FORMALIZED MATHEMATICS
Volume 13, Number 3, Pages 389–397
University of Białystok, 2005

Lines on Planes in n-Dimensional Euclidean Spaces

Akihiro Kubo
Shinshu University, Nagano, Japan

Summary. In the paper we introduce basic properties of lines in the plane on this space. Lines and planes are expressed by the vector equation and are the image of $\mathbb{R}$ and $\mathbb{R}^2$. By this, we can say that the properties of the classic Euclid geometry are satisfied also in $\mathcal{R}^n$ as we know them intuitively. Next, we define the metric between the point and the line of this space.

MML identifier: EUCLIDLP, version: 7.5.01 4.39.921

The notation and terminology used here are introduced in the following papers: [1], [5], [12], [4], [9], [14], [13], [8], [15], [6], [2], [3], [7], [11], and [10].

We follow the rules: a, a_1, a_2, a_3, b, b_1, b_2, b_3, r, s, t, u are real numbers, n is a natural number, and x_0, x, x_1, x_2, x_3, y_0, y, y_1, y_2, y_3 are elements of $\mathcal{R}^n$.

One can prove the following propositions:

(1) $\frac{s}{t} \cdot (u \cdot x) = \frac{s \cdot u}{t} \cdot x$ and $\frac{1}{t} \cdot (u \cdot x) = \frac{u}{t} \cdot x$.

(2) $x_1 + (x_2 + x_3) = (x_1 + x_2) + x_3$.

(3) $x - \langle \underbrace{0, \ldots, 0}_{n} \rangle = x$.

(4) $\langle \underbrace{0, \ldots, 0}_{n} \rangle - x = -x$.

(5) $x_1 - (x_2 + x_3) = x_1 - x_2 - x_3$.

(6) $x_1 - x_2 = x_1 + -x_2$.

(7) $x - x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$ and $x + -x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(8) $-a \cdot x = (-a) \cdot x$ and $-a \cdot x = a \cdot -x$.

(9) $x_1 - (x_2 - x_3) = (x_1 - x_2) + x_3$.

(10) $x_1 + (x_2 - x_3) = (x_1 + x_2) - x_3$.

ISSN 1426–2630

(11) $x_1 = x_2 + x_3$ iff $x_2 = x_1 - x_3$.

(12) $x = x_1 + x_2 + x_3$ iff $x - x_1 = x_2 + x_3$.

(13) $-(x_1 + x_2 + x_3) = -x_1 + -x_2 + -x_3$.

(14) $x_1 = x_2$ iff $x_1 - x_2 = \langle\underbrace{0, \ldots, 0}_{n}\rangle$.

(15) If $x_1 - x_0 = t \cdot x$ and $x_1 \neq x_0$, then $t \neq 0$.

(16) $(a-b) \cdot x = a \cdot x + (-b) \cdot x$ and $(a-b) \cdot x = a \cdot x + -b \cdot x$ and $(a-b) \cdot x = a \cdot x - b \cdot x$.

(17) $a \cdot (x-y) = a \cdot x + -a \cdot y$ and $a \cdot (x-y) = a \cdot x + (-a) \cdot y$ and $a \cdot (x-y) = a \cdot x - a \cdot y$.

(18) $(s - t - u) \cdot x = s \cdot x - t \cdot x - u \cdot x$.

(19) $x - (a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3) = x + ((-a_1) \cdot x_1 + (-a_2) \cdot x_2 + (-a_3) \cdot x_3)$.

(20) $x - (s + t + u) \cdot y = x + (-s) \cdot y + (-t) \cdot y + (-u) \cdot y$.

(21) $(x_1 + x_2) + (y_1 + y_2) = x_1 + y_1 + (x_2 + y_2)$.

(22) $(x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = x_1 + y_1 + (x_2 + y_2) + (x_3 + y_3)$.

(23) $(x_1 + x_2) - (y_1 + y_2) = (x_1 - y_1) + (x_2 - y_2)$.

(24) $(x_1 + x_2 + x_3) - (y_1 + y_2 + y_3) = (x_1 - y_1) + (x_2 - y_2) + (x_3 - y_3)$.

