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Basic Properties and Concept of Selected Subsequence of Zero Based Finite Sequences

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Summary. Here, we develop the theory of zero based finite sequences, which are sometimes, more useful in applications than normal one based finite sequences. The fundamental function Sgm is introduced as well as in case of normal finite sequences and other notions are also introduced. However, many theorems are a modification of old theorems of normal finite sequences, they are basically important and are necessary for applications. A new concept of selected subsequence is introduced. This concept came from the individual Ergodic theorem (see [7]) and it is the preparation for its proof.

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The articles [12], [1], [14], [5], [8], [2], [6], [4], [3], [13], [10], [9], and [11] provide the notation and terminology for this paper.

1. Preliminaries

In this paper D is a set.

One can prove the following proposition

(1) For every set x and for every natural number i such that $x \in i$ holds x is an element of $\mathbb{N}$.

Let us observe that every natural number is natural-membered.

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2. Additional Properties of Zero Based Finite Sequence

One can prove the following propositions:

(2) For every finite natural-membered set X_0 there exists a natural number m such that $X_0 \subseteq m$.

(3) Let p be a finite 0-sequence and b be a set. If $b \in \operatorname{rng} p$, then there exists an element i of $\mathbb{N}$ such that $i \in \operatorname{dom} p$ and $p(i) = b$.

(4) Let D be a set and p be a finite 0-sequence. Suppose that for every natural number i such that $i \in \operatorname{dom} p$ holds $p(i) \in D$. Then p is a finite 0-sequence of D.

The scheme *XSeqLambdaD* deals with a natural number $\mathcal{A}$, a non empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and states that:

There exists a finite 0-sequence z of $\mathcal{B}$ such that $\operatorname{len} z = \mathcal{A}$ and for every natural number j such that $j \in \mathcal{A}$ holds $z(j) = \mathcal{F}(j)$

for all values of the parameters.

One can prove the following proposition

(5) Let p, q be finite 0-sequences. Suppose $\operatorname{len} p = \operatorname{len} q$ and for every natural number j such that $j \in \operatorname{dom} p$ holds $p(j) = q(j)$. Then $p = q$.

Let f be a finite 0-sequence of $\mathbb{R}$ and let a be an element of $\mathbb{R}$. Then $f + a$ is a finite 0-sequence of $\mathbb{R}$.

We now state two propositions:

(6) Let f be a finite 0-sequence of $\mathbb{R}$ and a be an element of $\mathbb{R}$. Then $\operatorname{len}(f + a) = \operatorname{len} f$ and for every natural number i such that $i < \operatorname{len} f$ holds $(f + a)(i) = f(i) + a$.

(7) For all finite 0-sequences f_1, f_2 and for every natural number i such that $i < \operatorname{len} f_1$ holds $(f_1 \frown f_2)(i) = f_1(i)$.

Let f be a finite 0-sequence. The functor $\operatorname{Rev}(f)$ yielding a finite 0-sequence is defined as follows:

(Def. 1) $\operatorname{len} \operatorname{Rev}(f) = \operatorname{len} f$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} \operatorname{Rev}(f)$ holds $(\operatorname{Rev}(f))(i) = f(\operatorname{len} f - (i + 1))$.

We now state the proposition

(8) For every finite 0-sequence f holds $\operatorname{dom} f = \operatorname{dom} \operatorname{Rev}(f)$ and $\operatorname{rng} f = \operatorname{rng} \operatorname{Rev}(f)$.

Let D be a set and let f be a finite 0-sequence of D. Then $\operatorname{Rev}(f)$ is a finite 0-sequence of D.

We now state several propositions:

(9) For every finite 0-sequence p such that $p \neq \emptyset$ there exists a finite 0-sequence q and there exists a set x such that $p = q \frown \langle x \rangle$.

(10) For every natural number n and for every finite 0-sequence f such that $\operatorname{len} f \leq n$ holds $f \restriction n = f$.

(11) For every finite 0-sequence f and for all natural numbers n, m such that $n \leq \operatorname{len} f$ and $m \in n$ holds $(f \restriction n)(m) = f(m)$ and $m \in \operatorname{dom} f$.

