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Riemann Indefinite Integral of Functions of Real Variable[1]

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Summary. In this article we define the Riemann indefinite integral of functions of real variable and prove the linearity of that [1]. And we give some examples of the indefinite integral of some elementary functions. Furthermore, also the theorem about integral operation and uniform convergent sequence of functions is proved.

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The papers [24], [25], [3], [23], [5], [13], [2], [26], [7], [21], [8], [10], [4], [17], [16], [15], [14], [19], [20], [6], [9], [11], [18], [12], [27], and [22] provide the terminology and notation for this paper.

1. Preliminaries

For simplicity, we adopt the following rules: a, b, r are real numbers, A is a non empty set, X, x are sets, f, g, F, G are partial functions from $\mathbb{R}$ to $\mathbb{R}$, and n is an element of $\mathbb{N}$.

Next we state a number of propositions:

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(1) Let f, g be functions from A into $\mathbb{R}$. Suppose $\operatorname{rng} f$ is upper bounded and $\operatorname{rng} g$ is upper bounded and for every set x such that $x \in A$ holds $|f(x)-g(x)| \leq a$. Then $\sup \operatorname{rng} f - \sup \operatorname{rng} g \leq a$ and $\sup \operatorname{rng} g - \sup \operatorname{rng} f \leq a$.

(2) Let f, g be functions from A into $\mathbb{R}$. Suppose $\operatorname{rng} f$ is lower bounded and $\operatorname{rng} g$ is lower bounded and for every set x such that $x \in A$ holds $|f(x)-g(x)| \leq a$. Then $\inf \operatorname{rng} f - \inf \operatorname{rng} g \leq a$ and $\inf \operatorname{rng} g - \inf \operatorname{rng} f \leq a$.

(3) If $f \restriction X$ is bounded on X, then f is bounded on X.

(4) For every real number x such that $x \in X$ and $f \restriction X$ is differentiable in x holds f is differentiable in x.

(5) If $f \restriction X$ is differentiable on X, then f is differentiable on X.

(6) Suppose f is differentiable on X and g is differentiable on X. Then $f + g$ is differentiable on X and $f - g$ is differentiable on X and $f\,g$ is differentiable on X.

(7) If f is differentiable on X, then $r\,f$ is differentiable on X.

(8) Suppose for every set x such that $x \in X$ holds $g(x) \neq 0$ and f is differentiable on X and g is differentiable on X. Then $\frac{f}{g}$ is differentiable on X.

(9) If for every set x such that $x \in X$ holds $f(x) \neq 0$ and f is differentiable on X, then $\frac{1}{f}$ is differentiable on X.

(10) Suppose $a \leq b$ and $['a, b'] \subseteq X$ and F is differentiable on X and $F'_{\restriction X}$ is integrable on $['a, b']$ and $F'_{\restriction X}$ is bounded on $['a, b']$. Then $F(b) = \int\limits_a^b (F'_{\restriction X})(x)dx + F(a)$.

2. The Definition of Indefinite Integral

Let X be a set and let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. The functor IntegralFuncs(f, X) yields a set and is defined by the condition (Def. 1).

(Def. 1) $x \in \text{IntegralFuncs}(f, X)$ if and only if there exists a partial function F from $\mathbb{R}$ to $\mathbb{R}$ such that $x = F$ and F is differentiable on X and $F'_{\restriction X} = f \restriction X$.

Let X be a set and let F, f be partial functions from $\mathbb{R}$ to $\mathbb{R}$. We say that F is an integral of f on X if and only if:

(Def. 2) $F \in \text{IntegralFuncs}(f, X)$.

The following propositions are true:

(11) If F is an integral of f on X, then $X \subseteq \operatorname{dom} F$.

(12) Suppose F is an integral of f on X and G is an integral of g on X. Then $F + G$ is an integral of $f + g$ on X and $F - G$ is an integral of $f - g$ on X.

(13) If F is an integral of f on X, then $r\,F$ is an integral of $r\,f$ on X.

(14) If F is an integral of f on X and G is an integral of g on X, then $F\,G$ is an integral of $f\,G + F\,g$ on X.

(15) Suppose for every set x such that $x \in X$ holds $G(x) \neq 0$ and F is an integral of f on X and G is an integral of g on X. Then $\frac{F}{G}$ is an integral of $\frac{f\,G - F\,g}{G\,G}$ on X.