(25) $a \cdot (x_1 + x_2 + x_3) = a \cdot x_1 + a \cdot x_2 + a \cdot x_3$.

(26) $a \cdot (b_1 \cdot x_1 + b_2 \cdot x_2) = a \cdot b_1 \cdot x_1 + a \cdot b_2 \cdot x_2$.

(27) $a \cdot (b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3) = a \cdot b_1 \cdot x_1 + a \cdot b_2 \cdot x_2 + a \cdot b_3 \cdot x_3$.

(28) $a_1 \cdot x_1 + a_2 \cdot x_2 + (b_1 \cdot x_1 + b_2 \cdot x_2) = (a_1 + b_1) \cdot x_1 + (a_2 + b_2) \cdot x_2$.

(29) $a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3 + (b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3) = ((a_1 + b_1) \cdot x_1 + (a_2 + b_2) \cdot x_2) + (a_3 + b_3) \cdot x_3$.

(30) $(a_1 \cdot x_1 + a_2 \cdot x_2) - (b_1 \cdot x_1 + b_2 \cdot x_2) = (a_1 - b_1) \cdot x_1 + (a_2 - b_2) \cdot x_2$.

(31) $(a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3) - (b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3) = (a_1 - b_1) \cdot x_1 + (a_2 - b_2) \cdot x_2 + (a_3 - b_3) \cdot x_3$.

(32) If $a_1 + a_2 + a_3 = 1$, then $a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3 = x_1 + a_2 \cdot (x_2 - x_1) + a_3 \cdot (x_3 - x_1)$.

(33) If $x = x_1 + a_2 \cdot (x_2 - x_1) + a_3 \cdot (x_3 - x_1)$, then there exists a real number a_1 such that $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$ and $a_1 + a_2 + a_3 = 1$.

(34) For every natural number n such that $n \geq 1$ holds $1 * n \neq \langle\underbrace{0, \ldots, 0}_{n}\rangle$.

(35) For every subset A of $\mathcal{R}^n$ and for all x_1, x_2 such that A is a line and $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$ holds $A = \mathrm{Line}(x_1, x_2)$.

(36) For all elements x_1, x_2 of $\mathcal{R}^n$ such that $y_1 \in \mathrm{Line}(x_1, x_2)$ and $y_2 \in \mathrm{Line}(x_1, x_2)$ there exists a such that $y_2 - y_1 = a \cdot (x_2 - x_1)$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The predicate $x_1 \parallel x_2$ is defined as follows:

(Def. 1) $x_1 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $x_2 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and there exists r such that $x_1 = r\cdot x_2$.

One can prove the following proposition

(37) For all elements x_1, x_2 of $\mathcal{R}^n$ such that $x_1 \parallel x_2$ there exists a such that $a \neq 0$ and $x_1 = a \cdot x_2$.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. Let us note that the predicate $x_1 \parallel x_2$ is symmetric.

The following proposition is true

(38) If $x_1 \parallel x_2$ and $x_2 \parallel x_3$, then $x_1 \parallel x_3$.

Let n be a natural number and let x_1, x_2 be elements of $\mathcal{R}^n$. We say that x_1 and x_2 are linearly independent if and only if:

(Def. 2) For all real numbers a_1, a_2 such that $a_1 \cdot x_1 + a_2 \cdot x_2 = \langle\underbrace{0,\ldots,0}_{n}\rangle$ holds $a_1 = 0$ and $a_2 = 0$.

Let us note that the predicate x_1 and x_2 are linearly independent is symmetric.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. We introduce x_1 and x_2 are linearly dependent as an antonym of x_1 and x_2 are linearly independent.

Next we state a number of propositions:

(39) If x_1 and x_2 are linearly independent, then $x_1 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $x_2 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(40) For all x_1, x_2 such that x_1 and x_2 are linearly independent holds if $a_1 \cdot x_1 + a_2 \cdot x_2 = b_1 \cdot x_1 + b_2 \cdot x_2$, then $a_1 = b_1$ and $a_2 = b_2$.

