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Inferior Limit, Superior Limit and Convergence of Sequences of Extended Real Numbers

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Summary. In this article, we extended properties of sequences of real numbers to sequences of extended real numbers. We also introduced basic properties of the inferior limit, superior limit and convergence of sequences of extended real numbers.

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The notation and terminology used in this paper are introduced in the following articles: [18], [19], [1], [17], [20], [5], [21], [6], [7], [16], [2], [3], [8], [15], [13], [14], [12], [11], [22], [4], [10], and [9].

We adopt the following convention: n, m, k are elements of $\mathbb{N}$, X is a non empty subset of $\overline{\mathbb{R}}$, and Y is a non empty subset of $\mathbb{R}$.

Next we state four propositions:

(1) If $X = Y$ and Y is upper bounded, then X is upper bounded and $\sup X = \sup Y$.

(2) If $X = Y$ and X is upper bounded, then Y is upper bounded and $\sup X = \sup Y$.

(3) If $X = Y$ and Y is lower bounded, then X is lower bounded and $\inf X = \inf Y$.

(4) If $X = Y$ and X is lower bounded, then Y is lower bounded and $\inf X = \inf Y$.

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Let s_1 be a sequence of extended reals. The functor $\sup s_1$ yields an element of $\overline{\mathbb{R}}$ and is defined by:

(Def. 1) $\sup s_1 = \sup \operatorname{rng} s_1$.

The functor $\inf s_1$ yields an element of $\overline{\mathbb{R}}$ and is defined as follows:

(Def. 2) $\inf s_1 = \inf \operatorname{rng} s_1$.

Let s_1 be a sequence of extended reals. We say that s_1 is lower bounded if and only if:

(Def. 3) $\operatorname{rng} s_1$ is lower bounded.

We say that s_1 is upper bounded if and only if:

(Def. 4) $\operatorname{rng} s_1$ is upper bounded.

Let s_1 be a sequence of extended reals. We say that s_1 is bounded if and only if:

(Def. 5) s_1 is upper bounded and lower bounded.

In the sequel s_1 is a sequence of extended reals.

One can prove the following proposition

(5) For all s_1, n holds $\{s_1(k); k$ ranges over elements of $\mathbb{N}$: $n \leq k\}$ is a non empty subset of $\overline{\mathbb{R}}$.

Let s_1 be a sequence of extended reals. The inferior realsequence s_1 yields a sequence of extended reals and is defined by the condition (Def. 6).

(Def. 6) Let n be an element of $\mathbb{N}$. Then there exists a non empty subset Y of $\overline{\mathbb{R}}$ such that $Y = \{s_1(k); k$ ranges over elements of $\mathbb{N}$: $n \leq k\}$ and (the inferior realsequence $s_1)(n) = \inf Y$.

Let s_1 be a sequence of extended reals. The superior realsequence s_1 yields a sequence of extended reals and is defined by the condition (Def. 7).

(Def. 7) Let n be an element of $\mathbb{N}$. Then there exists a non empty subset Y of $\overline{\mathbb{R}}$ such that $Y = \{s_1(k); k$ ranges over elements of $\mathbb{N}$: $n \leq k\}$ and (the superior realsequence $s_1)(n) = \sup Y$.

We now state the proposition

(6) If s_1 is finite, then s_1 is a sequence of real numbers.

Let f be a partial function from $\mathbb{N}$ to $\overline{\mathbb{R}}$. We say that f is increasing if and only if:

(Def. 8) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m < n$ holds $f(m) < f(n)$.

We say that f is decreasing if and only if:

(Def. 9) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m < n$ holds $f(m) > f(n)$.

We say that f is non-decreasing if and only if:

(Def. 10) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m \leq n$ holds $f(m) \leq f(n)$.

We say that f is non-increasing if and only if:

(Def. 11) For all m, n such that $m \in \operatorname{dom} f$ and $n \in \operatorname{dom} f$ and $m \leq n$ holds $f(m) \geq f(n)$.

