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The Jordan-Hölder Theorem

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Summary. The goal of this article is to formalize the Jordan-Hölder theorem in the context of group with operators as in the book [5]. Accordingly, the article introduces the structure of group with operators and reformulates some theorems on a group already present in the Mizar Mathematical Library. Next, the article formalizes the Zassenhaus butterfly lemma and the Schreier refinement theorem, and defines the composition series.

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The terminology and notation used here are introduced in the following articles: [17], [25], [3], [26], [7], [27], [8], [9], [4], [10], [1], [12], [18], [2], [6], [21], [20], [22], [19], [15], [23], [11], [14], [16], [13], and [24].

1. Actions and Groups with Operators

Let O, E be sets. An action of O on E is a function from O into E^E.

Let O, E be sets, let A be an action of O on E, and let I_1 be a set. We say that I_1 is stable under the action of A if and only if:

(Def. 1) For every element o of O and for every function f from E into E such that $o \in O$ and $f = A(o)$ holds $f^\circ I_1 \subseteq I_1$.

Let O, E be sets, let A be an action of O on E, and let X be a subset of E. The stable subset generated by X yields a subset of E and is defined by the conditions (Def. 2).

(Def. 2)(i) $X \subseteq$ the stable subset generated by X,

(ii) the stable subset generated by X is stable under the action of A, and

(iii) for every subset Y of E such that Y is stable under the action of A and $X \subseteq Y$ holds the stable subset generated by $X \subseteq Y$.

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Let O, E be sets, let A be an action of O on E, and let F be a finite sequence of elements of O. The functor $\mathrm{Product}(F, A)$ yields a function from E into E and is defined by:

(Def. 3)(i) $\mathrm{Product}(F, A) = \mathrm{id}_E$ if $\mathrm{len}\, F = 0$,

(ii) there exists a finite sequence P_1 of elements of E^E such that $\mathrm{Product}(F, A) = P_1(\mathrm{len}\, F)$ and $\mathrm{len}\, P_1 = \mathrm{len}\, F$ and $P_1(1) = A(F(1))$ and for every natural number n such that $n \neq 0$ and $n < \mathrm{len}\, F$ there exist functions f, g from E into E such that $f = P_1(n)$ and $g = A(F(n+1))$ and $P_1(n+1) = f \cdot g$, otherwise.

Let O be a set, let G be a group, and let I_1 be an action of O on the carrier of G. We say that I_1 is distributive if and only if:

(Def. 4) For every element o of O such that $o \in O$ holds $I_1(o)$ is a homomorphism from G to G.

Let O be a set. We consider group structures with operators in O as extensions of groupoid as systems

⟨ a carrier, a multiplication, an action ⟩,

where the carrier is a set, the multiplication is a binary operation on the carrier, and the action is an action of O on the carrier.

Let O be a set. Observe that there exists a group structure with operators in O which is non empty.

Let O be a set and let I_1 be a non empty group structure with operators in O. We say that I_1 is distributive if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let G be a group and a be an action of O on the carrier of G. Suppose $a =$ the action of I_1 and the groupoid of $G =$ the groupoid of I_1. Then a is distributive.

Let O be a set. Observe that there exists a non empty group structure with operators in O which is strict, distributive, group-like, and associative.

Let O be a set. A group with operators in O is a distributive group-like associative non empty group structure with operators in O.

Let O be a set, let G be a group with operators in O, and let o be an element of O. The functor $G \frown o$ yields a homomorphism from G to G and is defined as follows:

(Def. 6) $G \frown o = \begin{cases} (\text{the action of } G)(o), & \text{if } o \in O, \\ \mathrm{id}_{\text{the carrier of } G}, & \text{otherwise.} \end{cases}$

Let O be a set and let G be a group with operators in O. A distributive group-like associative non empty group structure with operators in O is said to be a stable subgroup of G if:

(Def. 7) It is a subgroup of G and for every element o of O holds $\mathrm{it} \frown o = (G \frown o) \restriction$ the carrier of it.

Let O be a set and let G be a group with operators in O. Note that there exists a stable subgroup of G which is strict.

Let O be a set and let G be a group with operators in O. The functor $\{\mathbf{1}\}_G$ yields a strict stable subgroup of G and is defined by:

(Def. 8) The carrier of $\{\mathbf{1}\}_G = \{\mathbf{1}_G\}$.

Let O be a set and let G be a group with operators in O. The functor Ω_G yielding a strict stable subgroup of G is defined as follows:

(Def. 9) $\Omega_G =$ the group structure with operators of G.

Let O be a set, let G be a group with operators in O, and let I_1 be a stable subgroup of G. We say that I_1 is normal if and only if:

(Def. 10) For every strict subgroup H of G such that $H =$ the groupoid of I_1 holds H is normal.

Let O be a set and let G be a group with operators in O. Note that there exists a stable subgroup of G which is strict and normal.

Let O be a set, let G be a group with operators in O, and let H be a stable subgroup of G. Observe that there exists a stable subgroup of H which is normal.

Let O be a set and let G be a group with operators in O. Note that $\{\mathbf{1}\}_G$ is normal and Ω_G is normal.

Let O be a set and let G be a group with operators in O. The stable subgroups of G yields a set and is defined as follows:

(Def. 11) For every set x holds $x \in$ the stable subgroups of G iff x is a strict stable subgroup of G.

Let O be a set and let G be a group with operators in O. Observe that the stable subgroups of G is non empty.

Let I_1 be a group. We say that I_1 is simple if and only if:

(Def. 12) I_1 is not trivial and it is not true that there exists a strict normal subgroup H of I_1 such that $H \neq \Omega_{(I_1)}$ and $H \neq \{\mathbf{1}\}_{(I_1)}$.