(41) Let given x_1, x_2, y_1, y_1. Suppose y_1 and y_2 are linearly independent. Suppose $y_1 = a_1 \cdot x_1 + a_2 \cdot x_2$ and $y_2 = b_1 \cdot x_1 + b_2 \cdot x_2$. Then there exist real numbers c_1, c_2, d_1, d_2 such that $x_1 = c_1 \cdot y_1 + c_2 \cdot y_2$ and $x_2 = d_1 \cdot y_1 + d_2 \cdot y_2$.

(42) If x_1 and x_2 are linearly independent, then $x_1 \neq x_2$.

(43) If $x_2 - x_1$ and $x_3 - x_1$ are linearly independent, then $x_2 \neq x_3$.

(44) If x_1 and x_2 are linearly independent, then $x_1 + t \cdot x_2$ and x_2 are linearly independent.

(45) Suppose $x_1 - x_0$ and $x_3 - x_2$ are linearly independent and $y_0 \in \mathrm{Line}(x_0, x_1)$ and $y_1 \in \mathrm{Line}(x_0, x_1)$ and $y_0 \neq y_1$ and $y_2 \in \mathrm{Line}(x_2, x_3)$ and $y_3 \in \mathrm{Line}(x_2, x_3)$ and $y_2 \neq y_3$. Then $y_1 - y_0$ and $y_3 - y_2$ are linearly independent.

(46) If $x_1 \parallel x_2$, then x_1 and x_2 are linearly dependent and $x_1 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $x_2 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$.

(47) If x_1 and x_2 are linearly dependent, then $x_1 = \langle\underbrace{0,\ldots,0}_{n}\rangle$ or $x_2 = \langle\underbrace{0,\ldots,0}_{n}\rangle$ or $x_1 \parallel x_2$.

(48) For all elements x_1, x_2, y_1 of $\mathcal{R}^n$ there exists an element y_2 of $\mathcal{R}^n$ such that $y_2 \in \mathrm{Line}(x_1, x_2)$ and $x_1 - x_2$, $y_1 - y_2$ are orthogonal.

Let us consider n and let x_1, x_2 be elements of $\mathcal{R}^n$. The predicate $x_1 \perp x_2$ is defined by:

(Def. 3) $x_1 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and $x_2 \neq \langle\underbrace{0,\ldots,0}_{n}\rangle$ and x_1, x_2 are orthogonal.

Let us note that the predicate $x_1 \perp x_2$ is symmetric.

The following propositions are true:

(49) If $x \perp y_0$ and $y_0 \parallel y_1$, then $x \perp y_1$.

(50) If $x \perp y$, then x and y are linearly independent.

(51) If $x_1 \parallel x_2$, then $x_1 \not\perp x_2$.

(52) If $x_1 \perp x_2$, then $x_1 \nparallel x_2$.

Let us consider n. The functor $\mathrm{Lines}(\mathcal{R}^n)$ yields a family of subsets of $\mathcal{R}^n$ and is defined by:

(Def. 4) $\mathrm{Lines}(\mathcal{R}^n) = \{\mathrm{Line}(x_1, x_2)\}$.

Let us consider n. Note that $\mathrm{Lines}(\mathcal{R}^n)$ is non empty.

The following proposition is true

(53) $\mathrm{Line}(x_1, x_2) \in \mathrm{Lines}(\mathcal{R}^n)$.

In the sequel L, L_0, L_1, L_2 are elements of $\mathrm{Lines}(\mathcal{R}^n)$.

The following propositions are true:

(54) If $x_1 \in L$ and $x_2 \in L$, then $\mathrm{Line}(x_1, x_2) \subseteq L$.

(55) L_1 meets L_2 iff there exists x such that $x \in L_1$ and $x \in L_2$.

(56) If L_0 misses L_1 and $x \in L_0$, then $x \notin L_1$.

(57) There exist x_1, x_2 such that $L = \mathrm{Line}(x_1, x_2)$.

(58) There exists x such that $x \in L$.

(59) If $x_0 \in L$ and L is a line, then there exists x_1 such that $x_1 \neq x_0$ and $x_1 \in L$.

