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The Product Space of Real Normed Spaces and its Properties

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Summary. In this article, we define the product space of real linear spaces and real normed spaces. We also describe properties of these spaces.

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The terminology and notation used here are introduced in the following articles: [20], [9], [22], [2], [1], [19], [5], [23], [7], [10], [8], [4], [13], [12], [21], [14], [3], [6], [16], [11], [15], [17], and [18].

1. The Product Space of Real Linear Spaces

The following propositions are true:

(1) Let s, t be sequences of real numbers and g be a real number. Suppose that for every element n of $\mathbb{N}$ holds $t(n) = |s(n) - g|$. Then s is convergent and $\lim s = g$ if and only if t is convergent and $\lim t = 0$.

(2) Let x, y be finite sequences of elements of $\mathbb{R}$. Suppose $\operatorname{len} x = \operatorname{len} y$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{Seg} \operatorname{len} x$ holds $0 \leq x(i)$ and $x(i) \leq y(i)$. Then $|x| \leq |y|$.

(3) Let F be a finite sequence of elements of $\mathbb{R}$. If for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} F$ holds $F(i) = 0$, then $\sum F = 0$.

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Let f be a function and let X be a set. A function is called a multi-operation of X and f if:

(Def. 1) $\operatorname{dom} \mathrm{it} = \operatorname{dom} f$ and for every set i such that $i \in \operatorname{dom} f$ holds $\mathrm{it}(i)$ is a function from $[\!: X, f(i) :\!]$ into $f(i)$.

Let F be a sequence of non empty sets and let X be a set. Observe that every multi-operation of X and F is finite sequence-like.

We now state the proposition

(4) Let X be a set, F be a sequence of non empty sets, and p be a finite sequence. Then p is a multi-operation of X and F if and only if $\operatorname{len} p = \operatorname{len} F$ and for every set i such that $i \in \operatorname{dom} F$ holds $p(i)$ is a function from $[\!: X, F(i) :\!]$ into $F(i)$.

Let F be a sequence of non empty sets, let X be a set, let p be a multi-operation of X and F, and let i be an element of $\operatorname{dom} F$. Then $p(i)$ is a function from $[\!: X, F(i) :\!]$ into $F(i)$.

Next we state the proposition

(5) Let X be a non empty set, F be a sequence of non empty sets, and f, g be functions from $[\!: X, \prod F :\!]$ into $\prod F$. Suppose that for every element x of X and for every element d of $\prod F$ and for every element i of $\operatorname{dom} F$ holds $f(x, d)(i) = g(x, d)(i)$. Then $f = g$.

Let F be a sequence of non empty sets, let X be a non empty set, and let p be a multi-operation of X and F. The functor $\prod^{\circ} p$ yielding a function from $[\!: X, \prod F :\!]$ into $\prod F$ is defined as follows:

(Def. 2) For every element x of X and for every element d of $\prod F$ and for every element i of $\operatorname{dom} F$ holds $(\prod^{\circ} p)(x, d)(i) = p(i)(x, d(i))$.

Let R be a binary relation. We say that R is real-linear-space-yielding if and only if:

(Def. 3) For every set S such that $S \in \operatorname{rng} R$ holds S is a real linear space.

Let us note that there exists a finite sequence which is non empty and real-linear-space-yielding.

A real linear space-sequence is a non empty real-linear-space-yielding finite sequence.

Let G be a real linear space-sequence and let j be an element of $\operatorname{dom} G$. Then $G(j)$ is a real linear space.

Let G be a real linear space-sequence. The functor $\overline{G}$ yielding a sequence of non empty sets is defined by:

(Def. 4) $\operatorname{len} \overline{G} = \operatorname{len} G$ and for every element j of $\operatorname{dom} G$ holds $\overline{G}(j) =$ the carrier of $G(j)$.

Let G be a real linear space-sequence and let j be an element of $\operatorname{dom} \overline{G}$. Then $G(j)$ is a real linear space.

Let G be a real linear space-sequence. The functor $\langle +_{G_i} \rangle_i$ yielding a family of binary operations of $\overline{G}$ is defined as follows:

(Def. 5) $\mathrm{len}(\langle +_{G_i} \rangle_i) = \mathrm{len}\,\overline{G}$ and for every element j of $\mathrm{dom}\,\overline{G}$ holds $\langle +_{G_i} \rangle_i(j) =$ the addition of $G(j)$.

The functor $\langle -_{G_i} \rangle_i$ yields a family of unary operations of $\overline{G}$ and is defined as follows:

(Def. 6) $\mathrm{len}(\langle -_{G_i} \rangle_i) = \mathrm{len}\,\overline{G}$ and for every element j of $\mathrm{dom}\,\overline{G}$ holds $\langle -_{G_i} \rangle_i(j) = \mathrm{comp}\,G(j)$.

The functor $\langle 0_{G_i} \rangle_i$ yielding an element of $\prod \overline{G}$ is defined by:

(Def. 7) For every element j of $\mathrm{dom}\,\overline{G}$ holds $\langle 0_{G_i} \rangle_i(j) =$ the zero of $G(j)$.

The functor $\mathrm{multop}\,G$ yields a multi-operation of $\mathbb{R}$ and $\overline{G}$ and is defined by:

(Def. 8) $\mathrm{len}\,\mathrm{multop}\,G = \mathrm{len}\,\overline{G}$ and for every element j of $\mathrm{dom}\,\overline{G}$ holds $(\mathrm{multop}\,G)(j) =$ the external multiplication of $G(j)$.