Let us note that there exists a group which is strict and simple.

Let O be a set and let I_1 be a group with operators in O. We say that I_1 is simple if and only if:

(Def. 13) I_1 is not trivial and it is not true that there exists a strict normal stable subgroup H of I_1 such that $H \neq \Omega_{(I_1)}$ and $H \neq \{\mathbf{1}\}_{(I_1)}$.

Let O be a set. Observe that there exists a group with operators in O which is strict and simple.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. The functor $\operatorname{Cosets} N$ yields a set and is defined by:

(Def. 14) For every strict normal subgroup H of G such that $H =$ the groupoid of N holds $\operatorname{Cosets} N = \operatorname{Cosets} H$.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. The functor $\operatorname{CosOp} N$ yielding a binary operation on $\operatorname{Cosets} N$ is defined by:

(Def. 15) For every strict normal subgroup H of G such that $H =$ the groupoid of N holds $\operatorname{CosOp} N = \operatorname{CosOp} H$.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. The functor $\operatorname{CosAc} N$ yielding an action of O on $\operatorname{Cosets} N$ is defined as follows:

(Def. 16)(i) For every element o of O holds $(\operatorname{CosAc} N)(o) = \{\langle A, B\rangle; A$ ranges over elements of $\operatorname{Cosets} N, B$ ranges over elements of $\operatorname{Cosets} N$: $\bigvee_{g,h\,:\,\text{element of } G} (g \in A \ \wedge\ h \in B \ \wedge\ h = (G \frown o)(g))\}$ if O is not empty,

(ii) $\operatorname{CosAc} N = [\!: \emptyset, \{\mathrm{id}_{\operatorname{Cosets} N}\} :\!]$, otherwise.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. The functor ${}^{G}/_{N}$ yields a group structure with operators in O and is defined as follows:

(Def. 17) ${}^{G}/_{N} = \langle \operatorname{Cosets} N, \operatorname{CosOp} N, \operatorname{CosAc} N\rangle$.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. Note that ${}^{G}/_{N}$ is non empty and ${}^{G}/_{N}$ is distributive, group-like, and associative.

Let O be a set, let G, H be groups with operators in O, and let f be a function from G into H. We say that f is homomorphic if and only if:

(Def. 18) For every element o of O and for every element g of G holds $f((G \frown o)(g)) = (H \frown o)(f(g))$.

Let O be a set and let G, H be groups with operators in O. One can check that there exists a function from G into H which is multiplicative and homomorphic.

Let O be a set and let G, H be groups with operators in O. A homomorphism from G to H is a multiplicative homomorphic function from G into H.

Let O be a set, let G, H, I be groups with operators in O, let h be a homomorphism from G to H, and let h_1 be a homomorphism from H to I. Then $h_1 \cdot h$ is a homomorphism from G to I.

Let O be a set, let G, H be groups with operators in O, and let h be a homomorphism from G to H. We say that h is monomorphism if and only if:

(Def. 19) h is one-to-one.

We say that h is epimorphism if and only if:

(Def. 20) $\operatorname{rng} h =$ the carrier of H.

Let O be a set, let G, H be groups with operators in O, and let h be a homomorphism from G to H. We say that h is isomorphism if and only if:

(Def. 21) h is an epimorphism and a monomorphism.

Let O be a set and let G, H be groups with operators in O. We say that G and H are isomorphic if and only if:

(Def. 22) There exists a homomorphism from G to H which is an isomorphism.

Let us note that the predicate G and H are isomorphic is reflexive.

Let O be a set and let G, H be groups with operators in O. Let us note that the predicate G and H are isomorphic is symmetric.

Let O be a set, let G be a group with operators in O, and let N be a normal stable subgroup of G. The canonical homomorphism onto cosets of N yields a homomorphism from G to ${}^{G}/_{N}$ and is defined by the condition (Def. 23).

(Def. 23) Let H be a strict normal subgroup of G. Suppose $H =$ the groupoid of N. Then the canonical homomorphism onto cosets of $N =$ the canonical homomorphism onto cosets of H.

Let O be a set, let G, H be groups with operators in O, and let g be a homomorphism from G to H. The functor $\operatorname{Ker} g$ yields a strict stable subgroup of G and is defined as follows:

(Def. 24) The carrier of $\operatorname{Ker} g = \{a; a$ ranges over elements of G: $g(a) = \mathbf{1}_H\}$.

Let O be a set, let G, H be groups with operators in O, and let g be a homomorphism from G to H. Observe that $\operatorname{Ker} g$ is normal.

Let O be a set, let G, H be groups with operators in O, and let g be a homomorphism from G to H. The functor $\operatorname{Im} g$ yielding a strict stable subgroup of H is defined by:

(Def. 25) The carrier of $\operatorname{Im} g = g^\circ$(the carrier of G).

Let O be a set, let G be a group with operators in O, and let H be a stable subgroup of G. The functor $\overline{H}$ yielding a subset of G is defined as follows:

(Def. 26) $\overline{H} =$ the carrier of H.

Let O be a set, let G be a group with operators in O, and let H_1, H_2 be stable subgroups of G. The functor $H_1 \cdot H_2$ yields a subset of G and is defined as follows:

(Def. 27) $H_1 \cdot H_2 = \overline{H_1} \cdot \overline{H_2}$.

Let O be a set, let G be a group with operators in O, and let H_1, H_2 be stable subgroups of G. The functor $H_1 \cap H_2$ yielding a strict stable subgroup of G is defined by:

(Def. 28) The carrier of $H_1 \cap H_2 = \overline{H_1} \cap \overline{H_2}$.

Let us note that the functor $H_1 \cap H_2$ is commutative.