(60) If $x \notin L$ and L is a line, then there exist x_1, x_2 such that $L = \mathrm{Line}(x_1, x_2)$ and $x - x_1 \perp x_2 - x_1$.

(61) If $x \notin L$ and L is a line, then there exist x_1, x_2 such that $L = \mathrm{Line}(x_1, x_2)$ and $x - x_1$ and $x_2 - x_1$ are linearly independent.

Let n be a natural number, let x be an element of $\mathcal{R}^n$, and let L be an element of $\mathrm{Lines}(\mathcal{R}^n)$. The functor $\rho(x, L)$ yields a real number and is defined by:

(Def. 5) There exists a subset S of $\mathbb{R}$ such that $S = \{|x - x_0|; x_0$ ranges over elements of $\mathcal{R}^n$: $x_0 \in L\}$ and $\rho(x, L) = \inf S$.

Next we state three propositions:

(62) There exists x_0 such that $x_0 \in L$ and $|x - x_0| = \rho(x, L)$.

(63) $\rho(x, L) \geq 0$.

(64) $x \in L$ iff $\rho(x, L) = 0$.

Let us consider n and let us consider L_1, L_2. The predicate $L_1 \parallel L_2$ is defined as follows:

(Def. 6) There exist elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ such that $L_1 = \mathrm{Line}(x_1, x_2)$ and $L_2 = \mathrm{Line}(y_1, y_2)$ and $x_2 - x_1 \parallel y_2 - y_1$.

Let us note that the predicate $L_1 \parallel L_2$ is symmetric.

The following proposition is true

(65) If $L_0 \parallel L_1$ and $L_1 \parallel L_2$, then $L_0 \parallel L_2$.

Let us consider n and let us consider L_1, L_2. The predicate $L_1 \perp L_2$ is defined by:

(Def. 7) There exist elements x_1, x_2, y_1, y_2 of $\mathcal{R}^n$ such that $L_1 = \mathrm{Line}(x_1, x_2)$ and $L_2 = \mathrm{Line}(y_1, y_2)$ and $x_2 - x_1 \perp y_2 - y_1$.

Let us note that the predicate $L_1 \perp L_2$ is symmetric.

We now state a number of propositions:

(66) If $L_0 \perp L_1$ and $L_1 \parallel L_2$, then $L_0 \perp L_2$.

(67) If $x \notin L$ and L is a line, then there exists L_0 such that $x \in L_0$ and $L_0 \perp L$ and L_0 meets L.

(68) If L_1 misses L_2, then there exists x such that $x \in L_1$ and $x \notin L_2$.

(69) If $x_1 \in L$ and $x_2 \in L$ and $x_1 \neq x_2$, then $\mathrm{Line}(x_1, x_2) = L$ and L is a line.

(70) If L_1 is a line and L_2 is a line and $L_1 = L_2$, then $L_1 \parallel L_2$.

(71) If $L_1 \parallel L_2$, then L_1 is a line and L_2 is a line.

(72) If $L_1 \perp L_2$, then L_1 is a line and L_2 is a line.

(73) If $x \in L$ and $a \neq 1$ and $a \cdot x \in L$, then $\langle \underbrace{0, \ldots, 0}_{n} \rangle \in L$.

(74) If $x_1 \in L$ and $x_2 \in L$, then there exists x_3 such that $x_3 \in L$ and $x_3 - x_1 = a \cdot (x_2 - x_1)$.

(75) If $x_1 \in L$ and $x_2 \in L$ and $x_3 \in L$ and $x_1 \neq x_2$, then there exists a such that $x_3 - x_1 = a \cdot (x_2 - x_1)$.

(76) If $L_1 \parallel L_2$ and $L_1 \neq L_2$, then L_1 misses L_2.

(77) If $L_1 \parallel L_2$, then $L_1 = L_2$ or L_1 misses L_2.

(78) If $L_1 \parallel L_2$ and L_1 meets L_2, then $L_1 = L_2$.

(79) If L is a line, then there exists L_0 such that $x \in L_0$ and $L_0 \parallel L$.

