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# Several Higher Differentiation Formulas of Special Functions

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**Summary.** In this paper, we proved some basic properties of higher differentiation, and higher differentiation formulas of special functions [4].

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The notation and terminology used in this paper are introduced in the following articles: [16], [13], [2], [3], [5], [1], [7], [9], [12], [10], [8], [18], [14], [11], [6], [15], and [17].

For simplicity, we use the following convention: $x$, $r$, $a$, $x_0$, $p$ are real numbers, $n$, $i$, $m$ are elements of $\mathbb{N}$, $Z$ is an open subset of $\mathbb{R}$, and $f$, $f_1$, $f_2$ are partial functions from $\mathbb{R}$ to $\mathbb{R}$.

Next we state a number of propositions:

(1) For every function $f$ from $\mathbb{R}$ into $\mathbb{R}$ holds $\mathrm{dom}(f \restriction Z) = Z$.

(2) $(-f_1) - f_2 = f_1\, f_2$.

(3) If $n \geq 1$, then $\mathrm{dom}(\frac{1}{\square^n}) = \mathbb{R} \setminus \{0\}$ and $(\square^n)^{-1}(\{0\}) = \{0\}$.

(4) $(r \cdot p)\, \frac{1}{\square^n} = r\,(p\, \frac{1}{\square^n})$.

(5) For all elements $n$, $m$ of $\mathbb{R}$ holds $n\, f + m\, f = (n + m)\, f$.

(6) If $f \restriction Z$ is differentiable on $Z$, then $f$ is differentiable on $Z$.




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(7) If $n \geq 1$ and $f$ is differentiable $n$ times on $Z$, then $f$ is differentiable on $Z$.

(8) $\square^n$ is differentiable on $\mathbb{R}$.

(9) If $x \in Z$, then (the function sin)$'(Z)(2)(x) = -\sin x$.

(10) If $x \in Z$, then (the function sin)$'(Z)(3)(x) = -\cos x$.

(11) If $x \in Z$, then (the function sin)$'(Z)(n)(x) = \sin(x + \frac{n \cdot \pi}{2})$.

(12) If $x \in Z$, then (the function cos)$'(Z)(2)(x) = -\cos x$.

(13) If $x \in Z$, then (the function cos)$'(Z)(3)(x) = \sin x$.

(14) If $x \in Z$, then (the function cos)$'(Z)(n)(x) = \cos(x + \frac{n \cdot \pi}{2})$.

(15) If $f_1$ is differentiable $n$ times on $Z$ and $f_2$ is differentiable $n$ times on $Z$, then $(f_1 + f_2)'(Z)(n) = {f_1}'(Z)(n) + {f_2}'(Z)(n)$.

(16) If $f_1$ is differentiable $n$ times on $Z$ and $f_2$ is differentiable $n$ times on $Z$, then $(f_1 - f_2)'(Z)(n) = {f_1}'(Z)(n) - {f_2}'(Z)(n)$.

(17) If $f_1$ is differentiable $n$ times on $Z$ and $f_2$ is differentiable $n$ times on $Z$ and $i \leq n$, then $(f_1 + f_2)'(Z)(i) = {f_1}'(Z)(i) + {f_2}'(Z)(i)$.

(18) If $f_1$ is differentiable $n$ times on $Z$ and $f_2$ is differentiable $n$ times on $Z$ and $i \leq n$, then $(f_1 - f_2)'(Z)(i) = {f_1}'(Z)(i) - {f_2}'(Z)(i)$.

(19) If $f_1$ is differentiable $n$ times on $Z$ and $f_2$ is differentiable $n$ times on $Z$, then $f_1 + f_2$ is differentiable $n$ times on $Z$.

(20) If $f_1$ is differentiable $n$ times on $Z$ and $f_2$ is differentiable $n$ times on $Z$, then $f_1 - f_2$ is differentiable $n$ times on $Z$.

(21) If $f$ is differentiable $n$ times on $Z$, then $(r\,f)'(Z)(n) = r\,f'(Z)(n)$.

(22) If $f$ is differentiable $n$ times on $Z$, then $r\,f$ is differentiable $n$ times on $Z$.

(23) If $f$ is differentiable on $Z$, then $f'(Z)(1) = f'_{\restriction Z}$.

(24) If $n \geq 1$ and $f$ is differentiable $n$ times on $Z$, then $f'(Z)(1) = f'_{\restriction Z}$.

(25) If $x \in Z$, then $(r$ (the function sin)$)'(Z)(n)(x) = r \cdot \sin(x + \frac{n \cdot \pi}{2})$.

(26) If $x \in Z$, then $(r$ (the function cos)$)'(Z)(n)(x) = r \cdot \cos(x + \frac{n \cdot \pi}{2})$.

(27) If $x \in Z$, then $(r$ (the function exp)$)'(Z)(n)(x) = r \cdot \exp x$.