(16) Suppose that

(i) $a \leq b$,

(ii) $[{}'a, b'] \subseteq \mathrm{dom}\, f$,

(iii) f is continuous on $[{}'a, b']$,

(iv) $]a, b[\subseteq \mathrm{dom}\, F$, and

(v) for every real number x such that $x \in]a, b[$ holds $F(x) = \int\limits_{a}^{x} f(x)dx + F(a)$.

Then F is an integral of f on $]a, b[$.

(17) Let x, x_0 be real numbers. Suppose f is continuous on $[a, b]$ and $x \in]a, b[$ and $x_0 \in]a, b[$ and F is an integral of f on $]a, b[$. Then $F(x) = \int\limits_{x_0}^{x} f(x)dx + F(x_0)$.

(18) Suppose $a \leq b$ and $[{}'a, b'] \subseteq X$ and F is an integral of f on X and f is integrable on $[{}'a, b']$ and f is bounded on $[{}'a, b']$. Then $F(b) = \int\limits_{a}^{b} f(x)dx + F(a)$.

(19) Suppose $a \leq b$ and $[a, b] \subseteq X$ and f is continuous on X. Then f is continuous on $[{}'a, b']$ and f is integrable on $[{}'a, b']$ and f is bounded on $[{}'a, b']$.

(20) If $a \leq b$ and $[a, b] \subseteq X$ and f is continuous on X and F is an integral of f on X, then $F(b) = \int\limits_{a}^{b} f(x)dx + F(a)$.

(21) Suppose that $b \leq a$ and $[{}'b, a'] \subseteq X$ and f is integrable on $[{}'b, a']$ and g is integrable on $[{}'b, a']$ and f is bounded on $[{}'b, a']$ and g is bounded on $[{}'b, a']$ and $X \subseteq \mathrm{dom}\, f$ and $X \subseteq \mathrm{dom}\, g$ and F is an integral of f on X and G is an integral of g on X. Then $F(a) \cdot G(a) - F(b) \cdot G(b) = \int\limits_{b}^{a} (f\,G)(x)dx + \int\limits_{b}^{a} (F\,g)(x)dx$.

(22) Suppose that $b \leq a$ and $[b, a] \subseteq X$ and $X \subseteq \mathrm{dom}\, f$ and $X \subseteq \mathrm{dom}\, g$ and

f is continuous on X and g is continuous on X and F is an integral of f on X and G is an integral of g on X. Then $F(a) \cdot G(a) - F(b) \cdot G(b) = \int_b^a (f\, G)(x)dx + \int_b^a (F\, g)(x)dx$.

3. Examples of Indefinite Integral

We now state several propositions:

(23) The function sin is an integral of the function cos on $\mathbb{R}$.

(24) (The function sin)$(b)-$(the function sin)$(a) = \int_a^b$ (the function cos)$(x)dx$.

(25) (-1) (the function cos) is an integral of the function sin on $\mathbb{R}$.

(26) (The function cos)$(a)-$(the function cos)$(b) = \int_a^b$ (the function sin)$(x)dx$.

(27) The function exp is an integral of the function exp on $\mathbb{R}$.

(28) (The function exp)$(b)-$(the function exp)$(a) = \int_a^b$ (the function exp)$(x)dx$.

(29) $\frac{n+1}{\mathbb{Z}}$ is an integral of $(n+1)\,\frac{n}{\mathbb{Z}}$ on $\mathbb{R}$.

(30) $(\frac{n+1}{\mathbb{Z}})(b) - (\frac{n+1}{\mathbb{Z}})(a) = \int_a^b ((n+1)\,\frac{n}{\mathbb{Z}})(x)dx$.

4. Uniform Convergent Functional Sequence

We now state the proposition

(31) Let H be a sequence of partial functions from $\mathbb{R}$ into $\mathbb{R}$ and r_1 be a sequence of real numbers. Suppose that

(i) $a < b$,

(ii) for every element n of $\mathbb{N}$ holds $H(n)$ is integrable on $['a, b']$ and $H(n)$ is bounded on $['a, b']$ and $r_1(n) = \int_a^b H(n)(x)dx$, and

(iii) H is uniform-convergent on $['a, b']$.

Then $\lim_{['a,b']} H$ is bounded on $['a, b']$ and $\lim_{['a,b']} H$ is integrable on $['a, b']$ and r_1 is convergent and $\lim r_1 = \int_a^b \lim_{['a,b']} H(x)dx$.

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