One can prove the following two propositions:

(7)(i) s_1 is increasing iff for all elements n, m of $\mathbb{N}$ such that $m < n$ holds $s_1(m) < s_1(n)$,

(ii) s_1 is decreasing iff for all elements n, m of $\mathbb{N}$ such that $m < n$ holds $s_1(n) < s_1(m)$,

(iii) s_1 is non-decreasing iff for all elements n, m of $\mathbb{N}$ such that $m \leq n$ holds $s_1(m) \leq s_1(n)$, and

(iv) s_1 is non-increasing iff for all elements n, m of $\mathbb{N}$ such that $m \leq n$ holds $s_1(n) \leq s_1(m)$.

(8) (The inferior realsequence $s_1)(n) \leq s_1(n)$ and $s_1(n) \leq$ (the superior realsequence $s_1)(n)$.

Let us consider s_1. Observe that the superior realsequence s_1 is non-increasing and the inferior realsequence s_1 is non-decreasing.

Let s_1 be a sequence of extended reals. The functor $\limsup s_1$ yields an element of $\overline{\mathbb{R}}$ and is defined by:

(Def. 12) $\limsup s_1 = \inf$ (the superior realsequence s_1).

The functor $\liminf s_1$ yields an element of $\overline{\mathbb{R}}$ and is defined by:

(Def. 13) $\liminf s_1 = \sup$ (the inferior realsequence s_1).

In the sequel r_1 is a sequence of real numbers.

The following propositions are true:

(9) If $s_1 = r_1$ and r_1 is bounded, then the superior realsequence $s_1 =$ the superior realsequence r_1 and $\limsup s_1 = \limsup r_1$.

(10) If $s_1 = r_1$ and r_1 is bounded, then the inferior realsequence $s_1 =$ the inferior realsequence r_1 and $\liminf s_1 = \liminf r_1$.

(11) If s_1 is bounded, then s_1 is a sequence of real numbers.

(12) If $s_1 = r_1$, then s_1 is upper bounded iff r_1 is upper bounded.

(13) If $s_1 = r_1$, then s_1 is lower bounded iff r_1 is lower bounded.

(14) If $s_1 = r_1$ and r_1 is convergent, then s_1 is convergent to finite number and convergent and $\lim s_1 = \lim r_1$.

(15) If $s_1 = r_1$ and s_1 is convergent to finite number, then r_1 is convergent and $\lim s_1 = \lim r_1$.

(16) If $s_1 \uparrow k$ is convergent to finite number, then s_1 is convergent to finite number and convergent and $\lim s_1 = \lim(s_1 \uparrow k)$.

(17) If $s_1 \uparrow k$ is convergent, then s_1 is convergent and $\lim s_1 = \lim(s_1 \uparrow k)$.

(18) If $\limsup s_1 = \liminf s_1$ and $\liminf s_1 \in \mathbb{R}$, then there exists k such that $s_1 \uparrow k$ is bounded.

(19) If s_1 is convergent to finite number, then there exists k such that $s_1 \uparrow k$ is bounded.

(20) Suppose s_1 is convergent to finite number. Then $s_1 \uparrow k$ is convergent to finite number and $s_1 \uparrow k$ is convergent and $\lim s_1 = \lim(s_1 \uparrow k)$.

(21) If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim s_1 = \lim(s_1 \uparrow k)$.

(22) If s_1 is upper bounded, then $s_1 \uparrow k$ is upper bounded and if s_1 is lower bounded, then $s_1 \uparrow k$ is lower bounded.

(23) $\inf s_1 \leq s_1(n)$ and $s_1(n) \leq \sup s_1$.

(24) $\inf s_1 \leq \sup s_1$.

(25) If s_1 is non-increasing, then $s_1 \uparrow k$ is non-increasing and $\inf s_1 = \inf(s_1 \uparrow k)$.