Let O be a set, let G be a group with operators in O, and let A be a subset of G. The stable subgroup of A yielding a strict stable subgroup of G is defined by the conditions (Def. 29).

(Def. 29)(i) $A \subseteq$ the carrier of the stable subgroup of A, and

(ii) for every strict stable subgroup H of G such that $A \subseteq$ the carrier of H holds the stable subgroup of A is a stable subgroup of H.

Let O be a set, let G be a group with operators in O, and let H_1, H_2 be stable subgroups of G. The functor $H_1 \sqcup H_2$ yielding a strict stable subgroup of G is defined as follows:

(Def. 30) $H_1 \sqcup H_2 =$ the stable subgroup of $\overline{H_1} \cup \overline{H_2}$.

2. Some Theorems on Groups Reformulated for Groups with Operators

For simplicity, we follow the rules: x, O are sets, o is an element of O, G, H, I are groups with operators in O, A, B are subsets of G, N is a normal stable subgroup of G, H_1, H_2, H_3 are stable subgroups of G, g_1, g_2 are elements of G, h_1, h_2 are elements of H_1, and h is a homomorphism from G to H.

One can prove the following propositions:

(1) If $x \in H_1$, then $x \in G$.

(2) h_1 is an element of G.

(3) If $h_1 = g_1$ and $h_2 = g_2$, then $h_1 \cdot h_2 = g_1 \cdot g_2$.

(4) $\mathbf{1}_G = \mathbf{1}_{(H_1)}$.

(5) $\mathbf{1}_G \in H_1$.

(6) If $h_1 = g_1$, then ${h_1}^{-1} = {g_1}^{-1}$.

(7) If $g_1 \in H_1$ and $g_2 \in H_1$, then $g_1 \cdot g_2 \in H_1$.

(8) If $g_1 \in H_1$, then ${g_1}^{-1} \in H_1$.

(9) Suppose that

(i) $A \neq \emptyset$,

(ii) for all g_1, g_2 such that $g_1 \in A$ and $g_2 \in A$ holds $g_1 \cdot g_2 \in A$,

(iii) for every g_1 such that $g_1 \in A$ holds ${g_1}^{-1} \in A$, and

(iv) for all o, g_1 such that $g_1 \in A$ holds $(G \frown o)(g_1) \in A$.

Then there exists a strict stable subgroup H of G such that the carrier of $H = A$.

(10) G is a stable subgroup of G.

(11) Let G_1, G_2, G_3 be groups with operators in O. Suppose G_1 is a stable subgroup of G_2 and G_2 is a stable subgroup of G_3. Then G_1 is a stable subgroup of G_3.

(12) If the carrier of $H_1 \subseteq$ the carrier of H_2, then H_1 is a stable subgroup of H_2.

(13) If for every element g of G such that $g \in H_1$ holds $g \in H_2$, then H_1 is a stable subgroup of H_2.

(14) For all strict stable subgroups H_1, H_2 of G such that the carrier of $H_1 =$ the carrier of H_2 holds $H_1 = H_2$.

(15) $\{\mathbf{1}\}_G = \{\mathbf{1}\}_{(H_1)}$.

(16) $\{\mathbf{1}\}_G$ is a stable subgroup of H_1.

(17) If $\overline{H_1} \cdot \overline{H_2} = \overline{H_2} \cdot \overline{H_1}$, then there exists a strict stable subgroup H of G such that the carrier of $H = \overline{H_1} \cdot \overline{H_2}$.

(18)(i) For every stable subgroup H of G such that $H = H_1 \cap H_2$ holds the carrier of $H =$ (the carrier of H_1) $\cap$ (the carrier of H_2), and

(ii) for every strict stable subgroup H of G such that the carrier of $H =$ (the carrier of H_1) $\cap$ (the carrier of H_2) holds $H = H_1 \cap H_2$.

(19) For every strict stable subgroup H of G holds $H \cap H = H$.

(20) $(H_1 \cap H_2) \cap H_3 = H_1 \cap (H_2 \cap H_3)$.

(21) $\{\mathbf{1}\}_G \cap H_1 = \{\mathbf{1}\}_G$ and $H_1 \cap \{\mathbf{1}\}_G = \{\mathbf{1}\}_G$.

(22) $\bigcup \mathrm{Cosets}\, N =$ the carrier of G.

(23) Let N_1, N_2 be strict normal stable subgroups of G. Then there exists a strict normal stable subgroup N of G such that the carrier of $N = \overline{N_1} \cdot \overline{N_2}$.

(24) $g_1 \in$ the stable subgroup of A if and only if there exists a finite sequence F of elements of the carrier of G and there exists a finite sequence I of elements of $\mathbb{Z}$ and there exists a subset C of G such that $C =$ the stable subset generated by A and $\mathrm{len}\, F = \mathrm{len}\, I$ and $\mathrm{rng}\, F \subseteq C$ and $\prod(F^I) = g_1$.

(25) For every strict stable subgroup H of G holds the stable subgroup of $\overline{H} = H$.

(26) If $A \subseteq B$, then the stable subgroup of A is a stable subgroup of the stable subgroup of B.

The scheme *MeetSbgWOpEx* deals with a set $\mathcal{A}$, a group $\mathcal{B}$ with operators in $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

There exists a strict stable subgroup H of $\mathcal{B}$ such that the carrier of $H = \bigcap\{A; A$ ranges over subsets of $\mathcal{B}$: $\bigvee_{K:\ \text{strict stable subgroup of } \mathcal{B}} (A = \text{the carrier of } K \wedge \mathcal{P}[K])\}$

provided the parameters meet the following requirement:

- There exists a strict stable subgroup H of $\mathcal{B}$ such that $\mathcal{P}[H]$.

