FORMALIZED MATHEMATICS
2007, Vol. **15**, No. **3**, Pages 151–157

Some Properties of Line and Column Operations on Matrices

Xiquan Liang
Qingdao University of Science and Technology
China

Tao Sun
Qingdao University of Science and Technology
China

Dahai Hu
Qingdao University of Science and Technology
China

Summary. This article describes definitions of elementary operations about matrix and their main properties.

MML identifier: MATRIX12, version: 7.8.05 4.87.985

The articles [8], [13], [17], [11], [1], [18], [5], [6], [2], [7], [15], [16], [9], [10], [20], [4], [3], [21], [12], [14], and [19] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: j, k, l, n, m, i are natural numbers, K is a field, a is an element of K, M, M_1 are matrices over K of dimension $n \times m$, and A is a matrix over K of dimension n.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, and let l, k be natural numbers. The functor InterchangeLine(M, l, k) yielding a matrix over K of dimension $n \times m$ is defined by the conditions (Def. 1).

(Def. 1)(i) $\mathrm{len}\,\mathrm{InterchangeLine}(M, l, k) = \mathrm{len}\, M$, and

(ii) for all i, j such that $i \in \mathrm{dom}\, M$ and $j \in \mathrm{Seg}\,\mathrm{width}\, M$ holds if $i = l$, then $(\mathrm{InterchangeLine}(M, l, k))_{i,j} = M_{k,j}$ and if $i = k$, then $(\mathrm{InterchangeLine}(M, l, k))_{i,j} = M_{l,j}$ and if $i \neq l$ and $i \neq k$, then $(\mathrm{InterchangeLine}(M, l, k))_{i,j} = M_{i,j}$.

ISSN 1426–2630

The following three propositions are true:

(1) For all matrices M_1, M_2 over K of dimension $n \times m$ holds $\mathrm{width}\, M_1 = \mathrm{width}\, M_2$.

(2) Let given M, M_1, i such that $l \in \mathrm{dom}\, M$ and $k \in \mathrm{dom}\, M$ and $i \in \mathrm{dom}\, M$ and $M_1 = \mathrm{InterchangeLine}(M, l, k)$. Then

(i) if $i = l$, then $\mathrm{Line}(M_1, i) = \mathrm{Line}(M, k)$,

(ii) if $i = k$, then $\mathrm{Line}(M_1, i) = \mathrm{Line}(M, l)$, and

(iii) if $i \neq l$ and $i \neq k$, then $\mathrm{Line}(M_1, i) = \mathrm{Line}(M, i)$.

(3) For all a, i, j, M such that $i \in \mathrm{dom}\, M$ and $j \in \mathrm{Seg}\,\mathrm{width}\, M$ holds $(a \cdot \mathrm{Line}(M, i))(j) = a \cdot M_{i,j}$.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, let l be a natural number, and let a be an element of K. The functor $\mathrm{ScalarXLine}(M, l, a)$ yields a matrix over K of dimension $n \times m$ and is defined by the conditions (Def. 2).

(Def. 2)(i) $\mathrm{len}\, \mathrm{ScalarXLine}(M, l, a) = \mathrm{len}\, M$, and

(ii) for all i, j such that $i \in \mathrm{dom}\, M$ and $j \in \mathrm{Seg}\,\mathrm{width}\, M$ holds if $i = l$, then $(\mathrm{ScalarXLine}(M, l, a))_{i,j} = a \cdot M_{l,j}$ and if $i \neq l$, then $(\mathrm{ScalarXLine}(M, l, a))_{i,j} = M_{i,j}$.

We now state the proposition

(4) If $l \in \mathrm{dom}\, M$ and $i \in \mathrm{dom}\, M$ and $a \neq 0_K$ and $M_1 = \mathrm{ScalarXLine}(M, l, a)$, then if $i = l$, then $\mathrm{Line}(M_1, i) = a \cdot \mathrm{Line}(M, l)$ and if $i \neq l$, then $\mathrm{Line}(M_1, i) = \mathrm{Line}(M, i)$.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, let l, k be natural numbers, and let a be an element of K. Let us assume that $l \in \mathrm{dom}\, M$ and $k \in \mathrm{dom}\, M$. The functor $\mathrm{RlineXScalar}(M, l, k, a)$ yielding a matrix over K of dimension $n \times m$ is defined by the conditions (Def. 3).

(Def. 3)(i) $\mathrm{len}\, \mathrm{RlineXScalar}(M, l, k, a) = \mathrm{len}\, M$, and

(ii) for all i, j such that $i \in \mathrm{dom}\, M$ and $j \in \mathrm{Seg}\,\mathrm{width}\, M$ holds if $i = l$, then $(\mathrm{RlineXScalar}(M, l, k, a))_{i,j} = a \cdot M_{k,j} + M_{l,j}$ and if $i \neq l$, then $(\mathrm{RlineXScalar}(M, l, k, a))_{i,j} = M_{i,j}$.