(80) For all x, L such that $x \notin L$ and L is a line there exists L_0 such that $x \in L_0$ and $L_0 \parallel L$ and $L_0 \neq L$.

(81) For all x_0, x_1, y_0, y_1, L_1, L_2 such that $x_0 \in L_1$ and $x_1 \in L_1$ and $x_0 \neq x_1$ and $y_0 \in L_2$ and $y_1 \in L_2$ and $y_0 \neq y_1$ and $L_1 \perp L_2$ holds $x_1 - x_0 \perp y_1 - y_0$.

(82) For all L_1, L_2 such that $L_1 \perp L_2$ holds $L_1 \neq L_2$.

(83) For all x_1, x_2, L such that L is a line and $L = \mathrm{Line}(x_1, x_2)$ holds $x_1 \neq x_2$.

(84) If $x_0 \in L_1$ and $x_1 \in L_1$ and $x_0 \neq x_1$ and $y_0 \in L_2$ and $y_1 \in L_2$ and $y_0 \neq y_1$ and $L_1 \parallel L_2$, then $x_1 - x_0 \parallel y_1 - y_0$.

(85) Suppose $x_2 - x_1$ and $x_3 - x_1$ are linearly independent and $y_2 \in \mathrm{Line}(x_1, x_2)$ and $y_3 \in \mathrm{Line}(x_1, x_3)$ and $L_1 = \mathrm{Line}(x_2, x_3)$ and $L_2 = \mathrm{Line}(y_2, y_3)$. Then $L_1 \parallel L_2$ if and only if there exists a such that $a \neq 0$ and $y_2 - x_1 = a \cdot (x_2 - x_1)$ and $y_3 - x_1 = a \cdot (x_3 - x_1)$.

(86) For all L_1, L_2 such that L_1 is a line and L_2 is a line and $L_1 \neq L_2$ there exists x such that $x \in L_1$ and $x \notin L_2$.

(87) For all x, L_1, L_2 such that $L_1 \perp L_2$ and $x \in L_2$ there exists L_0 such that $x \in L_0$ and $L_0 \perp L_2$ and $L_0 \parallel L_1$.

(88) For all x, L_1, L_2 such that $x \in L_1$ and $x \in L_2$ and $L_1 \perp L_2$ there exists x_0 such that $x \neq x_0$ and $x_0 \in L_1$ and $x_0 \notin L_2$.

Let n be a natural number and let x_1, x_2, x_3 be elements of $\mathcal{R}^n$. The functor $\mathrm{Plane}(x_1, x_2, x_3)$ yielding a subset of $\mathcal{R}^n$ is defined as follows:

(Def. 8) $\mathrm{Plane}(x_1, x_2, x_3) = \{a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3 : a_1 + a_2 + a_3 = 1\}$.

Let n be a natural number and let x_1, x_2, x_3 be elements of $\mathcal{R}^n$. One can check that $\mathrm{Plane}(x_1, x_2, x_3)$ is non empty.

Let us consider n and let A be a subset of $\mathcal{R}^n$. We say that A is plane if and only if:

(Def. 9) There exist x_1, x_2, x_3 such that $x_2 - x_1$ and $x_3 - x_1$ are linearly independent and $A = \mathrm{Plane}(x_1, x_2, x_3)$.

One can prove the following propositions:

(89) $x_1 \in \mathrm{Plane}(x_1, x_2, x_3)$ and $x_2 \in \mathrm{Plane}(x_1, x_2, x_3)$ and $x_3 \in \mathrm{Plane}(x_1, x_2, x_3)$.

(90) If $x_1 \in \mathrm{Plane}(y_1, y_2, y_3)$ and $x_2 \in \mathrm{Plane}(y_1, y_2, y_3)$ and $x_3 \in \mathrm{Plane}(y_1, y_2, y_3)$, then $\mathrm{Plane}(x_1, x_2, x_3) \subseteq \mathrm{Plane}(y_1, y_2, y_3)$.

