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Partial Differentiation of Vector-Valued Functions on n-Dimensional Real Normed Linear Spaces

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Summary. In this article, we define and develop partial differentiation of vector-valued functions on n-dimensional real normed linear spaces (refer to [19] and [20]).

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The notation and terminology used in this paper have been introduced in the following papers: [7], [15], [2], [3], [24], [4], [5], [1], [11], [16], [6], [9], [12], [17], [18], [10], [8], [23], [14], [21], [13], and [22].

For simplicity, we use the following convention: n, m denote non empty elements of $\mathbb{N}$, i, j denote elements of $\mathbb{N}$, f denotes a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, g denotes a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, h denotes a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, x denotes a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, y denotes an element of $\mathcal{R}^m$, and X denotes a set.

We now state a number of propositions:

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(1) If $i \leq j$, then $\langle \underbrace{0, \ldots, 0}_{j} \rangle \restriction i = \langle \underbrace{0, \ldots, 0}_{i} \rangle$.

(2) If $i \leq j$, then $\langle \underbrace{0, \ldots, 0}_{j} \rangle \restriction (i -' 1) = \langle \underbrace{0, \ldots, 0}_{i -' 1} \rangle$.

(3) $\langle \underbrace{0, \ldots, 0}_{j} \rangle_{\lfloor i} = \langle \underbrace{0, \ldots, 0}_{j -' i} \rangle$.

(4) If $i \leq j$, then $\langle \underbrace{0, \ldots, 0}_{j} \rangle \restriction (i -' 1) = \langle \underbrace{0, \ldots, 0}_{i -' 1} \rangle$ and $\langle \underbrace{0, \ldots, 0}_{j} \rangle_{\lfloor i} = \langle \underbrace{0, \ldots, 0}_{j -' i} \rangle$.

(5) For every element x_1 of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ such that $1 \leq i \leq j$ holds $\|(\mathrm{reproj}(i, 0_{\langle \mathcal{E}^j, \| \cdot \| \rangle}))(x_1)\| = \|x_1\|$.

(6) Let m, i be elements of $\mathbb{N}$, x be an element of $\mathcal{R}^m$, and r be a real number. Then $(\mathrm{reproj}(i, x))(r) - x = (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(r - (\mathrm{proj}(i, m))(x))$ and $x - (\mathrm{reproj}(i, x))(r) = (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))((\mathrm{proj}(i, m))(x) - r)$.

(7) Let m, i be elements of $\mathbb{N}$, x be a point of $\langle \mathcal{E}^m, \| \cdot \| \rangle$, and p be a point of $\langle \mathcal{E}^1, \| \cdot \| \rangle$. Then $(\mathrm{reproj}(i, x))(p) - x = (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(p - (\mathrm{Proj}(i, m))(x))$ and $x - (\mathrm{reproj}(i, x))(p) = (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))((\mathrm{Proj}(i, m))(x) - p)$.

(8) Let m, n be non empty elements of $\mathbb{N}$, i be an element of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and Z be a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Suppose Z is open and $1 \leq i \leq m$. Then f is partially differentiable on Z w.r.t. i if and only if $Z \subseteq \mathrm{dom}\, f$ and for every point x of $\langle \mathcal{E}^m, \| \cdot \| \rangle$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. i.

(9) For all elements x, y of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds $\mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, x + y) = \mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, x) + \mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, y)$.

(10) For all elements x, a of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds $\mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, a \cdot x) = a \cdot \mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, x)$.

(11) For every element x of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ and $x \neq 0$ holds $\mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, x) \neq \langle \underbrace{0, \ldots, 0}_{m} \rangle$.

(12) Let x, y be elements of $\mathbb{R}$, z be an element of $\mathcal{R}^m$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq m$ and $y = (\mathrm{proj}(i, m))(z)$. Then $\mathrm{Replace}(z, i, x) - z = \mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, x - y)$ and $z - \mathrm{Replace}(z, i, x) = \mathrm{Replace}(\langle \underbrace{0, \ldots, 0}_{m} \rangle, i, y - x)$.

