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The Fundamental Group

Artur Korniłowicz[1]
University of Białystok

Yasunari Shidama
Shinshu University
Nagano

Adam Grabowski[2]
University of Białystok

Summary. This is the next article in a series devoted to the homotopy theory (following [11] and [12]). The concept of fundamental groups of pointed topological spaces has been introduced. Isomorphism of fundamental groups defined with respect to different points belonging to the same component has been stated. Triviality of fundamental group(s) of $\mathbb{R}^n$ has been shown.

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The articles [22], [7], [26], [27], [19], [4], [6], [5], [28], [2], [21], [1], [18], [20], [16], [8], [3], [15], [13], [17], [29], [9], [14], [24], [23], [10], [11], [25], and [12] provide the terminology and notation for this paper.

1. Preliminaries

We adopt the following convention: p, q, x, y are real numbers and n is a natural number.

Next we state a number of propositions:

(1) Let G, H be groups and h be a homomorphism from G to H. If $h \cdot h^{-1} = \mathrm{id}_H$ and $h^{-1} \cdot h = \mathrm{id}_G$, then h is an isomorphism.

(2) For every subset X of $\mathbb{I}$ and for every point a of $\mathbb{I}$ such that $X =]a, 1]$ holds $X^{\mathrm{c}} = [0, a]$.

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(3) For every subset X of $\mathbb{I}$ and for every point a of $\mathbb{I}$ such that $X = [0, a[$ holds $X^{c} = [a, 1]$.

(4) For every subset X of $\mathbb{I}$ and for every point a of $\mathbb{I}$ such that $X =]a, 1]$ holds X is open.

(5) For every subset X of $\mathbb{I}$ and for every point a of $\mathbb{I}$ such that $X = [0, a[$ holds X is open.

(6) For every element f of $\mathbb{R}^n$ holds $x \cdot -f = -x \cdot f$.

(7) For all elements f, g of $\mathbb{R}^n$ holds $x \cdot (f - g) = x \cdot f - x \cdot g$.

(8) For every element f of $\mathbb{R}^n$ holds $(x - y) \cdot f = x \cdot f - y \cdot f$.

(9) For all elements f, g, h, k of $\mathbb{R}^n$ holds $(f+g)-(h+k) = (f-h)+(g-k)$.

(10) For every element f of $\mathcal{R}^n$ such that $0 \leqslant x$ and $x \leqslant 1$ holds $|x \cdot f| \leqslant |f|$.

(11) For every element f of $\mathcal{R}^n$ and for every point p of $\mathbb{I}$ holds $|p \cdot f| \leqslant |f|$.

(12) Let e_1, e_2, e_3, e_4, e_5, e_6 be points of $\mathcal{E}^n$ and p_1, p_2, p_3, p_4 be points of $\mathcal{E}^n_{\rm T}$. Suppose $e_1 = p_1$ and $e_2 = p_2$ and $e_3 = p_3$ and $e_4 = p_4$ and $e_5 = p_1+p_3$ and $e_6 = p_2 + p_4$ and $\rho(e_1, e_2) < x$ and $\rho(e_3, e_4) < y$. Then $\rho(e_5, e_6) < x + y$.

(13) Let e_1, e_2, e_5, e_6 be points of $\mathcal{E}^n$ and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If $e_1 = p_1$ and $e_2 = p_2$ and $e_5 = y \cdot p_1$ and $e_6 = y \cdot p_2$ and $\rho(e_1, e_2) < x$ and $y \neq 0$, then $\rho(e_5, e_6) < |y| \cdot x$.

(14) Let e_1, e_2, e_3, e_4, e_5, e_6 be points of $\mathcal{E}^n$ and p_1, p_2, p_3, p_4 be points of $\mathcal{E}^n_{\rm T}$. Suppose $e_1 = p_1$ and $e_2 = p_2$ and $e_3 = p_3$ and $e_4 = p_4$ and $e_5 = x \cdot p_1 + y \cdot p_3$ and $e_6 = x \cdot p_2 + y \cdot p_4$ and $\rho(e_1, e_2) < p$ and $\rho(e_3, e_4) < q$ and $x \neq 0$ and $y \neq 0$. Then $\rho(e_5, e_6) < |x| \cdot p + |y| \cdot q$.

(16)[3] Let X be a non empty topological space and f, g be maps from X into $\mathcal{E}^n_{\rm T}$. Suppose f is continuous and for every point p of X holds $g(p) = y \cdot f(p)$. Then g is continuous.

(17) Let X be a non empty topological space and f_1, f_2, g be maps from X into $\mathcal{E}^n_{\rm T}$. Suppose f_1 is continuous and f_2 is continuous and for every point p of X holds $g(p) = x \cdot f_1(p) + y \cdot f_2(p)$. Then g is continuous.

