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The Sum and Product of Finite Sequences of Complex Numbers

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Summary. This article extends the [10]. We define the sum and the product of the sequence of complex numbers, and formalize these theorems. Our method refers to the [11].

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The notation and terminology used in this paper have been introduced in the following papers: [5], [7], [6], [4], [8], [13], [9], [2], [3], [15], [10], [12], and [14].

1. Auxiliary Theorems

Let F be a complex-valued binary relation. Then $\operatorname{rng} F$ is a subset of $\mathbb{C}$.

Let D be a non empty set, let F be a function from $\mathbb{C}$ into D, and let F_1 be a complex-valued finite sequence. Note that $F \cdot F_1$ is finite sequence-like.

For simplicity, we adopt the following rules: i, j denote natural numbers, x, x_1 denote elements of $\mathbb{C}$, c denotes a complex number, F, F_1, F_2 denote complex-valued finite sequences, and R, R_1 denote i-element finite sequences of elements of $\mathbb{C}$.

The unary operation sqrcomplex on $\mathbb{C}$ is defined as follows:

(Def. 1) For every c holds $(\text{sqrcomplex})(c) = c^{\mathbf{2}}$.

Next we state two propositions:

(1) sqrcomplex is distributive w.r.t. $\cdot_{\mathbb{C}}$.

(2) $\cdot_{\mathbb{C}}^{c}$ is distributive w.r.t. $+_{\mathbb{C}}$.

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2. Some Functors on the i-Tuples of Complex Numbers

Let us consider F_1, F_2. Then $F_1 + F_2$ is a finite sequence of elements of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 2) $F_1 + F_2 = (+_{\mathbb{C}})^{\circ}(F_1, F_2)$.

Let us observe that the functor $F_1 + F_2$ is commutative.

Let us consider i, R_1, R_2. Then $R_1 + R_2$ is an element of $\mathbb{C}^i$.

The following propositions are true:

(3) $(R_1 + R_2)(s) = R_1(s) + R_2(s)$.

(4) $\varepsilon_{\mathbb{C}} + F = \varepsilon_{\mathbb{C}}$.

(5) $\langle c_1 \rangle + \langle c_2 \rangle = \langle c_1 + c_2 \rangle$.

(6) $i \mapsto c_1 + i \mapsto c_2 = i \mapsto (c_1 + c_2)$.

Let us consider F. Then $-F$ is a finite sequence of elements of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 3) $-F = -_{\mathbb{C}} \cdot F$.

Let us consider i, R. Then $-R$ is an element of $\mathbb{C}^i$.

The following propositions are true:

(7) $-\langle c \rangle = \langle -c \rangle$.

(8) $-i \mapsto c = i \mapsto (-c)$.

(9) If $R_1 + R = R_2 + R$, then $R_1 = R_2$.

(10) $-(F_1 + F_2) = -F_1 + -F_2$.

Let us consider F_1, F_2. Then $F_1 - F_2$ is a finite sequence of elements of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 4) $F_1 - F_2 = (-_{\mathbb{C}})^{\circ}(F_1, F_2)$.

Let us consider i, R_1, R_2. Then $R_1 - R_2$ is an element of $\mathbb{C}^i$.

The following propositions are true:

(11) $(R_1 - R_2)(s) = R_1(s) - R_2(s)$.

(12) $\varepsilon_{\mathbb{C}} - F = \varepsilon_{\mathbb{C}}$ and $F - \varepsilon_{\mathbb{C}} = \varepsilon_{\mathbb{C}}$.

(13) $\langle c_1 \rangle - \langle c_2 \rangle = \langle c_1 - c_2 \rangle$.

(14) $i \mapsto c_1 - i \mapsto c_2 = i \mapsto (c_1 - c_2)$.

(15) $R - i \mapsto 0_{\mathbb{C}} = R$.

(16) $-(F_1 - F_2) = F_2 - F_1$.

(17) $-(F_1 - F_2) = -F_1 + F_2$.

(18) If $R_1 - R_2 = i \mapsto 0_{\mathbb{C}}$, then $R_1 = R_2$.

(19) $R_1 = (R_1 + R) - R$.

(20) $R_1 = (R_1 - R) + R$.

Let us consider F, c. We introduce $c \cdot F$ as a synonym of $c\,F$.

Let us consider F, c. Then $c \cdot F$ is a finite sequence of elements of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 5) $c \cdot F = \cdot_{\mathbb{C}}^{c} \cdot F$.

Let us consider i, R, c. Then $c \cdot R$ is an element of $\mathbb{C}^i$.