We now state the proposition

(5) If $l \in \mathrm{dom}\, M$ and $k \in \mathrm{dom}\, M$ and $i \in \mathrm{dom}\, M$ and $M_1 = \mathrm{RlineXScalar}(M, l, k, a)$, then if $i = l$, then $\mathrm{Line}(M_1, i) = a \cdot \mathrm{Line}(M, k) + \mathrm{Line}(M, l)$ and if $i \neq l$, then $\mathrm{Line}(M_1, i) = \mathrm{Line}(M, i)$.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, and let l, k be natural numbers. We introduce $\mathrm{ILine}(M, l, k)$ as a synonym of $\mathrm{InterchangeLine}(M, l, k)$.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, let l be a natural number, and let a be an element of K. We introduce $\mathrm{SXLine}(M, l, a)$ as a synonym of $\mathrm{ScalarXLine}(M, l, a)$.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, let l, k be natural numbers, and let a be an element of K. We introduce $\mathrm{RLineXS}(M, l, k, a)$ as a synonym of $\mathrm{RlineXScalar}(M, l, k, a)$.

We now state several propositions:

(6) If $l \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$ and $k \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$, then $\mathrm{ILine}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, k) \cdot A = \mathrm{ILine}(A, l, k)$.

(7) For all l, a, A such that $l \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$ and $a \neq 0_K$ holds $\mathrm{SXLine}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, a) \cdot A = \mathrm{SXLine}(A, l, a)$.

(8) If $l \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$ and $k \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$, then $\mathrm{RLineXS}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, k, a) \cdot A = \mathrm{RLineXS}(A, l, k, a)$.

(9) $\mathrm{ILine}(M, k, k) = M$.

(10) $\mathrm{ILine}(M, l, k) = \mathrm{ILine}(M, k, l)$.

(11) If $l \in \mathrm{dom}\, M$ and $k \in \mathrm{dom}\, M$, then $\mathrm{ILine}(\mathrm{ILine}(M, l, k), l, k) = M$.

(12) If $l \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$ and $k \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$, then $\mathrm{ILine}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, k)$ is invertible and $(\mathrm{ILine}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, k))^{\smile} = \mathrm{ILine}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, k)$.

(13) If $l \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$ and $k \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$ and $k \neq l$, then $\mathrm{RLineXS}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, k, a)$ is invertible and $(\mathrm{RLineXS}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, k, a))^{\smile} = \mathrm{RLineXS}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, k, -a)$.

(14) If $l \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$ and $a \neq 0_K$, then $\mathrm{SXLine}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, a)$ is invertible and $(\mathrm{SXLine}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, a))^{\smile} = \mathrm{SXLine}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, a^{-1})$.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, and let l, k be natural numbers. Let us assume that $l \in \mathrm{Seg}\,\mathrm{width}\,M$ and $k \in \mathrm{Seg}\,\mathrm{width}\,M$ and $n > 0$ and $m > 0$. The functor $\mathrm{InterchangeCol}(M, l, k)$ yields a matrix over K of dimension $n \times m$ and is defined by the conditions (Def. 4).

(Def. 4)(i) $\mathrm{len}\,\mathrm{InterchangeCol}(M, l, k) = \mathrm{len}\,M$, and

(ii) for all i, j such that $i \in \mathrm{dom}\,M$ and $j \in \mathrm{Seg}\,\mathrm{width}\,M$ holds if $j = l$, then $(\mathrm{InterchangeCol}(M, l, k))_{i,j} = M_{i,k}$ and if $j = k$, then $(\mathrm{InterchangeCol}(M, l, k))_{i,j} = M_{i,l}$ and if $j \neq l$ and $j \neq k$, then $(\mathrm{InterchangeCol}(M, l, k))_{i,j} = M_{i,j}$.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, let l be a natural number, and let a be an element of K. Let us assume that $l \in \mathrm{Seg}\,\mathrm{width}\,M$ and $n > 0$ and $m > 0$. The functor $\mathrm{ScalarXCol}(M, l, a)$ yielding a matrix over K of dimension $n \times m$ is defined by the conditions (Def. 5).

(Def. 5)(i) $\mathrm{len}\,\mathrm{ScalarXCol}(M, l, a) = \mathrm{len}\,M$, and

(ii) for all i, j such that $i \in \mathrm{dom}\,M$ and $j \in \mathrm{Seg}\,\mathrm{width}\,M$ holds if $j = l$, then $(\mathrm{ScalarXCol}(M, l, a))_{i,j} = a \cdot M_{i,l}$ and if $j \neq l$, then $(\mathrm{ScalarXCol}(M, l, a))_{i,j} = M_{i,j}$.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, let l, k be natural numbers, and let a be an element of K. Let us assume that $l \in \operatorname{Seg} \operatorname{width} M$ and $k \in \operatorname{Seg} \operatorname{width} M$ and $n > 0$ and $m > 0$. The functor $\operatorname{RcolXScalar}(M, l, k, a)$ yielding a matrix over K of dimension $n \times m$ is defined by the conditions (Def. 6).