(91) Let A be a subset of $\mathcal{R}^n$ and given x, x_1, x_2, x_3. Suppose $x \in \mathrm{Plane}(x_1, x_2, x_3)$ and there exist real numbers c_1, c_2, c_3 such that $c_1 + c_2 + c_3 = 0$ and $x = c_1 \cdot x_1 + c_2 \cdot x_2 + c_3 \cdot x_3$. Then $\langle \underbrace{0, \ldots, 0}_{n} \rangle \in \mathrm{Plane}(x_1, x_2, x_3)$.

(92) If $y_1 \in \mathrm{Plane}(x_1, x_2, x_3)$ and $y_2 \in \mathrm{Plane}(x_1, x_2, x_3)$, then $\mathrm{Line}(y_1, y_2) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$.

(93) For every subset A of $\mathcal{R}^n$ and for every x such that A is plane and $x \in A$ and there exists a such that $a \neq 1$ and $a \cdot x \in A$ holds $\langle \underbrace{0, \ldots, 0}_{n} \rangle \in A$.

(94) If $x_1 - x_1$ and $x_3 - x_1$ are linearly independent and $x \in \mathrm{Plane}(x_1, x_2, x_3)$ and $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$, then $a_1 + a_2 + a_3 = 1$ or $\langle \underbrace{0, \ldots, 0}_{n} \rangle \in \mathrm{Plane}(x_1, x_2, x_3)$.

(95) $x \in \mathrm{Plane}(x_1, x_2, x_3)$ iff there exist a_1, a_2, a_3 such that $a_1 + a_2 + a_3 = 1$ and $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$.

(96) Suppose that

(i) $x_2 - x_1$ and $x_3 - x_1$ are linearly independent,

(ii) $x \in \mathrm{Plane}(x_1, x_2, x_3)$,

(iii) $a_1 + a_2 + a_3 = 1$,

(iv) $x = a_1 \cdot x_1 + a_2 \cdot x_2 + a_3 \cdot x_3$,

(v) $b_1 + b_2 + b_3 = 1$, and

(vi) $x = b_1 \cdot x_1 + b_2 \cdot x_2 + b_3 \cdot x_3$.

Then $a_1 = b_1$ and $a_2 = b_2$ and $a_3 = b_3$.

Let us consider n. The functor $\mathrm{Planes}(\mathcal{R}^n)$ yielding a family of subsets of $\mathcal{R}^n$ is defined by:

(Def. 10) $\mathrm{Planes}(\mathcal{R}^n) = \{\mathrm{Plane}(x_1, x_2, x_3)\}$.

Let us consider n. Note that $\mathrm{Planes}(\mathcal{R}^n)$ is non empty.

The following proposition is true

(97) $\mathrm{Plane}(x_1, x_2, x_3) \in \mathrm{Planes}(\mathcal{R}^n)$.

In the sequel P, P_0, P_1, P_2 are elements of $\mathrm{Planes}(\mathcal{R}^n)$.

Next we state several propositions:

(98) If $x_1 \in P$ and $x_2 \in P$ and $x_3 \in P$, then $\mathrm{Plane}(x_1, x_2, x_3) \subseteq P$.

(99) If $x_1 \in P$ and $x_2 \in P$ and $x_3 \in P$ and $x_2 - x_1$ and $x_3 - x_1$ are linearly independent, then $P = \mathrm{Plane}(x_1, x_2, x_3)$.

(100) If P_1 is plane and $P_1 \subseteq P_2$, then $P_1 = P_2$.

(101) $\mathrm{Line}(x_1, x_2) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$ and $\mathrm{Line}(x_2, x_3) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$ and $\mathrm{Line}(x_3, x_1) \subseteq \mathrm{Plane}(x_1, x_2, x_3)$.

(102) If $x_1 \in P$ and $x_2 \in P$, then $\mathrm{Line}(x_1, x_2) \subseteq P$.

Let n be a natural number and let L_1, L_2 be elements of $\mathrm{Lines}(\mathcal{R}^n)$. We say that L_1 and L_2 are coplanar if and only if:

(Def. 11) There exist elements x_1, x_2, x_3 of $\mathcal{R}^n$ such that $L_1 \subseteq \mathrm{Plane}(x_1, x_2, x_3)$ and $L_2 \subseteq \mathrm{Plane}(x_1, x_2, x_3)$.