(28) $(\square^n)'_{\restriction Z} = (n\,(\square^{n-1})) \restriction Z$.

(29) If $x \neq 0$, then $\frac{1}{\square^n}$ is differentiable in $x$ and $(\frac{1}{\square^n})'(x) = -\frac{n \cdot x^{n-1}}{(x^n)^{\mathbf{2}}}$.

(30) If $n \geq 1$, then $(\square^n)'(Z)(2) = ((n \cdot (n-1))\,(\square^{n-2})) \restriction Z$.

(31) If $n \geq 2$, then $(\square^n)'(Z)(3) = ((n \cdot (n-1) \cdot (n-2))\,(\square^{n-3})) \restriction Z$.

(32) If $n > m$, then $(\square^n)'(Z)(m) = ((\binom{n}{m} \cdot m!)\,(\square^{n-m})) \restriction Z$.

(33) If $f$ is differentiable $n$ times on $Z$, then $(-f)'(Z)(n) = -f'(Z)(n)$ and $-f$ is differentiable $n$ times on $Z$.

(34) If $x_0 \in Z$, then (Taylor(the function $\sin, Z, x_0, x$))$(n) =$

$\frac{\sin(x_0+\frac{n\cdot\pi}{2})\cdot(x-x_0)^n}{n!}$ and (Taylor(the function cos, $Z, x_0, x$))($n$) = $\frac{\cos(x_0+\frac{n\cdot\pi}{2})\cdot(x-x_0)^n}{n!}$.

(35) If $r > 0$, then (Maclaurin(the function sin, $]-r, r[, x$))($n$) = $\frac{\sin(\frac{n\cdot\pi}{2})\cdot x^n}{n!}$ and (Maclaurin(the function cos, $]-r, r[, x$))($n$) = $\frac{\cos(\frac{n\cdot\pi}{2})\cdot x^n}{n!}$.

(36) If $n > m$ and $x \in Z$, then $(\square^n)'(Z)(m)(x) = \binom{n}{m} \cdot m! \cdot x^{n-m}$.

(37) If $x \in Z$, then $(\square^m)'(Z)(m)(x) = m!$.

(38) $\square^n$ is differentiable $n$ times on $Z$.

(39) If $x \in Z$ and $n > m$, then $(a\,(\square^n))'(Z)(m)(x) = a \cdot \binom{n}{m} \cdot m! \cdot x^{n-m}$.

(40) If $x \in Z$, then $(a\,(\square^n))'(Z)(n)(x) = a \cdot n!$.

(41) If $x_0 \in Z$ and $n > m$, then (Taylor($\square^n, Z, x_0, x$))($m$) = $\binom{n}{m} \cdot {x_0}^{n-m} \cdot (x - x_0)^m$ and (Taylor($\square^n, Z, x_0, x$))($n$) = $(x - x_0)^n$.

(42) Let $n$, $m$ be elements of $\mathbb{N}$ and $r$, $x$ be real numbers. If $n > m$ and $r > 0$, then (Maclaurin($\square^n, ]-r, r[, x$))($m$) = 0 and (Maclaurin($\square^n, ]-r, r[, x$))($n$) = $x^n$.

(43) $\frac{1}{\square^n}$ is differentiable on $]0, r[$.

(44) If $x_0 \in ]0, r[$, then $(\frac{1}{\square^n})'_{\restriction ]0,r[}(x_0) = -\frac{n}{(\square^{n+1})(x_0)}$.

(45) If $x \neq 0$, then $\frac{1}{\square^1}$ is differentiable in $x$ and $(\frac{1}{\square^1})'(x) = -\frac{1}{(x^1)^{\mathbf{2}}}$.

(46) If $]0, r[ \subseteq \mathrm{dom}(\frac{1}{\square^2})$, then $(\frac{1}{\square^1})'_{\restriction ]0,r[} = ((-1)\,\frac{1}{\square^2}) \restriction ]0, r[$.

(47) If $x \neq 0$, then $\frac{1}{\square^2}$ is differentiable in $x$ and $(\frac{1}{\square^2})'(x) = -\frac{2\cdot x^1}{(x^2)^{\mathbf{2}}}$.

(48) If $]0, r[ \subseteq \mathrm{dom}(\frac{1}{\square^3})$, then $(\frac{1}{\square^2})'_{\restriction ]0,r[} = ((-2)\,\frac{1}{\square^3}) \restriction ]0, r[$.

(49) If $n \geq 1$, then $(\frac{1}{\square^n})'_{\restriction ]0,r[} = ((-n)\,\frac{1}{\square^{n+1}}) \restriction ]0, r[$.

