FORMALIZED MATHEMATICS
Volume 13, Number 1, 2005
University of Białystok

Lebesgue Integral of Simple Valued Function[1]

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Summary. In this article, the authors introduce Lebesgue integral of simple valued function.

MML Identifier: `MESFUNC3`.

The terminology and notation used in this paper are introduced in the following papers: [23], [12], [25], [21], [26], [10], [11], [3], [22], [24], [7], [14], [1], [2], [20], [4], [5], [6], [8], [9], [19], [13], [15], [16], [17], and [18].

1. Integral of Simple Valued Function

The following propositions are true:

(1) Let n, m be natural numbers, a be a function from $[\!:\operatorname{Seg} n, \operatorname{Seg} m:\!]$ into $\mathbb{R}$, and p, q be finite sequences of elements of $\mathbb{R}$. Suppose that

(i) $\operatorname{dom} p = \operatorname{Seg} n$,

(ii) for every natural number i such that $i \in \operatorname{dom} p$ there exists a finite sequence r of elements of $\mathbb{R}$ such that $\operatorname{dom} r = \operatorname{Seg} m$ and $p(i) = \sum r$ and for every natural number j such that $j \in \operatorname{dom} r$ holds $r(j) = a(\langle i, j\rangle)$,

(iii) $\operatorname{dom} q = \operatorname{Seg} m$, and

(iv) for every natural number j such that $j \in \operatorname{dom} q$ there exists a finite sequence s of elements of $\mathbb{R}$ such that $\operatorname{dom} s = \operatorname{Seg} n$ and $q(j) = \sum s$ and for every natural number i such that $i \in \operatorname{dom} s$ holds $s(i) = a(\langle i, j\rangle)$.

Then $\sum p = \sum q$.

[1]This work has been partially supported by the MEXT grant Grant-in-Aid for Young Scientists (B)16700156.

ISSN 1426–2630

(2) Let F be a finite sequence of elements of $\overline{\mathbb{R}}$ and f be a finite sequence of elements of $\mathbb{R}$. If $F = f$, then $\sum F = \sum f$.

(3) Let X be a non empty set, S be a σ-field of subsets of X, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S. Then there exists a finite sequence F of separated subsets of S and there exists a finite sequence a of elements of $\overline{\mathbb{R}}$ such that

(i) $\operatorname{dom} f = \bigcup \operatorname{rng} F$,

(ii) $\operatorname{dom} F = \operatorname{dom} a$,

(iii) for every natural number n such that $n \in \operatorname{dom} F$ and for every set x such that $x \in F(n)$ holds $f(x) = a(n)$, and

(iv) for every set x such that $x \in \operatorname{dom} f$ there exists a finite sequence a_1 of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} a_1 = \operatorname{dom} a$ and for every natural number n such that $n \in \operatorname{dom} a_1$ holds $a_1(n) = a(n) \cdot \chi_{F(n),X}(x)$.

(4) Let X be a set and F be a finite sequence of elements of X. Then F is disjoint valued if and only if for all natural numbers i, j such that $i \in \operatorname{dom} F$ and $j \in \operatorname{dom} F$ and $i \neq j$ holds $F(i)$ misses $F(j)$.

(5) Let X be a non empty set, A be a set, S be a σ-field of subsets of X, F be a finite sequence of separated subsets of S, and G be a finite sequence of elements of S. Suppose $\operatorname{dom} G = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} G$ holds $G(i) = A \cap F(i)$. Then G is a finite sequence of separated subsets of S.

(6) Let X be a non empty set, A be a set, and F, G be finite sequences of elements of X. Suppose $\operatorname{dom} G = \operatorname{dom} F$ and for every natural number i such that $i \in \operatorname{dom} G$ holds $G(i) = A \cap F(i)$. Then $\bigcup \operatorname{rng} G = A \cap \bigcup \operatorname{rng} F$.

(7) Let X be a set, F be a finite sequence of elements of X, and i be a natural number. If $i \in \operatorname{dom} F$, then $F(i) \subseteq \bigcup \operatorname{rng} F$ and $F(i) \cap \bigcup \operatorname{rng} F = F(i)$.

(8) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and F be a finite sequence of separated subsets of S. Then $\operatorname{dom} F = \operatorname{dom}(M \cdot F)$.

(9) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, and F be a finite sequence of separated subsets of S. Then $M(\bigcup \operatorname{rng} F) = \sum (M \cdot F)$.

(10) Let F, G be finite sequences of elements of $\overline{\mathbb{R}}$ and a be an extended real number. Suppose that

(i) $a \neq +\infty$ and $a \neq -\infty$ or for every natural number i such that $i \in \operatorname{dom} F$ holds $F(i) < 0_{\overline{\mathbb{R}}}$ or for every natural number i such that $i \in \operatorname{dom} F$ holds $0_{\overline{\mathbb{R}}} < F(i)$,

(ii) $\operatorname{dom} F = \operatorname{dom} G$, and

(iii) for every natural number i such that $i \in \operatorname{dom} G$ holds $G(i) = a \cdot F(i)$.