(12) For every element i of $\mathbb{N}$ and for every finite 0-sequence q such that $i \leq \operatorname{len} q$ holds $\operatorname{len}(q \restriction i) = i$.

(13) For every element i of $\mathbb{N}$ and for every finite 0-sequence q holds $\operatorname{len}(q \restriction i) \leq i$.

(14) For every finite 0-sequence f and for every element n of $\mathbb{N}$ such that $\operatorname{len} f = n + 1$ holds $f = (f \restriction n) \frown \langle f(n) \rangle$.

Let f be a finite 0-sequence and let n be a natural number. The functor $f_{\lfloor n}$ yielding a finite 0-sequence is defined by:

(Def. 2) $\operatorname{len}(f_{\lfloor n}) = \operatorname{len} f -' n$ and for every natural number m such that $m \in \operatorname{dom}(f_{\lfloor n})$ holds $f_{\lfloor n}(m) = f(m + n)$.

One can prove the following three propositions:

(15) For every finite 0-sequence f and for every natural number n such that $n \geq \operatorname{len} f$ holds $f_{\lfloor n} = \emptyset$.

(16) For every finite 0-sequence f and for every natural number n such that $n < \operatorname{len} f$ holds $\operatorname{len}(f_{\lfloor n}) = \operatorname{len} f - n$.

(17) For every finite 0-sequence f and for all natural numbers n, m such that $m + n < \operatorname{len} f$ holds $f_{\lfloor n}(m) = f(m + n)$.

Let f be an one-to-one finite 0-sequence and let n be a natural number. Note that $f_{\lfloor n}$ is one-to-one.

We now state several propositions:

(18) For every finite 0-sequence f and for every natural number n holds $\operatorname{rng}(f_{\lfloor n}) \subseteq \operatorname{rng} f$.

(19) For every finite 0-sequence f holds $f_{\lfloor 0} = f$.

(20) For every natural number i and for all finite 0-sequences f, g holds $(f \frown g)_{\lfloor \operatorname{len} f + i} = g_{\lfloor i}$.

(21) For all finite 0-sequences f, g holds $(f \frown g)_{\lfloor \operatorname{len} f} = g$.

(22) For every finite 0-sequence f and for every element n of $\mathbb{N}$ holds $(f \restriction n) \frown (f_{\lfloor n}) = f$.

Let D be a set, let f be a finite 0-sequence of D, and let n be a natural number. Then $f_{\lfloor n}$ is a finite 0-sequence of D.

Let f be a finite 0-sequence and let k_1, k_2 be natural numbers. The functor $\operatorname{mid}(f, k_1, k_2)$ yields a finite 0-sequence and is defined as follows:

(Def. 3) For all elements k_{11}, k_{21} of $\mathbb{N}$ such that $k_{11} = k_1$ and $k_{21} = k_2$ holds $\operatorname{mid}(f, k_1, k_2) = (f \restriction k_{21})_{\lfloor k_{11} -' 1}$.

We now state several propositions:

(23) For every finite 0-sequence f and for all natural numbers k_1, k_2 such that $k_1 > k_2$ holds $\operatorname{mid}(f, k_1, k_2) = \emptyset$.

(24) For every finite 0-sequence f and for all natural numbers k_1, k_2 such that $1 \leq k_1$ and $k_2 \leq \operatorname{len} f$ holds $\operatorname{mid}(f, k_1, k_2) = f_{\lfloor k_1 -' 1} \restriction ((k_2 + 1) -' k_1)$.

(25) For every finite 0-sequence f and for every natural number k_2 holds $\operatorname{mid}(f, 1, k_2) = f \restriction k_2$.

(26) For every finite 0-sequence f of D and for every natural number k_2 such that $\operatorname{len} f \leq k_2$ holds $\operatorname{mid}(f, 1, k_2) = f$.

(27) For every finite 0-sequence f and for every element k_2 of $\mathbb{N}$ holds $\operatorname{mid}(f, 0, k_2) = \operatorname{mid}(f, 1, k_2)$.