The following propositions are true:

(27) The carrier of the stable subgroup of $A = \bigcap\{B; B$ ranges over subsets of G: $\bigvee_{H:\ \text{strict stable subgroup of } G} (B = \text{the carrier of } H \wedge A \subseteq \overline{H})\}$.

(28) For all strict normal stable subgroups N_1, N_2 of G holds $N_1 \cdot N_2 = N_2 \cdot N_1$.

(29) $H_1 \sqcup H_2 =$ the stable subgroup of $H_1 \cdot H_2$.

(30) If $H_1 \cdot H_2 = H_2 \cdot H_1$, then the carrier of $H_1 \sqcup H_2 = H_1 \cdot H_2$.

(31) For all strict normal stable subgroups N_1, N_2 of G holds the carrier of $N_1 \sqcup N_2 = N_1 \cdot N_2$.

(32) For all strict normal stable subgroups N_1, N_2 of G holds $N_1 \sqcup N_2$ is a normal stable subgroup of G.

(33) For every strict stable subgroup H of G holds $\{\mathbf{1}\}_G \sqcup H = H$ and $H \sqcup \{\mathbf{1}\}_G = H$.

(34) $\Omega_G \sqcup H_1 = \Omega_G$ and $H_1 \sqcup \Omega_G = \Omega_G$.

(35) H_1 is a stable subgroup of $H_1 \sqcup H_2$ and H_2 is a stable subgroup of $H_1 \sqcup H_2$.

(36) For every strict stable subgroup H_2 of G holds H_1 is a stable subgroup of H_2 iff $H_1 \sqcup H_2 = H_2$.

(37) Let H_3 be a strict stable subgroup of G. Suppose H_1 is a stable subgroup of H_3 and H_2 is a stable subgroup of H_3. Then $H_1 \sqcup H_2$ is a stable subgroup of H_3.

(38) Let H_2, H_3 be strict stable subgroups of G. Suppose H_1 is a stable subgroup of H_2. Then $H_1 \sqcup H_3$ is a stable subgroup of $H_2 \sqcup H_3$.

(39) For all stable subgroups X, Y of H_1 and for all stable subgroups X', Y' of G such that $X = X'$ and $Y = Y'$ holds $X' \cap Y' = X \cap Y$.

(40) If N is a stable subgroup of H_1, then N is a normal stable subgroup of H_1.

(41) $H_1 \cap N$ is a normal stable subgroup of H_1 and $N \cap H_1$ is a normal stable subgroup of H_1.

(42) For every strict group G with operators in O such that G is trivial holds $\{\mathbf{1}\}_G = G$.

(43) $\mathbf{1}_{{}^{G}/_{N}} = \overline{N}$.

(44) Let M, N be strict normal stable subgroups of G and M_1 be a normal stable subgroup of N. Suppose $M_1 = M$ and M is a stable subgroup of N. Then ${}^{N}/_{M_1}$ is a normal stable subgroup of ${}^{G}/_{M}$.

(45) $h(\mathbf{1}_G) = \mathbf{1}_H$.

(46) $h({g_1}^{-1}) = h(g_1)^{-1}$.

(47) $g_1 \in \operatorname{Ker} h$ iff $h(g_1) = \mathbf{1}_H$.

(48) For every strict normal stable subgroup N of G holds Ker (the canonical homomorphism onto cosets of N) $= N$.

(49) $\operatorname{rng} h =$ the carrier of $\operatorname{Im} h$.

(50) Im (the canonical homomorphism onto cosets of N) $= {}^{G}/_{N}$.

(51) Let H be a strict group with operators in O and h be a homomorphism from G to H. Then h is an epimorphism if and only if $\operatorname{Im} h = H$.

(52) Let H be a strict group with operators in O and h be a homomorphism from G to H. Suppose h is an epimorphism. Let c be an element of H. Then there exists an element a of G such that $h(a) = c$.

(53) The canonical homomorphism onto cosets of N is an epimorphism.

(54) The canonical homomorphism onto cosets of $\{\mathbf{1}\}_G$ is an isomorphism.

(55) If G and H are isomorphic and H and I are isomorphic, then G and I are isomorphic.

(56) For every strict group G with operators in O holds G and ${}^{G}/_{\{\mathbf{1}\}_G}$ are isomorphic.

(57) For every strict group G with operators in O holds ${}^{G}/_{\Omega_G}$ is trivial.

(58) Let G, H be strict groups with operators in O. If G and H are isomorphic and G is trivial, then H is trivial.

(59) ${}^{G}/_{\mathrm{Ker}\, h}$ and $\mathrm{Im}\, h$ are isomorphic.

(60) Let H, F_1, F_2 be strict stable subgroups of G. Suppose F_1 is a normal stable subgroup of F_2. Then $H \cap F_1$ is a normal stable subgroup of $H \cap F_2$.

3. Others Theorems on Actions and Groups with Operators

In the sequel E is a set, A is an action of O on E, C is a subset of G, and N_1 is a normal stable subgroup of H_1.

One can prove the following propositions:

(61) Ω_E is stable under the action of A.

(62) $[\!: O, \{\mathrm{id}_E\} :\!]$ is an action of O on E.

(63) Let O be a non empty set, E be a set, o be an element of O, and A be an action of O on E. Then $\mathrm{Product}(\langle o \rangle, A) = A(o)$.

(64) Let O be a non empty set, E be a set, F_1, F_2 be finite sequences of elements of O, and A be an action of O on E. Then $\mathrm{Product}(F_1 \frown F_2, A) = \mathrm{Product}(F_1, A) \cdot \mathrm{Product}(F_2, A)$.

(65) Let F be a finite sequence of elements of O and Y be a subset of E. If Y is stable under the action of A, then $(\mathrm{Product}(F, A))^{\circ} Y \subseteq Y$.

