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Partial Differentiation on Normed Linear Spaces $\mathcal{R}^n$

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Summary. In this article, we define the partial differentiation of functions of real variable and prove the linearity of this operator [18].

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The notation and terminology used here are introduced in the following papers: [21], [24], [25], [5], [26], [7], [6], [15], [13], [3], [1], [20], [11], [22], [23], [14], [8], [2], [4], [27], [28], [16], [9], [19], [17], [12], and [10].

1. Preliminaries

Let i, n be elements of $\mathbb{N}$. The functor $\mathrm{proj}(i,n)$ yielding a function from $\mathcal{R}^n$ into $\mathbb{R}$ is defined by:

(Def. 1) For every element x of $\mathcal{R}^n$ holds $(\mathrm{proj}(i,n))(x) = x(i)$.

Next we state two propositions:

(1) $\mathrm{dom}\,\mathrm{proj}(1,1) = \mathcal{R}^1$ and $\mathrm{rng}\,\mathrm{proj}(1,1) = \mathbb{R}$ and for every element x of $\mathbb{R}$ holds $(\mathrm{proj}(1,1))(\langle x \rangle) = x$ and $(\mathrm{proj}(1,1))^{-1}(x) = \langle x \rangle$.

(2)(i) $(\mathrm{proj}(1,1))^{-1}$ is a function from $\mathbb{R}$ into $\mathcal{R}^1$,

(ii) $(\mathrm{proj}(1,1))^{-1}$ is one-to-one,

(iii) $\mathrm{dom}((\mathrm{proj}(1,1))^{-1}) = \mathbb{R}$,

(iv) $\mathrm{rng}((\mathrm{proj}(1,1))^{-1}) = \mathcal{R}^1$, and

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(v) there exists a function g from $\mathbb{R}$ into $\mathcal{R}^1$ such that g is bijective and $(\mathrm{proj}(1,1))^{-1} = g$.

One can check that $\mathrm{proj}(1,1)$ is bijective.

Let g be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor $\langle g \rangle$ yields a partial function from $\mathcal{R}^1$ to $\mathcal{R}^1$ and is defined as follows:

(Def. 2) $\langle g \rangle = (\mathrm{proj}(1,1))^{-1} \cdot g \cdot \mathrm{proj}(1,1)$.

Let n be an element of $\mathbb{N}$ and let g be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$. The functor $\langle g \rangle$ yielding a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$ is defined as follows:

(Def. 3) $\langle g \rangle = (\mathrm{proj}(1,1))^{-1} \cdot g$.

Let i, n be elements of $\mathbb{N}$. The functor $\mathrm{Proj}(i,n)$ yielding a function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ is defined as follows:

(Def. 4) For every point x of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $(\mathrm{Proj}(i,n))(x) = \langle (\mathrm{proj}(i,n))(x) \rangle$.

Let i be an element of $\mathbb{N}$ and let x be a finite sequence of elements of $\mathbb{R}$. The functor $\mathrm{reproj}(i,x)$ yielding a function is defined as follows:

(Def. 5) $\mathrm{dom}\,\mathrm{reproj}(i,x) = \mathbb{R}$ and for every element r of $\mathbb{R}$ holds $(\mathrm{reproj}(i,x))(r) = \mathrm{Replace}(x,i,r)$.

Let n, i be elements of $\mathbb{N}$ and let x be an element of $\mathcal{R}^n$. Then $\mathrm{reproj}(i,x)$ is a function from $\mathbb{R}$ into $\mathcal{R}^n$.

Let n, i be elements of $\mathbb{N}$ and let x be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. The functor $\mathrm{reproj}(i,x)$ yielding a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is defined by the condition (Def. 6).

(Def. 6) Let r be an element of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Then there exists an element q of $\mathbb{R}$ and there exists an element y of $\mathcal{R}^n$ such that $r = \langle q \rangle$ and $y = x$ and $(\mathrm{reproj}(i,x))(r) = (\mathrm{reproj}(i,y))(q)$.

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. We say that f is differentiable in x if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and g is differentiable in y.

Let m, n be non empty elements of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. Let us assume that f is differentiable in x. The functor $f'(x)$ yields a function from $\mathcal{R}^m$ into $\mathcal{R}^n$ and is defined as follows:

(Def. 8) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and $f'(x) = g'(y)$.

