_1) \cdot v_1, f(v_2) \cdot v_2, f(v_3) \cdot v_3 \rangle$.

Let us consider V, L . The functor $\sum L$ yielding an element of V is defined by:

(Def. 23) There exists F such that F is one-to-one and $\text{rng } F = \text{the support of } L$ and $\sum L = \sum(L \cdot F)$.

Next we state several propositions:

- (18) $A \neq \emptyset$ and A is linearly closed iff for every l holds $\sum l \in A$.
 (19) $\sum \mathbb{Z}\text{-ZeroLCV} = 0_V$.
 (20) For every \mathbb{Z} -linear combination l of $\emptyset_{\text{the carrier of } V}$ holds $\sum l = 0_V$.
 (21) For every \mathbb{Z} -linear combination l of $\{v\}$ holds $\sum l = l(v) \cdot v$.
 (22) If $v_1 \neq v_2$, then for every \mathbb{Z} -linear combination l of $\{v_1, v_2\}$ holds $\sum l = l(v_1) \cdot v_1 + l(v_2) \cdot v_2$.
 (23) If the support of $L = \emptyset$, then $\sum L = 0_V$.
 (24) If the support of $L = \{v\}$, then $\sum L = L(v) \cdot v$.
 (25) If the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$, then $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

Let V be a non empty additive loop structure and let L_1, L_2 be \mathbb{Z} -linear combinations of V . Let us observe that $L_1 = L_2$ if and only if:

(Def. 24) For every element v of V holds $L_1(v) = L_2(v)$.

Let V be a non empty additive loop structure and let L_1, L_2 be \mathbb{Z} -linear combinations of V . Then $L_1 + L_2$ is a \mathbb{Z} -linear combination of V and it can be characterized by the condition:

(Def. 25) For every element v of V holds $(L_1 + L_2)(v) = L_1(v) + L_2(v)$.

Let us observe that the functor $L_1 + L_2$ is commutative.

The following propositions are true:

(26) The support of $L_1 + L_2 \subseteq$ (the support of L_1) \cup (the support of L_2).

(27) Suppose L_1 is a \mathbb{Z} -linear combination of A and L_2 is a \mathbb{Z} -linear combination of A . Then $L_1 + L_2$ is a \mathbb{Z} -linear combination of A .

(28) $L_1 + (L_2 + L_3) = (L_1 + L_2) + L_3$.

Let us consider V, a, L . Note that $L + \mathbb{Z}\text{-ZeroLC}V$ reduces to L .

The functor $a \cdot L$ yielding a \mathbb{Z} -linear combination of V is defined as follows:

(Def. 26) For every v holds $(a \cdot L)(v) = a \cdot L(v)$.

We now state several propositions:

(29) If $a \neq 0$, then the support of $a \cdot L =$ the support of L .

(30) $0 \cdot L = \mathbb{Z}\text{-ZeroLC}V$.

(31) If L is a \mathbb{Z} -linear combination of A , then $a \cdot L$ is a \mathbb{Z} -linear combination of A .

(32) $(a + b) \cdot L = a \cdot L + b \cdot L$.

(33) $a \cdot (L_1 + L_2) = a \cdot L_1 + a \cdot L_2$.

(34) $a \cdot (b \cdot L) = (a \cdot b) \cdot L$.

Let us consider V, L . One can check that $1 \cdot L$ reduces to L .

The functor $-L$ yielding a \mathbb{Z} -linear combination of V is defined as follows:

(Def. 27) $-L = (-1) \cdot L$.

Let us note that the functor $-L$ is involutive.

We now state four propositions:

(35) $(-L)(v) = -L(v)$.

(36) If $L_1 + L_2 = \mathbb{Z}\text{-ZeroLC}V$, then $L_2 = -L_1$.

(37) The support of $-L =$ the support of L .

(38) If L is a \mathbb{Z} -linear combination of A , then $-L$ is a \mathbb{Z} -linear combination of A .

Let us consider V, L_1, L_2 . The functor $L_1 - L_2$ yields a \mathbb{Z} -linear combination of V and is defined as follows:

(Def. 28) $L_1 - L_2 = L_1 + -L_2$.