We now state a number of propositions:

(103) L_1 and L_2 are coplanar iff there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$.

(104) If $L_1 \parallel L_2$, then L_1 and L_2 are coplanar.

(105) Suppose L_1 is a line and L_2 is a line and L_1 and L_2 are coplanar and L_1 misses L_2. Then there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(106) There exists P such that $x \in P$ and $L \subseteq P$.

(107) If $x \notin L$ and L is a line, then there exists P such that $x \in P$ and $L \subseteq P$ and P is plane.

(108) If $x \in P$ and $L \subseteq P$ and $x \notin L$ and L is a line, then P is plane.

(109) If $x \notin L$ and L is a line and $x \in P_0$ and $L \subseteq P_0$ and $x \in P_1$ and $L \subseteq P_1$, then $P_0 = P_1$.

(110) If L_1 is a line and L_2 is a line and L_1 and L_2 are coplanar and $L_1 \neq L_2$, then there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(111) For all L_1, L_2 such that L_1 is a line and L_2 is a line and $L_1 \neq L_2$ and L_1 meets L_2 there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(112) If L_1 is a line and L_2 is a line and $L_1 \neq L_2$ and L_1 meets L_2 and $L_1 \subseteq P_1$ and $L_2 \subseteq P_1$ and $L_1 \subseteq P_2$ and $L_2 \subseteq P_2$, then $P_1 = P_2$.

(113) If $L_1 \parallel L_2$ and $L_1 \neq L_2$, then there exists P such that $L_1 \subseteq P$ and $L_2 \subseteq P$ and P is plane.

(114) If $L_1 \perp L_2$ and L_1 meets L_2, then there exists P such that P is plane and $L_1 \subseteq P$ and $L_2 \subseteq P$.

(115) If $L_0 \subseteq P$ and $L_1 \subseteq P$ and $L_2 \subseteq P$ and $x \in L_0$ and $x \in L_1$ and $x \in L_2$ and $L_0 \perp L_2$ and $L_1 \perp L_2$, then $L_0 = L_1$.

(116) If L_1 and L_2 are coplanar and $L_1 \perp L_2$, then L_1 meets L_2.

(117) If $L_1 \subseteq P$ and $L_2 \subseteq P$ and $L_1 \perp L_2$ and $x \in P$ and $L_0 \parallel L_2$ and $x \in L_0$, then $L_0 \subseteq P$ and $L_0 \perp L_1$.

(118) If $L \subseteq P$ and $L_1 \subseteq P$ and $L_2 \subseteq P$ and $L \perp L_1$ and $L \perp L_2$, then $L_1 \parallel L_2$.

(119) Suppose $L_0 \subseteq P$ and $L_1 \subseteq P$ and $L_2 \subseteq P$ and $L_0 \parallel L_1$ and $L_1 \parallel L_2$ and $L_0 \neq L_1$ and $L_1 \neq L_2$ and $L_2 \neq L_0$ and L meets L_0 and L meets L_1. Then L meets L_2.

(120) If L_1 and L_2 are coplanar and L_1 is a line and L_2 is a line and L_1 misses L_2, then $L_1 \parallel L_2$.

(121) If $x_1 \in P$ and $x_2 \in P$ and $y_1 \in P$ and $y_2 \in P$ and $x_2 - x_1$ and $y_2 - y_1$ are linearly independent, then $\mathrm{Line}(x_1, x_2)$ meets $\mathrm{Line}(y_1, y_2)$.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[6] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[7] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[9] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.
[10] Akihiro Kubo. Lines in n-dimensional Euclidean spaces. *Formalized Mathematics*, 11(**4**):371–376, 2003.
[11] Yatsuka Nakamura, Andrzej Trybulec, and Czesław Byliński. Bounded domains and unbounded domains. *Formalized Mathematics*, 8(**1**):1–13, 1999.
[12] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[13] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[14] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received May 24, 2005