We now state four propositions:

(3) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$. Suppose $I = (\mathrm{proj}(1,1))^{-1}$. Then

(i) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ such that $x = I(y)$ holds $\|x\| = |y|$,

(ii) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $x = I(a)$ and $y = I(b)$ holds $x + y = I(a + b)$,

(iii) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ and for every real number a such that $x = I(y)$ holds $a \cdot x = I(a \cdot y)$,

(iv) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element a of $\mathbb{R}$ such that $x = I(a)$ holds $-x = I(-a)$, and

(v) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $x = I(a)$ and $y = I(b)$ holds $x - y = I(a - b)$.

(4) Let J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $J = \text{proj}(1, 1)$. Then

(i) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ such that $J(x) = y$ holds $\|x\| = |y|$,

(ii) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $J(x) = a$ and $J(y) = b$ holds $J(x + y) = a + b$,

(iii) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element y of $\mathbb{R}$ and for every real number a such that $J(x) = y$ holds $J(a \cdot x) = a \cdot y$,

(iv) for every vector x of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for every element a of $\mathbb{R}$ such that $J(x) = a$ holds $J(-x) = -a$, and

(v) for all vectors x, y of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ and for all elements a, b of $\mathbb{R}$ such that $J(x) = a$ and $J(y) = b$ holds $J(x - y) = a - b$.

(5) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$ and J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $I = (\text{proj}(1, 1))^{-1}$ and $J = \text{proj}(1, 1)$. Then

(i) for every rest R of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, $\langle \mathcal{E}^1, \|\cdot\| \rangle$ holds $J \cdot R \cdot I$ is a rest, and

(ii) for every linear operator L from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ holds $J \cdot L \cdot I$ is a linear function.

(6) Let I be a function from $\mathbb{R}$ into $\mathcal{R}^1$ and J be a function from $\mathcal{R}^1$ into $\mathbb{R}$. Suppose $I = (\text{proj}(1, 1))^{-1}$ and $J = \text{proj}(1, 1)$. Then

(i) for every rest R holds $I \cdot R \cdot J$ is a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and

(ii) for every linear function L holds $I \cdot L \cdot J$ is a bounded linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$.

In the sequel f is a partial function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, g is a partial function from $\mathbb{R}$ to $\mathbb{R}$, x is a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, and y is an element of $\mathbb{R}$.

We now state four propositions:

(7) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and f is differentiable in x, then g is differentiable in y and $g'(y) = (\text{proj}(1, 1) \cdot f'(x) \cdot (\text{proj}(1, 1))^{-1})(1)$.

(8) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and g is differentiable in y, then f is differentiable in x and $f'(x)(\langle 1 \rangle) = \langle g'(y) \rangle$.

(9) If $f = \langle g \rangle$ and $x = \langle y \rangle$, then f is differentiable in x iff g is differentiable in y.

(10) If $f = \langle g \rangle$ and $x = \langle y \rangle$ and f is differentiable in x, then $f'(x)(\langle 1 \rangle) = \langle g'(y) \rangle$.

2. Partial Differentiation

For simplicity, we adopt the following rules: m, n are non empty elements of $\mathbb{N}$, i, j are elements of $\mathbb{N}$, f is a partial function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to $\langle \mathcal{E}^1, \|\cdot\| \rangle$, g is a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, x is a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and y is an element of $\mathcal{R}^n$.

Let n, m be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. We say that f is partially differentiable in x w.r.t. i if and only if:

(Def. 9) $f \cdot \mathrm{reproj}(i, x)$ is differentiable in $(\mathrm{Proj}(i, m))(x)$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. The functor $\mathrm{partdiff}(f, x, i)$ yielding a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ is defined as follows:

(Def. 10) $\mathrm{partdiff}(f, x, i) = (f \cdot \mathrm{reproj}(i, x))'((\mathrm{Proj}(i, m))(x))$.

Let n be a non empty element of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. We say that f is partially differentiable in x w.r.t. i if and only if:

(Def. 11) $f \cdot \mathrm{reproj}(i, x)$ is differentiable in $(\mathrm{proj}(i, n))(x)$.

Let n be a non empty element of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^n$ to $\mathbb{R}$, and let x be an element of $\mathcal{R}^n$. The functor $\mathrm{partdiff}(f, x, i)$ yields a real number and is defined by:

(Def. 12) $\mathrm{partdiff}(f, x, i) = (f \cdot \mathrm{reproj}(i, x))'((\mathrm{proj}(i, n))(x))$.

We now state several propositions:

(11) $\mathrm{Proj}(i, n) = (\mathrm{proj}(1, 1))^{-1} \cdot \mathrm{proj}(i, n)$.