The following four propositions are true:

- (39) $(L_1 - L_2)(v) = L_1(v) - L_2(v)$.
- (40) The support of $L_1 - L_2 \subseteq (\text{the support of } L_1) \cup (\text{the support of } L_2)$.
- (41) Suppose L_1 is a \mathbb{Z} -linear combination of A and L_2 is a \mathbb{Z} -linear combination of A . Then $L_1 - L_2$ is a \mathbb{Z} -linear combination of A .
- (42) $L - L = \mathbb{Z}\text{-ZeroLC } V$.

Let us consider V . The functor LC_V yielding a set is defined by:

(Def. 29) $x \in \text{LC}_V$ iff x is a \mathbb{Z} -linear combination of V .

Let us consider V . One can verify that LC_V is non empty.

In the sequel e, e_1, e_2 denote elements of LC_V .

Let us consider V, e . The functor ${}^{\textcircled{a}}e$ yielding a \mathbb{Z} -linear combination of V is defined by:

(Def. 30) ${}^{\textcircled{a}}e = e$.

Let us consider V, L . The functor ${}^{\textcircled{a}}L$ yielding an element of LC_V is defined by:

(Def. 31) ${}^{\textcircled{a}}L = L$.

Let us consider V . The functor $+_{\text{LC}_V}$ yields a binary operation on LC_V and is defined as follows:

(Def. 32) For all e_1, e_2 holds $+_{\text{LC}_V}(e_1, e_2) = ({}^{\textcircled{a}}e_1) + {}^{\textcircled{a}}e_2$.

Let us consider V . The functor \cdot_{LC_V} yields a function from $\mathbb{Z} \times \text{LC}_V$ into LC_V and is defined by:

(Def. 33) For all a, e holds $\cdot_{\text{LC}_V}(\langle a, e \rangle) = a \cdot ({}^{\textcircled{a}}e)$.

Let us consider V . The functor $\text{LC-}\mathbb{Z}\text{-Module } V$ yielding a \mathbb{Z} -module structure is defined as follows:

(Def. 34) $\text{LC-}\mathbb{Z}\text{-Module } V = \langle \text{LC}_V, {}^{\textcircled{a}}\mathbb{Z}\text{-ZeroLC } V, +_{\text{LC}_V}, \cdot_{\text{LC}_V} \rangle$.

Let us consider V . One can check that $\text{LC-}\mathbb{Z}\text{-Module } V$ is strict and non empty.

Let us consider V . Observe that $\text{LC-}\mathbb{Z}\text{-Module } V$ is Abelian, add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, and scalar unital.

Next we state several propositions:

- (43) The carrier of $\text{LC-}\mathbb{Z}\text{-Module } V = \text{LC}_V$.
- (44) $0_{\text{LC-}\mathbb{Z}\text{-Module } V} = \mathbb{Z}\text{-ZeroLC } V$.
- (45) The addition of $\text{LC-}\mathbb{Z}\text{-Module } V = +_{\text{LC}_V}$.
- (46) The external multiplication of $\text{LC-}\mathbb{Z}\text{-Module } V = \cdot_{\text{LC}_V}$.
- (47) $L_1^{\text{LC-}\mathbb{Z}\text{-Module } V} + L_2^{\text{LC-}\mathbb{Z}\text{-Module } V} = L_1 + L_2$.
- (48) $a \cdot L^{\text{LC-}\mathbb{Z}\text{-Module } V} = a \cdot L$.
- (49) $-L^{\text{LC-}\mathbb{Z}\text{-Module } V} = -L$.
- (50) $L_1^{\text{LC-}\mathbb{Z}\text{-Module } V} - L_2^{\text{LC-}\mathbb{Z}\text{-Module } V} = L_1 - L_2$.

Let us consider V, A . The functor LC- \mathbb{Z} -Module A yielding a strict submodule of LC- \mathbb{Z} -Module V is defined by:

(Def. 35) The carrier of LC- \mathbb{Z} -Module $A = \{l\}$.