(66) Let E be a non empty set, A be an action of O on E, X be a subset of E, and a be an element of E. Suppose X is not empty. Then $a \in$ the stable subset generated by X if and only if there exists a finite sequence F of elements of O and there exists an element x of X such that $(\mathrm{Product}(F, A))(x) = a$.

(67) For every strict group G there exists a strict group H with operators in O such that $G =$ the groupoid of H.

(68) The groupoid of H_1 is a strict subgroup of G.

(69) The groupoid of N is a strict normal subgroup of G.

(70) If $g_1 \in H_1$, then $(G \frown o)(g_1) \in H_1$.

(71) Let O be a set, G, H be groups with operators in O, G' be a strict stable subgroup of G, and f be a homomorphism from G to H. Then there exists a strict stable subgroup H' of H such that the carrier of $H' = f^{\circ}$(the carrier of G').

(72) If B is empty, then the stable subgroup of $B = \{\mathbf{1}\}_G$.

(73) If $B =$ the carrier of $\mathrm{gr}(C)$, then the stable subgroup of $C =$ the stable subgroup of B.

(74) Let N' be a normal subgroup of G. Suppose N' = the groupoid of N. Then ${}^{G}/_{N'}$ = the groupoid of ${}^{G}/_{N}$ and $\mathbf{1}_{{}^{G}/_{N'}} = \mathbf{1}_{{}^{G}/_{N}}$.

(75) Suppose the carrier of H_1 = the carrier of H_2. Then the group structure with operators of H_1 = the group structure with operators of H_2.

(76) Suppose ${}^{H_1}/_{N_1}$ is trivial. Then the group structure with operators of H_1 = the group structure with operators of N_1.

(77) If the carrier of H_1 = the carrier of N_1, then ${}^{H_1}/_{N_1}$ is trivial.

(78) Let G, H be groups with operators in O, N be a stable subgroup of G, H' be a strict stable subgroup of H, and f be a homomorphism from G to H. Suppose $N = \operatorname{Ker} f$. Then there exists a strict stable subgroup G' of G such that
  (i) the carrier of $G' = f^{-1}$(the carrier of H'), and
  (ii) if H' is normal, then N is a normal stable subgroup of G' and G' is normal.

(79) Let G, H be groups with operators in O, N be a stable subgroup of G, G' be a strict stable subgroup of G, and f be a homomorphism from G to H. Suppose $N = \operatorname{Ker} f$. Then there exists a strict stable subgroup H' of H such that
  (i) the carrier of $H' = f^{\circ}$(the carrier of G'),
  (ii) f^{-1}(the carrier of H') = the carrier of $G' \sqcup N$, and
  (iii) if f is an epimorphism and G' is normal, then H' is normal.

(80) Let G be a strict group with operators in O, N be a strict normal stable subgroup of G, and H be a strict stable subgroup of ${}^{G}/_{N}$. Suppose the carrier of G = (the canonical homomorphism onto cosets of $N)^{-1}$(the carrier of H). Then $H = \Omega_{{}^{G}/_{N}}$.

(81) Let G be a strict group with operators in O, N be a strict normal stable subgroup of G, and H be a strict stable subgroup of ${}^{G}/_{N}$. Suppose the carrier of N = (the canonical homomorphism onto cosets of $N)^{-1}$(the carrier of H). Then $H = \{\mathbf{1}\}_{{}^{G}/_{N}}$.

(82) Let G, H be strict groups with operators in O. If G and H are isomorphic and G is simple, then H is simple.

(83) Let G be a group with operators in O, H be a stable subgroup of G, F_3 be a finite sequence of elements of the carrier of G, F_4 be a finite sequence of elements of the carrier of H, and I be a finite sequence of elements of $\mathbb{Z}$. If $F_3 = F_4$ and $\operatorname{len} F_3 = \operatorname{len} I$, then $\prod({F_3}^{I}) = \prod({F_4}^{I})$.

(84) Let O, E_1, E_2 be sets, A_1 be an action of O on E_1, A_2 be an action of O on E_2, and F be a finite sequence of elements of O. Suppose that
  (i) $E_1 \subseteq E_2$, and
  (ii) for every element o of O and for every function f_1 from E_1 into E_1 and for every function f_2 from E_2 into E_2 such that $f_1 = A_1(o)$ and $f_2 = A_2(o)$

holds $f_1 = f_2 \restriction E_1$.

Then $\mathrm{Product}(F, A_1) = \mathrm{Product}(F, A_2) \restriction E_1$.

(85) Let N_1, N_2 be strict stable subgroups of H_1 and N_1', N_2' be strict stable subgroups of G. If $N_1 = N_1'$ and $N_2 = N_2'$, then $N_1' \cdot N_2' = N_1 \cdot N_2$.

(86) Let N_1, N_2 be strict stable subgroups of H_1 and N_1', N_2' be strict stable subgroups of G. If $N_1 = N_1'$ and $N_2 = N_2'$, then $N_1' \sqcup N_2' = N_1 \sqcup N_2$.

(87) Let N_1, N_2 be strict stable subgroups of G. Suppose N_1 is a normal stable subgroup of H_1 and N_2 is a normal stable subgroup of H_1. Then $N_1 \sqcup N_2$ is a normal stable subgroup of H_1.

(88) Let f be a homomorphism from G to H and g be a homomorphism from H to I. Then the carrier of $\mathrm{Ker}(g \cdot f) = f^{-1}$(the carrier of $\mathrm{Ker}\, g$).

(89) Let G' be a stable subgroup of G, H' be a stable subgroup of H, and f be a homomorphism from G to H. Suppose the carrier of $H' = f^{\circ}$(the carrier of G') or the carrier of $G' = f^{-1}$(the carrier of H'). Then $f\restriction$the carrier of G' is a homomorphism from G' to H'.