(12) If $x = y$, then $\mathrm{reproj}(i, y) \cdot \mathrm{proj}(1, 1) = \mathrm{reproj}(i, x)$.

(13) If $f = \langle g \rangle$ and $x = y$, then $\langle g \cdot \mathrm{reproj}(i, y) \rangle = f \cdot \mathrm{reproj}(i, x)$.

(14) Suppose $f = \langle g \rangle$ and $x = y$. Then f is partially differentiable in x w.r.t. i if and only if g is partially differentiable in y w.r.t. i.

(15) If $f = \langle g \rangle$ and $x = y$ and f is partially differentiable in x w.r.t. i, then $(\mathrm{partdiff}(f, x, i))(\langle 1 \rangle) = \langle \mathrm{partdiff}(g, y, i) \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. We say that f is partially differentiable in x w.r.t. i if and only if the condition (Def. 13) is satisfied.

(Def. 13) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and g is partially differentiable in y w.r.t. i.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let x be an element of $\mathcal{R}^m$. Let us assume that f is partially differentiable in x w.r.t. i. The functor partdiff(f, x, i) yielding an element of $\mathcal{R}^n$ is defined as follows:

(Def. 14) There exists a partial function g from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and there exists a point y of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $f = g$ and $x = y$ and partdiff$(f, x, i) = ($partdiff$(g, y, i))(\langle 1 \rangle)$.

One can prove the following four propositions:

(16) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$. Then F is partially differentiable in x w.r.t. i if and only if G is partially differentiable in y w.r.t. i.

(17) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$ and F is partially differentiable in x w.r.t. i. Then $($partdiff$(F, x, i))(\langle 1 \rangle) =$ partdiff(G, y, i).

(18) Let g_1 be a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$. Suppose $g_1 = \langle g \rangle$. Then g_1 is partially differentiable in y w.r.t. i if and only if g is partially differentiable in y w.r.t. i.

(19) Let g_1 be a partial function from $\mathcal{R}^n$ to $\mathcal{R}^1$. Suppose $g_1 = \langle g \rangle$ and g_1 is partially differentiable in y w.r.t. i. Then partdiff$(g_1, y, i) = \langle$partdiff$(g, y, i)\rangle$.

3. Linearity of Partial Differential Operator

For simplicity, we use the following convention: X is a set, r is a real number, f, f_1, f_2 are partial functions from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, g, g_1, g_2 are partial functions from $\mathcal{R}^n$ to $\mathbb{R}$, h is a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x is a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$, y is an element of $\mathcal{R}^n$, and z is an element of $\mathcal{R}^m$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$. We say that f is partially differentiable in x w.r.t. i and j if and only if:

(Def. 15) $\mathrm{Proj}(j, n) \cdot f \cdot \mathrm{reproj}(i, x)$ is differentiable in $(\mathrm{Proj}(i, m))(x)$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and let x be a point of $\langle \mathcal{E}^m, \|\cdot\| \rangle$.

The functor $\mathrm{partdiff}(f, x, i, j)$ yields a point of the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $\langle \mathcal{E}^1, \| \cdot \| \rangle$ and is defined by:

(Def. 16) $\mathrm{partdiff}(f, x, i, j) = (\mathrm{Proj}(j, n) \cdot f \cdot \mathrm{reproj}(i, x))'((\mathrm{Proj}(i, m))(x))$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let h be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let z be an element of $\mathcal{R}^m$. We say that h is partially differentiable in z w.r.t. i and j if and only if:

(Def. 17) $\mathrm{proj}(j, n) \cdot h \cdot \mathrm{reproj}(i, z)$ is differentiable in $(\mathrm{proj}(i, m))(z)$.

Let m, n be non empty elements of $\mathbb{N}$, let i, j be elements of $\mathbb{N}$, let h be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, and let z be an element of $\mathcal{R}^m$. The functor $\mathrm{partdiff}(h, z, i, j)$ yielding a real number is defined as follows:

(Def. 18) $\mathrm{partdiff}(h, z, i, j) = (\mathrm{proj}(j, n) \cdot h \cdot \mathrm{reproj}(i, z))'((\mathrm{proj}(i, m))(z))$.

The following propositions are true:

(20) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \| \cdot \| \rangle$, and y be an element of $\mathcal{R}^m$. Suppose $F = G$ and $x = y$. Then F is differentiable in x if and only if G is differentiable in y.