(13) For all elements x, y of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds $(\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(x + y) = (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(x) + (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(y)$.

(14) For all points x, y of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds $(\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(x + y) = (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(x) + (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(y)$.

(15) For all elements x, a of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds $(\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(a \cdot x) = a \cdot (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(x)$.

(16) Let x be a point of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, a be an element of $\mathbb{R}$, and i be an element of $\mathbb{N}$. If $1 \leq i \leq m$, then $(\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(a \cdot x) = a \cdot (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(x)$.

(17) For every element x of $\mathbb{R}$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ and $x \neq 0$ holds $(\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(x) \neq \langle \underbrace{0, \ldots, 0}_{m} \rangle$.

(18) For every point x of $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ and $x \neq 0_{\langle \mathcal{E}^1, \| \cdot \| \rangle}$ holds $(\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(x) \neq 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}$.

(19) Let x, y be elements of $\mathbb{R}$, z be an element of $\mathcal{R}^m$, and i be an element of $\mathbb{N}$. Suppose $1 \leq i \leq m$ and $y = (\mathrm{proj}(i, m))(z)$. Then $(\mathrm{reproj}(i, z))(x) - z = (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(x - y)$ and $z - (\mathrm{reproj}(i, z))(x) = (\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(y - x)$.

(20) Let x, y be points of $\langle \mathcal{E}^1, \| \cdot \| \rangle$, i be an element of $\mathbb{N}$, and z be a point of $\langle \mathcal{E}^m, \| \cdot \| \rangle$. Suppose $1 \leq i \leq m$ and $y = (\mathrm{Proj}(i, m))(z)$. Then $(\mathrm{reproj}(i, z))(x) - z = (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(x - y)$ and $z - (\mathrm{reproj}(i, z))(x) = (\mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(y - x)$.

(21) Suppose f is differentiable in x and $1 \leq i \leq m$. Then f is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(f, x, i) = f'(x) \cdot \mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle})$.

(22) Suppose g is differentiable in y and $1 \leq i \leq m$. Then g is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(g, y, i) = (g'(y) \cdot \mathrm{reproj}(i, 0_{\langle \mathcal{E}^m, \| \cdot \| \rangle}))(\langle 1 \rangle)$.

Let n be a non empty element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. We say that f is differentiable in x if and only if:

(Def. 1) $\langle f \rangle$ is differentiable in x.

Let n be a non empty element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. The functor $f'(x)$ yielding a function from $\mathcal{R}^n$ into $\mathbb{R}$ is defined as follows:

(Def. 2) $f'(x) = \mathrm{proj}(1, 1) \cdot \langle f \rangle'(x)$.

Next we state several propositions:

(23) Suppose h is differentiable in y and $1 \leq i \leq m$. Then h is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(h, y, i) = (h \cdot \mathrm{reproj}(i, y))'((\mathrm{proj}(i, m))(y))$ and $\mathrm{partdiff}(h, y, i) = h'(y)((\mathrm{reproj}(i, \langle \underbrace{0, \ldots, 0}_{m} \rangle))(1))$.

(24) Let m be a non empty element of $\mathbb{N}$ and v, w, u be finite sequences of elements of $\mathcal{R}^m$. If $\mathrm{dom}\, v = \mathrm{dom}\, w$ and $u = v + w$, then $\sum u = \sum v + \sum w$.

(25) Let m be a non empty element of $\mathbb{N}$, r be a real number, and w, u be finite sequences of elements of $\mathcal{R}^m$. If $u = r\, w$, then $\sum u = r \cdot \sum w$.

(26) Let n be a non empty element of $\mathbb{N}$ and h, g be finite sequences of elements of $\mathcal{R}^n$. Suppose $\mathrm{len}\, h = \mathrm{len}\, g + 1$ and for every natural number i such that $i \in \mathrm{dom}\, g$ holds $g_i = h_i - h_{i+1}$. Then $h_1 - h_{\mathrm{len}\, h} = \sum g$.