One can prove the following four propositions:

(21) $c \cdot \langle c_1 \rangle = \langle c \cdot c_1 \rangle$.

(22) $c_1 \cdot (i \mapsto c_2) = i \mapsto (c_1 \cdot c_2)$.

(23) $(c_1 + c_2) \cdot F = c_1 \cdot F + c_2 \cdot F$.

(24) $0_{\mathbb{C}} \cdot R = i \mapsto 0_{\mathbb{C}}$.

Let us consider F_1, F_2. We introduce $F_1 \bullet F_2$ as a synonym of $F_1\, F_2$.

Let us consider F_1, F_2. Then $F_1 \bullet F_2$ is a finite sequence of elements of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 6) $F_1 \bullet F_2 = (\cdot_{\mathbb{C}})^{\circ}(F_1, F_2)$.

Let us note that the functor $F_1 \bullet F_2$ is commutative.

Let us consider i, R_1, R_2. Then $R_1 \bullet R_2$ is an element of $\mathbb{C}^i$.

Next we state four propositions:

(25) $\varepsilon_{\mathbb{C}} \bullet F = \varepsilon_{\mathbb{C}}$.

(26) $\langle c_1 \rangle \bullet \langle c_2 \rangle = \langle c_1 \cdot c_2 \rangle$.

(27) $i \mapsto c \bullet R = c \cdot R$.

(28) $i \mapsto c_1 \bullet i \mapsto c_2 = i \mapsto (c_1 \cdot c_2)$.

3. Finite Sum of Finite Sequence of Complex Numbers

One can prove the following propositions:

(29) $\sum(\varepsilon_{\mathbb{C}}) = 0_{\mathbb{C}}$.

(30) $\sum\langle c \rangle = c$.

(31) $\sum(F \frown \langle c \rangle) = \sum F + c$.

(32) $\sum(F_1 \frown F_2) = \sum F_1 + \sum F_2$.

(33) $\sum(\langle c \rangle \frown F) = c + \sum F$.

(34) $\sum\langle c_1, c_2 \rangle = c_1 + c_2$.

(35) $\sum\langle c_1, c_2, c_3 \rangle = c_1 + c_2 + c_3$.

(36) $\sum(i \mapsto c) = i \cdot c$.

(37) $\sum(i \mapsto 0_{\mathbb{C}}) = 0_{\mathbb{C}}$.

(38) $\sum(c \cdot F) = c \cdot \sum F$.

(39) $\sum(-F) = -\sum F$.

(40) $\sum(R_1 + R_2) = \sum R_1 + \sum R_2$.

(41) $\sum(R_1 - R_2) = \sum R_1 - \sum R_2$.

4. The Product of Finite Sequences of Complex Numbers

One can prove the following propositions:

(42) $\prod(\varepsilon_{\mathbb{C}}) = 1$.

(43) $\prod(\langle c \rangle \frown F) = c \cdot \prod F$.

(44) For every element R of $\mathbb{C}^0$ holds $\prod R = 1$.

(45) $\prod((i + j) \mapsto c) = \prod(i \mapsto c) \cdot \prod(j \mapsto c)$.

(46) $\prod((i \cdot j) \mapsto c) = \prod(j \mapsto \prod(i \mapsto c))$.

(47) $\prod(i \mapsto (c_1 \cdot c_2)) = \prod(i \mapsto c_1) \cdot \prod(i \mapsto c_2)$.

(48) $\prod(R_1 \bullet R_2) = \prod R_1 \cdot \prod R_2$.

(49) $\prod(c \cdot R) = \prod(i \mapsto c) \cdot \prod R$.

5. Modified Part of [1]

We now state several propositions:

(50) For every complex-valued finite sequence x holds $\mathrm{len}(-x) = \mathrm{len}\, x$.

(51) For all complex-valued finite sequences x_1, x_2 such that $\mathrm{len}\, x_1 = \mathrm{len}\, x_2$ holds $\mathrm{len}(x_1 + x_2) = \mathrm{len}\, x_1$.

(52) For all complex-valued finite sequences x_1, x_2 such that $\mathrm{len}\, x_1 = \mathrm{len}\, x_2$ holds $\mathrm{len}(x_1 - x_2) = \mathrm{len}\, x_1$.

(53) For every real number a and for every complex-valued finite sequence x holds $\mathrm{len}(a \cdot x) = \mathrm{len}\, x$.

(54) For all complex-valued finite sequences x, y, z such that $\mathrm{len}\, x = \mathrm{len}\, y = \mathrm{len}\, z$ holds $(x + y) \bullet z = x \bullet z + y \bullet z$.

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