(21) Let m, n be non empty elements of $\mathbb{N}$, F be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, G be a partial function from $\mathcal{R}^m$ to $\mathcal{R}^n$, x be a point of $\langle \mathcal{E}^m, \| \cdot \| \rangle$, and y be an element of $\mathcal{R}^m$. If $F = G$ and $x = y$ and F is differentiable in x, then $F'(x) = G'(y)$.

(22) If $f = h$ and $x = z$, then $\mathrm{Proj}(j, n) \cdot f \cdot \mathrm{reproj}(i, x) = \langle \mathrm{proj}(j, n) \cdot h \cdot \mathrm{reproj}(i, z) \rangle$.

(23) Suppose $f = h$ and $x = z$. Then f is partially differentiable in x w.r.t. i and j if and only if h is partially differentiable in z w.r.t. i and j.

(24) If $f = h$ and $x = z$ and f is partially differentiable in x w.r.t. i and j, then $(\mathrm{partdiff}(f, x, i, j))(\langle 1 \rangle) = \langle \mathrm{partdiff}(h, z, i, j) \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and let X be a set. We say that f is partially differentiable on X w.r.t. i if and only if:

(Def. 19) $X \subseteq \mathrm{dom}\, f$ and for every point x of $\langle \mathcal{E}^m, \| \cdot \| \rangle$ such that $x \in X$ holds $f{\restriction}X$ is partially differentiable in x w.r.t. i.

We now state the proposition

(25) If f is partially differentiable on X w.r.t. i, then X is a subset of $\langle \mathcal{E}^m, \| \cdot \| \rangle$.

Let m, n be non empty elements of $\mathbb{N}$, let i be an element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to $\langle \mathcal{E}^n, \| \cdot \| \rangle$, and let us consider X. Let us assume that f is partially differentiable on X w.r.t. i. The functor $f{\restriction}^i X$ yielding a partial function from $\langle \mathcal{E}^m, \| \cdot \| \rangle$ to the real norm space of bounded linear operators from $\langle \mathcal{E}^1, \| \cdot \| \rangle$ into $\langle \mathcal{E}^n, \| \cdot \| \rangle$ is defined by:

(Def. 20) $\mathrm{dom}(f{\upharpoonright}^i X) = X$ and for every point x of $\langle \mathcal{E}^m, \|\cdot\| \rangle$ such that $x \in X$ holds $(f{\upharpoonright}^i X)_x = \mathrm{partdiff}(f, x, i)$.

The following propositions are true:

(26) $(f_1 + f_2) \cdot \mathrm{reproj}(i, x) = f_1 \cdot \mathrm{reproj}(i, x) + f_2 \cdot \mathrm{reproj}(i, x)$ and $(f_1 - f_2) \cdot \mathrm{reproj}(i, x) = f_1 \cdot \mathrm{reproj}(i, x) - f_2 \cdot \mathrm{reproj}(i, x)$.

(27) $r\,(f \cdot \mathrm{reproj}(i, x)) = (r\,f) \cdot \mathrm{reproj}(i, x)$.

(28) Suppose f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i. Then $f_1 + f_2$ is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(f_1 + f_2, x, i) = \mathrm{partdiff}(f_1, x, i) + \mathrm{partdiff}(f_2, x, i)$.

(29) Suppose g_1 is partially differentiable in y w.r.t. i and g_2 is partially differentiable in y w.r.t. i. Then $g_1 + g_2$ is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(g_1 + g_2, y, i) = \mathrm{partdiff}(g_1, y, i) + \mathrm{partdiff}(g_2, y, i)$.

(30) Suppose f_1 is partially differentiable in x w.r.t. i and f_2 is partially differentiable in x w.r.t. i. Then $f_1 - f_2$ is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(f_1 - f_2, x, i) = \mathrm{partdiff}(f_1, x, i) - \mathrm{partdiff}(f_2, x, i)$.

(31) Suppose g_1 is partially differentiable in y w.r.t. i and g_2 is partially differentiable in y w.r.t. i. Then $g_1 - g_2$ is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(g_1 - g_2, y, i) = \mathrm{partdiff}(g_1, y, i) - \mathrm{partdiff}(g_2, y, i)$.

(32) Suppose f is partially differentiable in x w.r.t. i. Then $r\,f$ is partially differentiable in x w.r.t. i and $\mathrm{partdiff}(r\,f, x, i) = r \cdot \mathrm{partdiff}(f, x, i)$.

(33) Suppose g is partially differentiable in y w.r.t. i. Then $r\,g$ is partially differentiable in y w.r.t. i and $\mathrm{partdiff}(r\,g, y, i) = r \cdot \mathrm{partdiff}(g, y, i)$.

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