(50) Suppose $f_1$ is differentiable 2 times on $Z$ and $f_2$ is differentiable 2 times on $Z$. Then $(f_1\,f_2)'(Z)(2) = {f_1}'(Z)(2)\,f_2 + 2\,((f_1)'_{\restriction Z}\,(f_2)'_{\restriction Z}) + f_1\,{f_2}'(Z)(2)$.

(51) If $Z \subseteq$ dom (the function ln) and $Z \subseteq \mathrm{dom}(\frac{1}{\square^1})$, then (the function ln)$'_{\restriction Z} = \frac{1}{\square^1} \restriction Z$.

(52) If $n \geq 1$ and $x_0 \in ]0, r[$, then $(\frac{1}{\square^n})'(]0, r[)(2)(x_0) = n \cdot (n+1) \cdot (\frac{1}{\square^{n+2}})(x_0)$.

(53) ((The function sin) (the function sin))$'(Z)(2) = 2\,$(((the function cos) (the function cos))$\restriction Z) + (-2)\,$(((the function sin) (the function sin))$\restriction Z$).

(54) ((The function cos) (the function cos))$'(Z)(2) = 2\,$(((the function sin) (the function sin))$\restriction Z) + (-2)\,$(((the function cos) (the function cos))$\restriction Z$).

(55) ((The function sin) (the function cos))$'(Z)(2) =$ 
$4\,$(((−the function sin) (the function cos))$\restriction Z$).

(56) Suppose $Z \subseteq$ dom (the function tan). Then the function tan is differentiable on $Z$ and (the function tan)$'_{\restriction Z} = (\frac{1}{\text{the function cos}}\,\frac{1}{\text{the function cos}}) \restriction Z$.

(57) Suppose $Z \subseteq$ dom (the function tan). Then $\frac{1}{\text{the function cos}}$ is differentiable on $Z$ and $(\frac{1}{\text{the function cos}})'_{\restriction Z} = (\frac{1}{\text{the function cos}}$ (the function tan)$) \restriction Z$.

(58) Suppose $Z \subseteq \operatorname{dom}$ (the function tan). Then (the function tan)$'(Z)(2) = 2\,(((\text{the function tan})\ \frac{1}{\text{the function cos}}\ \frac{1}{\text{the function cos}}) \restriction Z)$.

(59) Suppose $Z \subseteq \operatorname{dom}$ (the function cot). Then
 (i) the function cot is differentiable on $Z$, and
 (ii) (the function cot)$'_{\restriction Z} = ((-1)\,(\frac{1}{\text{the function sin}}\ \frac{1}{\text{the function sin}})) \restriction Z$.

(60) Suppose $Z \subseteq \operatorname{dom}$ (the function cot). Then
 (i) $\frac{1}{\text{the function sin}}$ is differentiable on $Z$, and
 (ii) $(\frac{1}{\text{the function sin}})'_{\restriction Z} = (-\frac{1}{\text{the function sin}}\ (\text{the function cot})) \restriction Z$.

(61) Suppose $Z \subseteq \operatorname{dom}$ (the function cot). Then (the function cot)$'(Z)(2) = 2\,(((\text{the function cot})\ \frac{1}{\text{the function sin}}\ \frac{1}{\text{the function sin}}) \restriction Z)$.

(62) ((The function exp) (the function sin))$'(Z)(2) = 2\,(((\text{the function exp})\ (\text{the function cos})) \restriction Z)$.

(63) ((The function exp) (the function cos))$'(Z)(2) = 2\,(((\text{the function exp})\ -\text{the function sin}) \restriction Z)$.

(64) Suppose $f_1$ is differentiable 3 times on $Z$ and $f_2$ is differentiable 3 times on $Z$. Then $(f_1\,f_2)'(Z)(3) = f_1{}'(Z)(3)\,f_2 + (3\,(f_1{}'(Z)(2)\,(f_2)'_{\restriction Z}) + 3\,((f_1)'_{\restriction Z}\,f_2{}'(Z)(2))) + f_1\,f_2{}'(Z)(3)$.

(65) ((The function sin) (the function sin))$'(Z)(3) = (-8)\,(((\text{the function cos})\ (\text{the function sin})) \restriction Z)$.

(66) If $f$ is differentiable 2 times on $Z$, then $(f\,f)'(Z)(2) = 2\,(f\,f'(Z)(2)) + 2\,(f'_{\restriction Z}\,f'_{\restriction Z})$.

(67) Suppose $f$ is differentiable 2 times on $Z$ and for every $x_0$ such that $x_0 \in Z$ holds $f(x_0) \neq 0$. Then $(\frac{1}{f})'(Z)(2) = \frac{2\,f'_{\restriction Z}\,f'_{\restriction Z} - f'(Z)(2)\,f}{f\,(f\,f)}$.

(68) ((The function exp) (the function sin))$'(Z)(3) = (2\,((\text{the function exp})\ (-\text{the function sin} + \text{the function cos}))) \restriction Z$.

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