(28) For all finite 0-sequences f, g holds $\operatorname{mid}(f ^\frown g, \operatorname{len} f + 1, \operatorname{len} f + \operatorname{len} g) = g$.

Let D be a set, let f be a finite 0-sequence of D, and let k_1, k_2 be natural numbers. Then $\operatorname{mid}(f, k_1, k_2)$ is a finite 0-sequence of D.

Let f be a finite 0-sequence of $\mathbb{R}$. The functor $\sum f$ yields an element of $\mathbb{R}$ and is defined by the condition (Def. 4).

(Def. 4) There exists a finite 0-sequence g of $\mathbb{R}$ such that $\operatorname{len} f = \operatorname{len} g$ and $f(0) = g(0)$ and for every natural number i such that $i+1 < \operatorname{len} f$ holds $g(i+1) = g(i) + f(i+1)$ and $\sum f = g(\operatorname{len} f -' 1)$.

Let f be an empty finite 0-sequence of $\mathbb{R}$. Observe that $\sum f$ is zero.

We now state two propositions:

(29) For every empty finite 0-sequence f of $\mathbb{R}$ holds $\sum f = 0$.

(30) For all finite 0-sequences h_1, h_2 of $\mathbb{R}$ holds $\sum h_1 ^\frown h_2 = (\sum h_1) + \sum h_2$.

3. Selected Subsequences

Let X be a finite natural-membered set. The functor $\operatorname{Sgm}_0 X$ yields a finite 0-sequence of $\mathbb{N}$ and is defined as follows:

(Def. 5) $\operatorname{rng} \operatorname{Sgm}_0 X = X$ and for all natural numbers l, m, k_1, k_2 such that $l < m < \operatorname{len} \operatorname{Sgm}_0 X$ and $k_1 = (\operatorname{Sgm}_0 X)(l)$ and $k_2 = (\operatorname{Sgm}_0 X)(m)$ holds $k_1 < k_2$.

Let A be a finite natural-membered set. Note that $\operatorname{Sgm}_0 A$ is one-to-one.

Next we state three propositions:

(31) For every finite natural-membered set A holds $\operatorname{len} \operatorname{Sgm}_0 A = \overline{\overline{A}}$.

(32) For all finite natural-membered sets X, Y such that $X \subseteq Y$ and $X \neq \emptyset$ holds $(\operatorname{Sgm}_0 Y)(0) \leq (\operatorname{Sgm}_0 X)(0)$.

(33) For every natural number n holds $(\operatorname{Sgm}_0\{n\})(0) = n$.

Let B_1, B_2 be sets. The predicate $B_1 < B_2$ is defined by:

(Def. 6) For all natural numbers n, m such that $n \in B_1$ and $m \in B_2$ holds $n < m$.

Let B_1, B_2 be sets. The predicate $B_1 \leq B_2$ is defined by:

(Def. 7) For all natural numbers n, m such that $n \in B_1$ and $m \in B_2$ holds $n \leq m$.

The following propositions are true:

(34) For all sets B_1, B_2 such that $B_1 < B_2$ holds $B_1 \cap B_2 \cap \mathbb{N} = \emptyset$.

(35) For all finite natural-membered sets B_1, B_2 such that $B_1 < B_2$ holds B_1 misses B_2.

(36) For all sets A, B_1, B_2 such that $B_1 < B_2$ holds $A \cap B_1 < A \cap B_2$.

(37) For all finite natural-membered sets X, Y such that $Y \neq \emptyset$ and there exists a set x such that $x \in X$ and $\{x\} \leq Y$ holds $(\mathrm{Sgm}_0\, X)(0) \leq (\mathrm{Sgm}_0\, Y)(0)$.

(38) Let X_0, Y_0 be finite natural-membered sets and i be a natural number. If $X_0 < Y_0$ and $i < \mathrm{card}\, X_0$, then $\mathrm{rng}(\mathrm{Sgm}_0(X_0 \cup Y_0) \restriction \mathrm{card}\, X_0) = X_0$ and $(\mathrm{Sgm}_0(X_0 \cup Y_0) \restriction \mathrm{card}\, X_0)(i) = (\mathrm{Sgm}_0(X_0 \cup Y_0))(i)$.