(90) Let G, H be strict groups with operators in O, N, L, G' be strict stable subgroups of G, and f be a homomorphism from G to H. Suppose $N = \mathrm{Ker}\, f$ and L is a strict normal stable subgroup of G'. Then
 (i) $L \sqcup G' \cap N$ is a normal stable subgroup of G',
 (ii) $L \sqcup N$ is a normal stable subgroup of $G' \sqcup N$, and
 (iii) for every strict normal stable subgroup N_1 of $G' \sqcup N$ and for every strict normal stable subgroup N_2 of G' such that $N_1 = L \sqcup N$ and $N_2 = L \sqcup G' \cap N$ holds ${}^{(G' \sqcup N)}/_{N_1}$ and ${}^{G'}/_{N_2}$ are isomorphic.

4. The Zassenhaus Butterfly Lemma

The following propositions are true:

(91) Let H, K, H', K' be strict stable subgroups of G, J_1 be a normal stable subgroup of $H' \sqcup H \cap K$, and H_4 be a normal stable subgroup of $H \cap K$. Suppose H' is a normal stable subgroup of H and K' is a normal stable subgroup of K and $J_1 = H' \sqcup H \cap K'$ and $H_4 = H' \cap K \sqcup K' \cap H$. Then ${}^{(H' \sqcup H \cap K)}/_{J_1}$ and ${}^{(H \cap K)}/_{H_4}$ are isomorphic.

(92) Let H, K, H', K' be strict stable subgroups of G. Suppose H' is a normal stable subgroup of H and K' is a normal stable subgroup of K. Then $H' \sqcup H \cap K'$ is a normal stable subgroup of $H' \sqcup H \cap K$.

(93) Let H, K, H', K' be strict stable subgroups of G, J_1 be a normal stable subgroup of $H' \sqcup H \cap K$, and J_2 be a normal stable subgroup of $K' \sqcup K \cap H$. Suppose $J_1 = H' \sqcup H \cap K'$ and $J_2 = K' \sqcup K \cap H'$ and H' is a normal stable subgroup of H and K' is a normal stable subgroup of K. Then ${}^{(H' \sqcup H \cap K)}/_{J_1}$ and ${}^{(K' \sqcup K \cap H)}/_{J_2}$ are isomorphic.

5. Composition Series

Let O be a set, let G be a group with operators in O, and let I_1 be a finite sequence of elements of the stable subgroups of G. We say that I_1 is composition series if and only if the conditions (Def. 31) are satisfied.

(Def. 31)(i) $I_1(1) = \Omega_G$,

(ii) $I_1(\operatorname{len} I_1) = \{\mathbf{1}\}_G$, and

(iii) for every natural number i such that $i \in \operatorname{dom} I_1$ and $i+1 \in \operatorname{dom} I_1$ and for all stable subgroups H_1, H_2 of G such that $H_1 = I_1(i)$ and $H_2 = I_1(i+1)$ holds H_2 is a normal stable subgroup of H_1.

Let O be a set and let G be a group with operators in O. One can verify that there exists a finite sequence of elements of the stable subgroups of G which is composition series.

Let O be a set and let G be a group with operators in O. A composition series of G is a composition series finite sequence of elements of the stable subgroups of G.

Let O be a set, let G be a group with operators in O, and let s_1, s_2 be composition series of G. We say that s_1 is finer than s_2 if and only if:

(Def. 32) There exists a set x such that $x \subseteq \operatorname{dom} s_1$ and $s_2 = s_1 \cdot \operatorname{Sgm} x$.

Let us note that the predicate s_1 is finer than s_2 is reflexive.

Let O be a set, let G be a group with operators in O, and let I_1 be a composition series of G. We say that I_1 is strictly decreasing if and only if the condition (Def. 33) is satisfied.

(Def. 33) Let i be a natural number. Suppose $i \in \operatorname{dom} I_1$ and $i+1 \in \operatorname{dom} I_1$. Let H be a stable subgroup of G and N be a normal stable subgroup of H. If $H = I_1(i)$ and $N = I_1(i+1)$, then ${}^{H}/_{N}$ is not trivial.

Let O be a set, let G be a group with operators in O, and let I_1 be a composition series of G. We say that I_1 is Jordan-Hölder if and only if the conditions (Def. 34) are satisfied.

(Def. 34)(i) I_1 is strictly decreasing, and

(ii) it is not true that there exists a composition series s of G such that $s \neq I_1$ and s is strictly decreasing and finer than I_1.

Let O be a set, let G_1, G_2 be groups with operators in O, let s_1 be a composition series of G_1, and let s_2 be a composition series of G_2. We say that s_1 is equivalent with s_2 if and only if the conditions (Def. 35) are satisfied.

(Def. 35)(i) $\operatorname{len} s_1 = \operatorname{len} s_2$, and

(ii) for every natural number n such that $n+1 = \operatorname{len} s_1$ there exists a permutation p of $\operatorname{Seg} n$ such that for every stable subgroup H_1 of G_1 and for every stable subgroup H_2 of G_2 and for every normal stable subgroup N_1 of H_1 and for every normal stable subgroup N_2 of H_2 and for all natural numbers i, j such that $1 \leq i$ and $i \leq n$ and $j = p(i)$ and $H_1 = s_1(i)$ and

$H_2 = s_2(j)$ and $N_1 = s_1(i+1)$ and $N_2 = s_2(j+1)$ holds ${}^{H_1}/_{N_1}$ and ${}^{H_2}/_{N_2}$ are isomorphic.