3. LINEARLY INDEPENDENT SUBSET OF \mathbb{Z} -MODULE

For simplicity, we use the following convention: W, W_1, W_2, W_3 are submodules of V , v, v_1 are vectors of V , C is a subset of V , T is a finite subset of V , L, L_1, L_2 are \mathbb{Z} -linear combinations of V , l is a \mathbb{Z} -linear combination of A , and G is a finite sequence of elements of the carrier of V .

One can prove the following propositions:

$$(51) \quad f \cdot (F \wedge G) = (f \cdot F) \wedge (f \cdot G).$$

$$(52) \quad \sum(L_1 + L_2) = \sum L_1 + \sum L_2.$$

$$(53) \quad \sum(a \cdot L) = a \cdot \sum L.$$

$$(54) \quad \sum(-L) = -\sum L.$$

$$(55) \quad \sum(L_1 - L_2) = \sum L_1 - \sum L_2.$$

Let us consider V, A . We say that A is linearly independent if and only if:

(Def. 36) For every l such that $\sum l = 0_V$ holds the support of $l = \emptyset$.

Let us consider V, A . We introduce A is linearly dependent as an antonym of A is linearly independent.

We now state three propositions:

(56) If $A \subseteq B$ and B is linearly independent, then A is linearly independent.

(57) If A is linearly independent, then $0_V \notin A$.

(58) $\emptyset_{\text{the carrier of } V}$ is linearly independent.

Let us consider V . Observe that there exists a subset of V which is linearly independent.

One can prove the following proposition

(59) If V inherits cancelable on multiplication, then $\{v\}$ is linearly independent iff $v \neq 0_V$.

Let us consider V . Note that $\{0_V\}$ is linearly dependent as a subset of V .

One can prove the following propositions:

(60) If $\{v_1, v_2\}$ is linearly independent, then $v_1 \neq 0_V$.

(61) $\{v, 0_V\}$ is linearly dependent.

(62) Suppose V inherits cancelable on multiplication. Then $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if $v_2 \neq 0_V$ and for all a, b such that $b \neq 0$ holds $b \cdot v_1 \neq a \cdot v_2$.

(63) Suppose V inherits cancelable on multiplication. Then $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent if and only if for all a, b such that $a \cdot v_1 + b \cdot v_2 = 0_V$ holds $a = 0$ and $b = 0$.

Let us consider V , A . The functor $\text{Lin}(A)$ yielding a strict submodule of V is defined as follows:

(Def. 37) The carrier of $\text{Lin}(A) = \{\sum l\}$.

The following propositions are true:

- (64) $x \in \text{Lin}(A)$ iff there exists l such that $x = \sum l$.
- (65) If $x \in A$, then $x \in \text{Lin}(A)$.
- (66) $x \in \mathbf{0}_V$ iff $x = 0_V$.
- (67) $\text{Lin}(\emptyset_{\text{the carrier of } V}) = \mathbf{0}_V$.
- (68) If $\text{Lin}(A) = \mathbf{0}_V$, then $A = \emptyset$ or $A = \{0_V\}$.
- (69) For every strict \mathbb{Z} -module V and for every subset A of V such that $A = \text{the carrier of } V$ holds $\text{Lin}(A) = V$.
- (70) If $A \subseteq B$, then $\text{Lin}(A)$ is a submodule of $\text{Lin}(B)$.
- (71) For every strict \mathbb{Z} -module V and for all subsets A, B of V such that $\text{Lin}(A) = V$ and $A \subseteq B$ holds $\text{Lin}(B) = V$.
- (72) $\text{Lin}(A \cup B) = \text{Lin}(A) + \text{Lin}(B)$.
- (73) $\text{Lin}(A \cap B)$ is a submodule of $\text{Lin}(A) \cap \text{Lin}(B)$.

4. THEOREMS RELATED TO SUBMODULE

One can prove the following propositions:

- (74) If W_1 is a submodule of W_3 , then $W_1 \cap W_2$ is a submodule of W_3 .
- (75) If W_1 is a submodule of W_2 and a submodule of W_3 , then W_1 is a submodule of $W_2 \cap W_3$.
- (76) If W_1 is a submodule of W_3 and W_2 is a submodule of W_3 , then $W_1 + W_2$ is a submodule of W_3 .
- (77) If W_1 is a submodule of W_2 , then W_1 is a submodule of $W_2 + W_3$.

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