(27) Let n be a non empty element of $\mathbb{N}$ and h, g, j be finite sequences of elements of $\mathcal{R}^n$. Suppose $\mathrm{len}\, h = \mathrm{len}\, j$ and $\mathrm{len}\, g = \mathrm{len}\, j$ and for every natural number i such that $i \in \mathrm{dom}\, j$ holds $j_i = h_i - g_i$. Then $\sum j = \sum h - \sum g$.

(28) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x, y be elements of $\mathcal{R}^m$. Then there exists a finite sequence h of elements of $\mathcal{R}^m$ and there exists a finite sequence g of elements of $\mathcal{R}^n$ such that

(i) $\mathrm{len}\, h = m + 1$,

(ii) $\mathrm{len}\, g = m$,

(iii) for every natural number i such that $i \in \mathrm{dom}\, h$ holds $h_i = (y {\restriction} ((m + 1) -' i)) \frown \langle \underbrace{0, \ldots, 0}_{i -' 1} \rangle$,

(iv) for every natural number i such that $i \in \mathrm{dom}\, g$ holds $g_i = f_{x+h_i} - f_{x+h_{i+1}}$,

(v) for every natural number i and for every element h_1 of $\mathcal{R}^m$ such that $i \in \mathrm{dom}\, h$ and $h_i = h_1$ holds $|h_1| \leq |y|$, and

(vi) $f_{x+y} - f_x = \sum g$.

(29) Let m be a non empty element of $\mathbb{N}$ and f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^1$. Then there exists a partial function f_0 from $\mathcal{R}^m$ to $\mathbb{R}$ such that $f = \langle f_0 \rangle$.

(30) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, f_0 be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x be an element of $\mathcal{R}^m$, and x_0 be an element of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. If $x \in \mathrm{dom}\, f$ and $x = x_0$ and $f = f_0$, then $f_x = (f_0)_{x_0}$.

Let m be a non empty element of $\mathbb{N}$ and let X be a subset of $\mathcal{R}^m$. We say that X is open if and only if:

(Def. 3) There exists a subset X_0 of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $X_0 = X$ and X_0 is open.

The following proposition is true

(31) Let m be a non empty element of $\mathbb{N}$ and X be a subset of $\mathcal{R}^m$. Then X is open if and only if for every element x of $\mathcal{R}^m$ such that $x \in X$ there exists a real number r such that $r > 0$ and $\{y \in \mathcal{R}^m \colon |y - x| < r\} \subseteq X$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let X be a set. We say that f is partially differentiable on X w.r.t. i if and only if:

(Def. 4) $X \subseteq \operatorname{dom} f$ and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $f{\restriction}X$ is partially differentiable in x w.r.t. i.

One can prove the following propositions:

(32) Let m, n be non empty elements of $\mathbb{N}$ and f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$. Suppose f is partially differentiable on X w.r.t. i. Then X is a subset of $\mathcal{R}^m$.

(33) Let m, n be non empty elements of $\mathbb{N}$, i be an element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and Z be a set. Suppose $f = g$. Then f is partially differentiable on Z w.r.t. i if and only if g is partially differentiable on Z w.r.t. i.

(34) Let m, n be non empty elements of $\mathbb{N}$, i be an element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and Z be a subset of $\mathcal{R}^m$. Suppose Z is open and $1 \leq i \leq m$. Then f is partially differentiable on Z w.r.t. i if and only if $Z \subseteq \operatorname{dom} f$ and for every element x of $\mathcal{R}^m$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. i.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let us consider X. Let us assume that f is partially differentiable on X w.r.t. i. The functor $f{\restriction}^i X$ yielding a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$ is defined as follows:

(Def. 5) $\operatorname{dom}(f{\restriction}^i X) = X$ and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $(f{\restriction}^i X)_x = \operatorname{partdiff}(f, x, i)$.

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x_0 be an element of $\mathcal{R}^m$. We say that f is continuous in x_0 if and only if:

(Def. 6) There exists a point y_0 of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ and there exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $x_0 = y_0$ and $f = g$ and g is continuous in y_0.