Let O be a set, let G be a group with operators in O, and let s be a composition series of G. The series of quotients of s yielding a finite sequence is defined as follows:

(Def. 36)(i) $\operatorname{len} s = \operatorname{len}$ (the series of quotients of s) $+ 1$ and for every natural number i such that $i \in \operatorname{dom}$ (the series of quotients of s) and for every stable subgroup H of G and for every normal stable subgroup N of H such that $H = s(i)$ and $N = s(i+1)$ holds (the series of quotients of $s)(i) = {}^{H}/_{N}$ if $\operatorname{len} s > 1$,

(ii) the series of quotients of $s = \emptyset$, otherwise.

Let O be a set, let f_1, f_2 be finite sequences, and let p be a permutation of $\operatorname{dom} f_1$. We say that f_1 and f_2 are equivalent under p in O if and only if the conditions (Def. 37) are satisfied.

(Def. 37)(i) $\operatorname{len} f_1 = \operatorname{len} f_2$, and

(ii) for all groups H_1, H_2 with operators in O and for all natural numbers i, j such that $i \in \operatorname{dom} f_1$ and $j = p^{-1}(i)$ and $H_1 = f_1(i)$ and $H_2 = f_2(j)$ holds H_1 and H_2 are isomorphic.

For simplicity, we follow the rules: y is a set, s_1, s'_1, s_2, s'_2 are composition series of G, f_3 is a finite sequence of elements of the stable subgroups of G, f_1, f_2 are finite sequences, and i, j, n are natural numbers.

We now state a number of propositions:

(94) If $i \in \operatorname{dom} s_1$ and $i+1 \in \operatorname{dom} s_1$ and $s_1(i) = s_1(i+1)$ and $f_3 = (s_1)_{\restriction i}$, then f_3 is composition series.

(95) If s_1 is finer than s_2, then there exists n such that $\operatorname{len} s_1 = \operatorname{len} s_2 + n$.

(96) If $\operatorname{len} s_2 = \operatorname{len} s_1$ and s_2 is finer than s_1, then $s_1 = s_2$.

(97) If s_1 is not empty and s_2 is finer than s_1, then s_2 is not empty.

(98) If s_1 is finer than s_2 and Jordan-Hölder and s_2 is Jordan-Hölder, then $s_1 = s_2$.

(99) If $i \in \operatorname{dom} s_1$ and $i+1 \in \operatorname{dom} s_1$ and $s_1(i) = s_1(i+1)$ and $s'_1 = (s_1)_{\restriction i}$ and s_2 is Jordan-Hölder and s_1 is finer than s_2, then s'_1 is finer than s_2.

(100) Suppose $\operatorname{len} s_1 > 1$ and $s_2 \neq s_1$ and s_2 is strictly decreasing and finer than s_1. Then there exist i, j such that $i \in \operatorname{dom} s_1$ and $i \in \operatorname{dom} s_2$ and $i+1 \in \operatorname{dom} s_1$ and $i+1 \in \operatorname{dom} s_2$ and $j \in \operatorname{dom} s_2$ and $i+1 < j$ and $s_1(i) = s_2(i)$ and $s_1(i+1) \neq s_2(i+1)$ and $s_1(i+1) = s_2(j)$.

(101) If $i \in \operatorname{dom} s_1$ and $j \in \operatorname{dom} s_1$ and $i \leq j$ and $H_1 = s_1(i)$ and $H_2 = s_1(j)$, then H_2 is a stable subgroup of H_1.

(102) If $y \in \operatorname{rng}$ (the series of quotients of s_1), then y is a strict group with operators in O.

(103) Suppose $i \in \operatorname{dom}$ (the series of quotients of s_1) and for every H such that $H =$ (the series of quotients of s_1)(i) holds H is trivial. Then $i \in \operatorname{dom} s_1$ and $i+1 \in \operatorname{dom} s_1$ and $s_1(i) = s_1(i+1)$.

(104) Suppose $i \in \operatorname{dom} s_1$ and $i+1 \in \operatorname{dom} s_1$ and $s_1(i) = s_1(i+1)$ and $s_2 = (s_1)_{\restriction i}$. Then the series of quotients of $s_2 =$ (the series of quotients of $s_1)_{\restriction i}$.

(105) Suppose $f_1 =$ the series of quotients of s_1 and $i \in \operatorname{dom} f_1$ and for every H such that $H = f_1(i)$ holds H is trivial. Then $(s_1)_{\restriction i}$ is a composition series of G and for every s_2 such that $s_2 = (s_1)_{\restriction i}$ holds the series of quotients of $s_2 = (f_1)_{\restriction i}$.

(106) Suppose that

(i) $f_1 =$ the series of quotients of s_1,

(ii) $f_2 =$ the series of quotients of s_2,

(iii) $i \in \operatorname{dom} f_1$,

(iv) for every H such that $H = f_1(i)$ holds H is trivial, and

(v) there exists a permutation p of $\operatorname{dom} f_1$ such that f_1 and f_2 are equivalent under p in O and $j = p^{-1}(i)$.

Then there exists a permutation p' of $\operatorname{dom}((f_1)_{\restriction i})$ such that $(f_1)_{\restriction i}$ and $(f_2)_{\restriction j}$ are equivalent under p' in O.

(107) Let G_1, G_2 be groups with operators in O, s_1 be a composition series of G_1, and s_2 be a composition series of G_2. If s_1 is empty and s_2 is empty, then s_1 is equivalent with s_2.

(108) Let G_1, G_2 be groups with operators in O, s_1 be a composition series of G_1, and s_2 be a composition series of G_2. Suppose s_1 is not empty and s_2 is not empty. Then s_1 is equivalent with s_2 if and only if there exists a permutation p of dom (the series of quotients of s_1) such that the series of quotients of s_1 and the series of quotients of s_2 are equivalent under p in O.