(39) For all finite natural-membered sets X, Y and for every natural number i such that $X < Y$ and $i \in \overline{\overline{X}}$ holds $(\mathrm{Sgm}_0(X \cup Y))(i) \in X$.

(40) Let X, Y be finite natural-membered sets and i be a natural number. If $X < Y$ and $i < \mathrm{len}\, \mathrm{Sgm}_0\, X$, then $(\mathrm{Sgm}_0\, X)(i) = (\mathrm{Sgm}_0(X \cup Y))(i)$.

(41) Let X_0, Y_0 be finite natural-membered sets and i be a natural number. If $X_0 < Y_0$ and $i < \mathrm{card}\, Y_0$, then $\mathrm{rng}((\mathrm{Sgm}_0(X_0 \cup Y_0))_{\lfloor \mathrm{card}\, X_0}) = Y_0$ and $(\mathrm{Sgm}_0(X_0 \cup Y_0))_{\lfloor \mathrm{card}\, X_0}(i) = (\mathrm{Sgm}_0(X_0 \cup Y_0))(i + \mathrm{card}\, X_0)$.

(42) Let X, Y be finite natural-membered sets and i be a natural number. If $X < Y$ and $i < \mathrm{len}\, \mathrm{Sgm}_0\, Y$, then $(\mathrm{Sgm}_0\, Y)(i) = (\mathrm{Sgm}_0(X \cup Y))(i + \mathrm{len}\, \mathrm{Sgm}_0\, X)$.

(43) For all finite natural-membered sets X, Y such that $Y \neq \emptyset$ and $X < Y$ holds $(\mathrm{Sgm}_0\, Y)(0) = (\mathrm{Sgm}_0(X \cup Y))(\mathrm{len}\, \mathrm{Sgm}_0\, X)$.

(44) Let l, m, n, k be natural numbers and X be a finite natural-membered set. If $k < l$ and $m < \mathrm{len}\, \mathrm{Sgm}_0\, X$ and $(\mathrm{Sgm}_0\, X)(m) = k$ and $(\mathrm{Sgm}_0\, X)(n) = l$, then $m < n$.

(45) For all finite natural-membered sets X, Y such that $X \neq \emptyset$ and $X < Y$ holds $(\mathrm{Sgm}_0\, X)(0) = (\mathrm{Sgm}_0(X \cup Y))(0)$.

(46) For all finite natural-membered sets X, Y holds $X < Y$ iff $\mathrm{Sgm}_0(X \cup Y) = (\mathrm{Sgm}_0\, X) \frown \mathrm{Sgm}_0\, Y$.

Let f be a finite 0-sequence and let B be a set. The B-subsequence of f yields a finite 0-sequence and is defined as follows:

(Def. 8) The B-subsequence of $f = f \cdot \mathrm{Sgm}_0(B \cap \mathrm{len}\, f)$.

One can prove the following proposition

(47) Let f be a finite 0-sequence and B be a set. Then

(i) len (the B-subsequence of f) $= \overline{\overline{B \cap \mathrm{len}\, f}}$, and

(ii) for every natural number i such that $i <$ len (the B-subsequence of f) holds (the B-subsequence of f)$(i) = f((\mathrm{Sgm}_0(B \cap \mathrm{len}\, f))(i))$.

Let D be a set, let f be a finite 0-sequence of D, and let B be a set. Then the B-subsequence of f is a finite 0-sequence of D.

Let f be a finite 0-sequence. One can verify that the $\emptyset$-subsequence of f is empty.

Let B be a set. Observe that the B-subsequence of $\emptyset$ is empty.

We now state the proposition

(48) Let B_1, B_2 be finite natural-membered sets and f be a finite 0-sequence of $\mathbb{R}$. Suppose $B_1 < B_2$. Then $\sum$ the $B_1 \cup B_2$-subsequence of $f = (\sum$ the B_1-subsequence of $f) + \sum$ the B_2-subsequence of f.

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