The following propositions are true:

(35) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, x be an element of $\mathcal{R}^m$, and y be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $f = g$ and $x = y$. Then f is continuous in x if and only if g is continuous in y.

(36) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and x_0 be an element of $\mathcal{R}^m$. Then f is continuous in x_0 if and

only if the following conditions are satisfied:

(i) $x_0 \in \operatorname{dom} f$, and

(ii) for every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every element x_2 of $\mathcal{R}^m$ such that $x_2 \in \operatorname{dom} f$ and $|x_2 - x_0| < s$ holds $|f_{x_2} - f_{x_0}| < r$.

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let us consider X. We say that f is continuous on X if and only if:

(Def. 7) $X \subseteq \operatorname{dom} f$ and for every element x_0 of $\mathcal{R}^m$ such that $x_0 \in X$ holds $f \restriction X$ is continuous in x_0.

Next we state a number of propositions:

(37) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, g be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and X be a set. If $f = g$, then f is continuous on X iff g is continuous on X.

(38) Let m, n be non empty elements of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and X be a set. Then f is continuous on X if and only if the following conditions are satisfied:

(i) $X \subseteq \operatorname{dom} f$, and

(ii) for every element x_0 of $\mathcal{R}^m$ and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_2 of $\mathcal{R}^m$ such that $x_2 \in X$ and $|x_2 - x_0| < s$ holds $|f_{x_2} - f_{x_0}| < r$.

(39) Let m be a non empty element of $\mathbb{N}$, x, y be elements of $\mathcal{R}^m$, i be an element of $\mathbb{N}$, and x_1 be a real number. If $1 \leq i \leq m$ and $y = (\operatorname{reproj}(i, x))(x_1)$, then $(\operatorname{proj}(i, m))(y) = x_1$.

(40) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, x, y be elements of $\mathcal{R}^m$, i be an element of $\mathbb{N}$, and x_1 be a real number. If $1 \leq i \leq m$ and $y = (\operatorname{reproj}(i, x))(x_1)$, then $\operatorname{reproj}(i, x) = \operatorname{reproj}(i, y)$.

(41) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, g be a partial function from $\mathbb{R}$ to $\mathbb{R}$, x, y be elements of $\mathcal{R}^m$, i be an element of $\mathbb{N}$, and x_1 be a real number. If $1 \leq i \leq m$ and $y = (\operatorname{reproj}(i, x))(x_1)$ and $g = f \cdot \operatorname{reproj}(i, x)$, then $g'(x_1) = \operatorname{partdiff}(f, y, i)$.

(42) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, p, q be real numbers, x be an element of $\mathcal{R}^m$, and i be an element of $\mathbb{N}$. Suppose that

(i) $1 \leq i$,

(ii) $i \leq m$,

(iii) $p < q$,

(iv) for every real number h such that $h \in [p, q]$ holds $(\operatorname{reproj}(i, x))(h) \in \operatorname{dom} f$, and

(v) for every real number h such that $h \in [p,q]$ holds f is partially differentiable in $(\mathrm{reproj}(i,x))(h)$ w.r.t. i.
Then there exists a real number r and there exists an element y of $\mathcal{R}^m$ such that $r \in]p,q[$ and $y = (\mathrm{reproj}(i,x))(r)$ and $f_{(\mathrm{reproj}(i,x))(q)} - f_{(\mathrm{reproj}(i,x))(p)} = (q-p) \cdot \mathrm{partdiff}(f,y,i)$.

(43) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, p, q be real numbers, x be an element of $\mathcal{R}^m$, and i be an element of $\mathbb{N}$. Suppose that
(i) $1 \leq i$,
(ii) $i \leq m$,
(iii) $p \leq q$,
(iv) for every real number h such that $h \in [p,q]$ holds $(\mathrm{reproj}(i,x))(h) \in \mathrm{dom}\, f$, and
(v) for every real number h such that $h \in [p,q]$ holds f is partially differentiable in $(\mathrm{reproj}(i,x))(h)$ w.r.t. i.
Then there exists a real number r and there exists an element y of $\mathcal{R}^m$ such that $r \in [p,q]$ and $y = (\mathrm{reproj}(i,x))(r)$ and $f_{(\mathrm{reproj}(i,x))(q)} - f_{(\mathrm{reproj}(i,x))(p)} = (q-p) \cdot \mathrm{partdiff}(f,y,i)$.