Let G be a real linear space-sequence. The functor $\prod G$ yielding a strict non empty RLS structure is defined by:

(Def. 9) $\prod G = \langle \prod \overline{G}, \langle 0_{G_i} \rangle_i, \prod^\circ(\langle +_{G_i} \rangle_i), \prod^\circ \mathrm{multop}\,G \rangle$.

Let G be a real linear space-sequence. One can check that $\prod G$ is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

2. The Product Space of Real Normed Spaces

Let R be a binary relation. We say that R is real-norm-space-yielding if and only if:

(Def. 10) For every set x such that $x \in \mathrm{rng}\,R$ holds x is a real normed space.

One can check that there exists a finite sequence which is non empty and real-norm-space-yielding.

A real norm space-sequence is a non empty real-norm-space-yielding finite sequence.

Let G be a real norm space-sequence and let j be an element of $\mathrm{dom}\,G$. Then $G(j)$ is a real normed space.

Let us note that every finite sequence which is real-norm-space-yielding is also real-linear-space-yielding.

Let G be a real norm space-sequence and let x be an element of $\prod \overline{G}$. The functor $\mathrm{normsequence}(G, x)$ yields an element of $\mathcal{R}^{\mathrm{len}\,G}$ and is defined as follows:

(Def. 11) $\mathrm{len}\,\mathrm{normsequence}(G, x) = \mathrm{len}\,G$ and for every element j of $\mathrm{dom}\,G$ holds $(\mathrm{normsequence}(G, x))(j) = (\text{the norm of } G(j))(x(j))$.

Let G be a real norm space-sequence. The functor $\mathrm{productnorm}\,G$ yields a function from $\prod \overline{(G \textbf{ qua } \text{real linear space-sequence})}$ into $\mathbb{R}$ and is defined by:

(Def. 12) For every element x of $\prod \overline{G}$ holds $(\mathrm{productnorm}\,G)(x) = |\,\mathrm{normsequence}(G, x)|$.

Let G be a real norm space-sequence. The functor $\prod G$ yielding a strict non empty normed structure is defined as follows:

(Def. 13) The RLS structure of $\prod G = \prod(G$ **qua** real linear space-sequence) and the norm of $\prod G = \operatorname{productnorm} G$.

In the sequel G is a real norm space-sequence.

We now state four propositions:

(6) $\prod G = \langle \prod \overline{G}, \langle 0_{G_i} \rangle_i, \prod^\circ(\langle +_{G_i} \rangle_i), \prod^\circ \operatorname{multop} G, \operatorname{productnorm} G \rangle$.

(7) For every vector x of $\prod G$ and for every element y of $\prod \overline{G}$ such that $x = y$ holds $\|x\| = |\operatorname{normsequence}(G, y)|$.

(8) For all elements x, y, z of $\prod \overline{G}$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} x$ and $z = (\prod^\circ(\langle +_{G_i} \rangle_i))(x, y)$ holds $(\operatorname{normsequence}(G, z))(i) \leq (\operatorname{normsequence}(G, x) + \operatorname{normsequence}(G, y))(i)$.

(9) For every element x of $\prod \overline{G}$ and for every element i of $\mathbb{N}$ such that $i \in \operatorname{dom} x$ holds $0 \leq (\operatorname{normsequence}(G, x))(i)$.

Let G be a real norm space-sequence. Observe that $\prod G$ is real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following propositions:

(10) Let G be a real norm space-sequence, i be an element of $\operatorname{dom} G$, x be a point of $\prod G$, y be an element of $\prod \overline{G}$, and x_1 be a point of $G(i)$. If $y = x$ and $x_1 = y(i)$, then $\|x_1\| \leq \|x\|$.

(11) Let G be a real norm space-sequence, i be an element of $\operatorname{dom} G$, x, y be points of $\prod G$, x_1, y_1 be points of $G(i)$, and z_1, z_2 be elements of $\prod \overline{G}$. If $x_1 = z_1(i)$ and $z_1 = x$ and $y_1 = z_2(i)$ and $z_2 = y$, then $\|y_1 - x_1\| \leq \|y - x\|$.

(12) Let G be a real norm space-sequence, s_1 be a sequence of $\prod G$, x_0 be a point of $\prod G$, and y_0 be an element of $\prod \overline{G}$. Suppose $x_0 = y_0$ and s_1 is convergent and $\lim s_1 = x_0$. Let i be an element of $\operatorname{dom} G$. Then there exists a sequence s_2 of $G(i)$ such that s_2 is convergent and $y_0(i) = \lim s_2$ and for every element m of $\mathbb{N}$ there exists an element s_3 of $\prod \overline{G}$ such that $s_3 = s_1(m)$ and $s_2(m) = s_3(i)$.

(13) Let G be a real norm space-sequence, s_1 be a sequence of $\prod G$, x_0 be a point of $\prod G$, and y_0 be an element of $\prod \overline{G}$. Suppose that

(i) $x_0 = y_0$, and

(ii) for every element i of $\operatorname{dom} G$ there exists a sequence s_2 of $G(i)$ such that s_2 is convergent and $y_0(i) = \lim s_2$ and for every element m of $\mathbb{N}$ there exists an element s_3 of $\prod \overline{G}$ such that $s_3 = s_1(m)$ and $s_2(m) = s_3(i)$.

Then s_1 is convergent and $\lim s_1 = x_0$.

(14) For every real norm space-sequence G such that for every element i of $\operatorname{dom} G$ holds $G(i)$ is complete holds $\prod G$ is complete.

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