(Def. 6)(i) $\operatorname{len} \operatorname{RcolXScalar}(M, l, k, a) = \operatorname{len} M$, and

(ii) for all i, j such that $i \in \operatorname{dom} M$ and $j \in \operatorname{Seg} \operatorname{width} M$ holds if $j = l$, then $(\operatorname{RcolXScalar}(M, l, k, a))_{i,j} = a \cdot M_{i,k} + M_{i,l}$ and if $j \neq l$, then $(\operatorname{RcolXScalar}(M, l, k, a))_{i,j} = M_{i,j}$.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, and let l, k be natural numbers. We introduce $\operatorname{ICol}(M, l, k)$ as a synonym of $\operatorname{InterchangeCol}(M, l, k)$.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, let l be a natural number, and let a be an element of K. We introduce $\operatorname{SXCol}(M, l, a)$ as a synonym of $\operatorname{ScalarXCol}(M, l, a)$.

Let us consider n, m, let us consider K, let M be a matrix over K of dimension $n \times m$, let l, k be natural numbers, and let a be an element of K. We introduce $\operatorname{RColXS}(M, l, k, a)$ as a synonym of $\operatorname{RcolXScalar}(M, l, k, a)$.

We now state several propositions:

(15) If $l \in \operatorname{Seg} \operatorname{width} M$ and $k \in \operatorname{Seg} \operatorname{width} M$ and $n > 0$ and $m > 0$ and $M_1 = M^{\mathrm{T}}$, then $(\operatorname{ILine}(M_1, l, k))^{\mathrm{T}} = \operatorname{ICol}(M, l, k)$.

(16) If $l \in \operatorname{Seg} \operatorname{width} M$ and $a \neq 0_K$ and $n > 0$ and $m > 0$ and $M_1 = M^{\mathrm{T}}$, then $(\operatorname{SXLine}(M_1, l, a))^{\mathrm{T}} = \operatorname{SXCol}(M, l, a)$.

(17) If $l \in \operatorname{Seg} \operatorname{width} M$ and $k \in \operatorname{Seg} \operatorname{width} M$ and $n > 0$ and $m > 0$ and $M_1 = M^{\mathrm{T}}$, then $(\operatorname{RLineXS}(M_1, l, k, a))^{\mathrm{T}} = \operatorname{RColXS}(M, l, k, a)$.

(18) If $l \in \operatorname{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$ and $k \in \operatorname{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$ and $n > 0$, then $A \cdot \operatorname{ICol}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, k) = \operatorname{ICol}(A, l, k)$.

(19) If $l \in \operatorname{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n})$ and $a \neq 0_K$ and $n > 0$, then $A \cdot \operatorname{SXCol}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n \times n}, l, a) = \operatorname{SXCol}(A, l, a)$.

(20) If $l \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})$ and $k \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})$ and $n > 0$, then $A \cdot \mathrm{RColXS}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}, l, k, a) = \mathrm{RColXS}(A, l, k, a)$.

(21) If $l \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})$ and $k \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})$ and $n > 0$, then $(\mathrm{ICol}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}, l, k))^{\smile} = \mathrm{ICol}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}, l, k)$.

(22) If $l \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})$ and $k \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})$ and $k \neq l$ and $n > 0$, then $(\mathrm{RColXS}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}, l, k, a))^{\smile} = \mathrm{RColXS}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}, l, k, -a)$.

(23) If $l \in \mathrm{dom}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})$ and $a \neq 0_K$ and $n > 0$, then $(\mathrm{SXCol}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}, l, a))^{\smile} = \mathrm{SXCol}(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}, l, a^{-1})$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.

[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.
[9] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[10] Michał Muzalewski. Construction of rings and left-, right-, and bi-modules over a ring. *Formalized Mathematics*, 2(**1**):3–11, 1991.
[11] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[12] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[14] Wojciech A. Trybulec. Binary operations on finite sequences. *Formalized Mathematics*, 1(**5**):979–981, 1990.
[15] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[16] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[17] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[19] Xiaopeng Yue, Xiquan Liang, and Zhongpin Sun. Some properties of some special matrices. *Formalized Mathematics*, 13(**4**):541–547, 2005.
[20] Katarzyna Zawadzka. The sum and product of finite sequences of elements of a field. *Formalized Mathematics*, 3(**2**):205–211, 1992.
[21] Katarzyna Zawadzka. The product and the determinant of matrices with entries in a field. *Formalized Mathematics*, 4(**1**):1–8, 1993.

Received August 13, 2007