(44) Let m be a non empty element of $\mathbb{N}$, x, y, z, w be elements of $\mathcal{R}^m$, i be an element of $\mathbb{N}$, and d, p, q, r be real numbers. Suppose $1 \leq i \leq m$ and $|y-x| < d$ and $|z-x| < d$ and $p = (\mathrm{proj}(i,m))(y)$ and $z = (\mathrm{reproj}(i,y))(q)$ and $r \in [p,q]$ and $w = (\mathrm{reproj}(i,y))(r)$. Then $|w-x| < d$.

(45) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, X be a subset of $\mathcal{R}^m$, x, y, z be elements of $\mathcal{R}^m$, i be an element of $\mathbb{N}$, and d, p, q be real numbers. Suppose that $1 \leq i \leq m$ and X is open and $x \in X$ and $|y-x| < d$ and $|z-x| < d$ and $X \subseteq \mathrm{dom}\, f$ and for every element x of $\mathcal{R}^m$ such that $x \in X$ holds f is partially differentiable in x w.r.t. i and $0 < d$ and for every element z of $\mathcal{R}^m$ such that $|z-x| < d$ holds $z \in X$ and $z = (\mathrm{reproj}(i,y))(p)$ and $q = (\mathrm{proj}(i,m))(y)$. Then there exists an element w of $\mathcal{R}^m$ such that $|w-x| < d$ and f is partially differentiable in w w.r.t. i and $f_z - f_y = (p-q) \cdot \mathrm{partdiff}(f,w,i)$.

(46) Let m be a non empty element of $\mathbb{N}$, h be a finite sequence of elements of $\mathcal{R}^m$, y, x be elements of $\mathcal{R}^m$, and j be an element of $\mathbb{N}$. Suppose $\mathrm{len}\, h = m+1$ and $1 \leq j \leq m$ and for every natural number i such that $i \in \mathrm{dom}\, h$ holds $h_i = (y \restriction ((m+1) -' i)) \frown \langle \underbrace{0, \ldots, 0}_{i -' 1} \rangle$. Then $x + h_j = (\mathrm{reproj}((m+1) -' j, x + h_{j+1}))((\mathrm{proj}((m+1) -' j, m))(x+y))$.

(47) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^1$, X be a subset of $\mathcal{R}^m$, and x be an element of $\mathcal{R}^m$. Suppose that
(i) X is open,
(ii) $x \in X$, and

(iii) for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\restriction}^i X$ is continuous on X.

Then

(iv) f is differentiable in x, and

(v) for every element h of $\mathcal{R}^m$ there exists a finite sequence w of elements of $\mathcal{R}^1$ such that $\operatorname{dom} w = \operatorname{Seg} m$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{Seg} m$ holds $w(i) = (\operatorname{proj}(i,m))(h) \cdot \operatorname{partdiff}(f,x,i)$ and $f'(x)(h) = \sum w$.

(48) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, X be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose that

(i) X is open,

(ii) $x \in X$, and

(iii) for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\restriction}^i X$ is continuous on X.

Then

(iv) f is differentiable in x, and

(v) for every point h of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ there exists a finite sequence w of elements of $\mathcal{R}^1$ such that $\operatorname{dom} w = \operatorname{Seg} m$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{Seg} m$ holds $w(i) = (\operatorname{partdiff}(f,x,i))(\langle (\operatorname{proj}(i,m))(h) \rangle)$ and $f'(x)(h) = \sum w$.

(49) Let m be a non empty element of $\mathbb{N}$, f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and X be a subset of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose X is open. Then for every element i of $\mathbb{N}$ such that $1 \leq i \leq m$ holds f is partially differentiable on X w.r.t. i and $f{\restriction}^i X$ is continuous on X if and only if f is differentiable on X and $f'_{\restriction X}$ is continuous on X.

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