Then $\sum G = a \cdot \sum F$.

(11) Every finite sequence of elements of $\mathbb{R}$ is a finite sequence of elements of $\overline{\mathbb{R}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, let f be a partial function from X to $\overline{\mathbb{R}}$, let F be a finite sequence of separated subsets of S, and let a be a finite sequence of elements of $\overline{\mathbb{R}}$. We say that F and a are re-presentation of f if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) $\operatorname{dom} f = \bigcup \operatorname{rng} F$,

(ii) $\operatorname{dom} F = \operatorname{dom} a$, and

(iii) for every natural number n such that $n \in \operatorname{dom} F$ and for every set x such that $x \in F(n)$ holds $f(x) = a(n)$.

One can prove the following propositions:

(12) Let X be a non empty set, S be a σ-field of subsets of X, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S. Then there exists a finite sequence F of separated subsets of S and there exists a finite sequence a of elements of $\overline{\mathbb{R}}$ such that F and a are re-presentation of f.

(13) Let X be a non empty set, S be a σ-field of subsets of X, and F be a finite sequence of separated subsets of S. Then there exists a finite sequence G of separated subsets of S such that

(i) $\bigcup \operatorname{rng} F = \bigcup \operatorname{rng} G$, and

(ii) for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) \neq \emptyset$ and there exists a natural number m such that $m \in \operatorname{dom} F$ and $F(m) = G(n)$.

(14) Let X be a non empty set, S be a σ-field of subsets of X, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose f is simple function in S and for every set x such that $x \in \operatorname{dom} f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$. Then there exists a finite sequence F of separated subsets of S and there exists a finite sequence a of elements of $\overline{\mathbb{R}}$ such that

(i) F and a are re-presentation of f,

(ii) $a(1) = 0_{\overline{\mathbb{R}}}$, and

(iii) for every natural number n such that $2 \leq n$ and $n \in \operatorname{dom} a$ holds $0_{\overline{\mathbb{R}}} < a(n)$ and $a(n) < +\infty$.

(15) Let X be a non empty set, S be a σ-field of subsets of X, f be a partial function from X to $\overline{\mathbb{R}}$, F be a finite sequence of separated subsets of S, a be a finite sequence of elements of $\overline{\mathbb{R}}$, and x be an element of X. Suppose F and a are re-presentation of f and $x \in \operatorname{dom} f$. Then there exists a finite sequence a_1 of elements of $\overline{\mathbb{R}}$ such that $\operatorname{dom} a_1 = \operatorname{dom} a$ and for every natural number n such that $n \in \operatorname{dom} a_1$ holds $a_1(n) = a(n) \cdot \chi_{F(n),X}(x)$ and $f(x) = \sum a_1$.

(16) Let p be a finite sequence of elements of $\overline{\mathbb{R}}$ and q be a finite sequence of elements of $\mathbb{R}$. If $p = q$, then $\sum p = \sum q$.

(17) Let p be a finite sequence of elements of $\overline{\mathbb{R}}$. Suppose for every natural number n such that $n \in \operatorname{dom} p$ holds $0_{\overline{\mathbb{R}}} \leq p(n)$ and there exists a natural number n such that $n \in \operatorname{dom} p$ and $p(n) = +\infty$. Then $\sum p = +\infty$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let f be a partial function from X to $\overline{\mathbb{R}}$. Let us assume that f is simple function in S and $\operatorname{dom} f \neq \emptyset$ and for every set x such that $x \in \operatorname{dom} f$ holds $0_{\overline{\mathbb{R}}} \leq f(x)$. The functor integral$(X, S, M, f)$ yielding an element of $\overline{\mathbb{R}}$ is defined by the condition (Def. 2).

(Def. 2) There exists a finite sequence F of separated subsets of S and there exist finite sequences a, x of elements of $\overline{\mathbb{R}}$ such that
- (i) F and a are re-presentation of f,
- (ii) $a(1) = 0_{\overline{\mathbb{R}}}$,
- (iii) for every natural number n such that $2 \leq n$ and $n \in \operatorname{dom} a$ holds $0_{\overline{\mathbb{R}}} < a(n)$ and $a(n) < +\infty$,
- (iv) $\operatorname{dom} x = \operatorname{dom} F$,
- (v) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = a(n) \cdot (M \cdot F)(n)$, and
- (vi) integral$(X, S, M, f) = \sum x$.

2. Additional Lemma

We now state the proposition

(18) Let a be a finite sequence of elements of $\overline{\mathbb{R}}$ and p, N be elements of $\overline{\mathbb{R}}$. Suppose $N = \operatorname{len} a$ and for every natural number n such that $n \in \operatorname{dom} a$ holds $a(n) = p$. Then $\sum a = N \cdot p$.

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Received September 5, 2004