(26) If s_1 is non-decreasing, then $s_1 \uparrow k$ is non-decreasing and $\sup s_1 = \sup(s_1 \uparrow k)$.

(27) (The superior realsequence $s_1)(n) = \sup(s_1 \uparrow n)$ and (the inferior realsequence $s_1)(n) = \inf(s_1 \uparrow n)$.

(28) Let s_1 be a sequence of extended reals and j be an element of $\mathbb{N}$. Then the superior realsequence $s_1 \uparrow j =$ (the superior realsequence $s_1) \uparrow j$ and $\limsup(s_1 \uparrow j) = \limsup s_1$.

(29) Let s_1 be a sequence of extended reals and j be an element of $\mathbb{N}$. Then the inferior realsequence $s_1 \uparrow j =$ (the inferior realsequence $s_1) \uparrow j$ and $\liminf(s_1 \uparrow j) = \liminf s_1$.

(30) Let s_1 be a sequence of extended reals and k be an element of $\mathbb{N}$. Suppose s_1 is non-increasing and $-\infty < s_1(k)$ and $s_1(k) < +\infty$. Then $s_1 \uparrow k$ is upper bounded and $\sup(s_1 \uparrow k) = s_1(k)$.

(31) Let s_1 be a sequence of extended reals and k be an element of $\mathbb{N}$. Suppose s_1 is non-decreasing and $-\infty < s_1(k)$ and $s_1(k) < +\infty$. Then $s_1 \uparrow k$ is lower bounded and $\inf(s_1 \uparrow k) = s_1(k)$.

(32) Let s_1 be a sequence of extended reals. Suppose that for every element n of $\mathbb{N}$ holds $+\infty \leq s_1(n)$. Then s_1 is convergent to $+\infty$.

(33) Let s_1 be a sequence of extended reals. Suppose that for every element n of $\mathbb{N}$ holds $s_1(n) \leq -\infty$. Then s_1 is convergent to $-\infty$.

(34) Let s_1 be a sequence of extended reals. Suppose s_1 is non-increasing and $-\infty = \inf s_1$. Then s_1 is convergent to $-\infty$ and $\lim s_1 = -\infty$.

(35) Let s_1 be a sequence of extended reals. Suppose s_1 is non-decreasing and $+\infty = \sup s_1$. Then s_1 is convergent to $+\infty$ and $\lim s_1 = +\infty$.

(36) For every sequence s_1 of extended reals such that s_1 is non-increasing holds s_1 is convergent and $\lim s_1 = \inf s_1$.

(37) For every sequence s_1 of extended reals such that s_1 is non-decreasing holds s_1 is convergent and $\lim s_1 = \sup s_1$.

(38) Let s_2, s_3 be sequences of extended reals. Suppose s_2 is convergent and s_3 is convergent and for every element n of $\mathbb{N}$ holds $s_2(n) \leq s_3(n)$. Then $\lim s_2 \leq \lim s_3$.

(39) For every sequence s_1 of extended reals holds $\liminf s_1 \leq \limsup s_1$.

(40) For every sequence s_1 of extended reals holds s_1 is convergent iff $\liminf s_1 = \limsup s_1$.

(41) For every sequence s_1 of extended reals such that s_1 is convergent holds $\lim s_1 = \liminf s_1$ and $\lim s_1 = \limsup s_1$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):163–171, 1991.
[3] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.
[4] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[8] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.
[9] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(**2**):53–70, 2006.
[10] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(**3**):495–500, 2001.
[11] Adam Grabowski. On the Kuratowski limit operators. *Formalized Mathematics*, 11(**4**):399–409, 2003.
[12] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[13] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[14] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[15] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[16] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.
[17] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[20] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[21] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[22] Bo Zhang, Hiroshi Yamazaki, and Yatsuka Nakamura. Inferior limit and superior limit of sequences of real numbers. *Formalized Mathematics*, 13(**3**):375–381, 2005.

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