(109) Suppose s_1 is finer than s_2 and s_2 is Jordan-Hölder and $\operatorname{len} s_1 > \operatorname{len} s_2$. Then there exists i such that $i \in \operatorname{dom}$ (the series of quotients of s_1) and for every H such that $H =$ (the series of quotients of s_1)(i) holds H is trivial.

(110) Suppose $\operatorname{len} s_1 > 1$. Then s_1 is Jordan-Hölder if and only if for every i such that $i \in \operatorname{dom}$ (the series of quotients of s_1) holds (the series of quotients of s_1)(i) is a strict simple group with operators in O.

(111) Suppose $1 \leq i$ and $i \leq \operatorname{len} s_1 - 1$. Then $s_1(i)$ is a strict stable subgroup of G and $s_1(i+1)$ is a strict stable subgroup of G.

(112) If $1 \leq i$ and $i \leq \operatorname{len} s_1 - 1$ and $H_1 = s_1(i)$ and $H_2 = s_1(i+1)$, then H_2 is a normal stable subgroup of H_1.

(113) s_1 is equivalent with s_1.

(114) If $\operatorname{len} s_1 \leq 1$ or $\operatorname{len} s_2 \leq 1$ and if $\operatorname{len} s_1 \leq \operatorname{len} s_2$, then s_2 is finer than s_1.

(115) If s_1 is equivalent with s_2 and Jordan-Hölder, then s_2 is Jordan-Hölder.

6. The Schreier Refinement Theorem

Let us consider O, G, s_1, s_2. Let us assume that $\operatorname{len} s_1 > 1$ and $\operatorname{len} s_2 > 1$. The Schreier series of s_1 and s_2 yielding a composition series of G is defined by the condition (Def. 38).

(Def. 38) Let k, i, j be natural numbers and H_1, H_2, H_3 be stable subgroups of G. Then

(i) if $k = (i-1) \cdot (\operatorname{len} s_2 - 1) + j$ and $1 \leq i$ and $i \leq \operatorname{len} s_1 - 1$ and $1 \leq j$ and $j \leq \operatorname{len} s_2 - 1$ and $H_1 = s_1(i+1)$ and $H_2 = s_1(i)$ and $H_3 = s_2(j)$, then (the Schreier series of s_1 and $s_2)(k) = H_1 \sqcup H_2 \cap H_3$,

(ii) if $k = (\operatorname{len} s_1 - 1) \cdot (\operatorname{len} s_2 - 1) + 1$, then (the Schreier series of s_1 and $s_2)(k) = \{\mathbf{1}\}_G$, and

(iii) len (the Schreier series of s_1 and $s_2) = (\operatorname{len} s_1 - 1) \cdot (\operatorname{len} s_2 - 1) + 1$.

Next we state three propositions:

(116) If $\operatorname{len} s_1 > 1$ and $\operatorname{len} s_2 > 1$, then the Schreier series of s_1 and s_2 is finer than s_1.

(117) If $\operatorname{len} s_1 > 1$ and $\operatorname{len} s_2 > 1$, then the Schreier series of s_1 and s_2 is equivalent with the Schreier series of s_2 and s_1.

(118) There exist s_1', s_2' such that s_1' is finer than s_1 and s_2' is finer than s_2 and s_1' is equivalent with s_2'.

7. The Jordan-Hölder Theorem

One can prove the following proposition

(119) If s_1 is Jordan-Hölder and s_2 is Jordan-Hölder, then s_1 is equivalent with s_2.

8. Appendix

Next we state several propositions:

(120) For all binary relations P, R holds $P = \operatorname{rng} P \upharpoonright R$ iff $P^{\smile} = R^{\smile} \upharpoonright \operatorname{dom}(P^{\smile})$.

(121) For every set X and for all binary relations P, R holds $P \cdot (R \upharpoonright X) = (X \upharpoonright P) \cdot R$.

(122) Let n be a natural number, X be a set, and f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. If $X \subseteq \operatorname{Seg} n$ and $X \subseteq \operatorname{dom} f$ and f is increasing on X and $f^{\circ}X \subseteq \mathbb{N} \setminus \{0\}$, then $\operatorname{Sgm}(f^{\circ}X) = f \cdot \operatorname{Sgm} X$.

(123) Let y be a set and i, n be natural numbers. Suppose $y \subseteq \mathrm{Seg}(n+1)$ and $i \in \mathrm{Seg}(n+1)$ and $i \notin y$. Then there exists x such that $\mathrm{Sgm}\, x = (\mathrm{Sgm}(\mathrm{Seg}(n+1) \setminus \{i\}))^{-1} \cdot \mathrm{Sgm}\, y$ and $x \subseteq \mathrm{Seg}\, n$.

(124) Let D be a non empty set, f be a finite sequence of elements of D, p be an element of D, and n be an element of $\mathbb{N}$. If $n \in \mathrm{dom}\, f$, then $f = (\mathrm{Ins}(f, n, p))\restriction_{n+1}$.

(125) Let G, H be groups, F_1 be a finite sequence of elements of the carrier of G, F_2 be a finite sequence of elements of the carrier of H, I be a finite sequence of elements of $\mathbb{Z}$, and f be a homomorphism from G to H. Suppose for every element k of $\mathbb{N}$ such that $k \in \mathrm{Seg}\,\mathrm{len}\, F_1$ holds $F_2(k) = f(F_1(k))$ and $\mathrm{len}\, F_1 = \mathrm{len}\, I$ and $\mathrm{len}\, F_2 = \mathrm{len}\, I$. Then $f(\prod(F_1{}^I)) = \prod(F_2{}^I)$.

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