(18) Let F be a map from $[\!: \mathcal{E}^n_{\rm T}, \mathbb{I} :\!]$ into $\mathcal{E}^n_{\rm T}$. Suppose that for every point x of $\mathcal{E}^n_{\rm T}$ and for every point i of $\mathbb{I}$ holds $F(x, i) = (1 - i) \cdot x$. Then F is continuous.

(19) Let F be a map from $[\!: \mathcal{E}^n_{\rm T}, \mathbb{I} :\!]$ into $\mathcal{E}^n_{\rm T}$. Suppose that for every point x of $\mathcal{E}^n_{\rm T}$ and for every point i of $\mathbb{I}$ holds $F(x, i) = i \cdot x$. Then F is continuous.

2. Paths

For simplicity, we follow the rules: X denotes a non empty topological space, a, b, c, d, e, f denote points of X, T denotes a non empty arcwise connected

[3]The proposition (15) has been removed.

topological space, and a_1, b_1, c_1, d_1, e_1, f_1 denote points of T.

One can prove the following propositions:

(20) Suppose a, b are connected and b, c are connected. Let A be a path from a to b and B be a path from b to c. Then A, $A + B + -B$ are homotopic.

(21) For every path A from a_1 to b_1 and for every path B from b_1 to c_1 holds A, $A + B + -B$ are homotopic.

(22) Suppose a, b are connected and c, b are connected. Let A be a path from a to b and B be a path from c to b. Then A, $A + -B + B$ are homotopic.

(23) For every path A from a_1 to b_1 and for every path B from c_1 to b_1 holds A, $A + -B + B$ are homotopic.

(24) Suppose a, b are connected and c, a are connected. Let A be a path from a to b and B be a path from c to a. Then A, $-B + B + A$ are homotopic.

(25) For every path A from a_1 to b_1 and for every path B from c_1 to a_1 holds A, $-B + B + A$ are homotopic.

(26) Suppose a, b are connected and a, c are connected. Let A be a path from a to b and B be a path from a to c. Then A, $B + -B + A$ are homotopic.

(27) For every path A from a_1 to b_1 and for every path B from a_1 to c_1 holds A, $B + -B + A$ are homotopic.

(28) Suppose a, b are connected and c, b are connected. Let A, B be paths from a to b and C be a path from b to c. If $A + C$, $B + C$ are homotopic, then A, B are homotopic.

(29) Let A, B be paths from a_1 to b_1 and C be a path from b_1 to c_1. If $A+C$, $B + C$ are homotopic, then A, B are homotopic.

(30) Suppose a, b are connected and a, c are connected. Let A, B be paths from a to b and C be a path from c to a. If $C + A$, $C + B$ are homotopic, then A, B are homotopic.

(31) Let A, B be paths from a_1 to b_1 and C be a path from c_1 to a_1. If $C+A$, $C + B$ are homotopic, then A, B are homotopic.

(32) Suppose a, b are connected and b, c are connected and c, d are connected and d, e are connected. Let A be a path from a to b, B be a path from b to c, C be a path from c to d, and D be a path from d to e. Then $A + B + C + D$, $A + (B + C) + D$ are homotopic.

(33) Let A be a path from a_1 to b_1, B be a path from b_1 to c_1, C be a path from c_1 to d_1, and D be a path from d_1 to e_1. Then $A + B + C + D$, $A + (B + C) + D$ are homotopic.

(34) Suppose a, b are connected and b, c are connected and c, d are connected and d, e are connected. Let A be a path from a to b, B be a path from b to c, C be a path from c to d, and D be a path from d to e. Then $(A + B + C) + D$, $A + (B + C + D)$ are homotopic.

(35) Let A be a path from a_1 to b_1, B be a path from b_1 to c_1, C be a path from c_1 to d_1, and D be a path from d_1 to e_1. Then $(A + B + C) + D$, $A + (B + C + D)$ are homotopic.

(36) Suppose a, b are connected and b, c are connected and c, d are connected and d, e are connected. Let A be a path from a to b, B be a path from b to c, C be a path from c to d, and D be a path from d to e. Then $(A + (B + C)) + D$, $A + B + (C + D)$ are homotopic.

(37) Let A be a path from a_1 to b_1, B be a path from b_1 to c_1, C be a path from c_1 to d_1, and D be a path from d_1 to e_1. Then $(A + (B + C)) + D$, $A + B + (C + D)$ are homotopic.

(38) Suppose a, b are connected and b, c are connected and b, d are connected. Let A be a path from a to b, B be a path from d to b, and C be a path from b to c. Then $A + -B + B + C$, $A + C$ are homotopic.

(39) Let A be a path from a_1 to b_1, B be a path from d_1 to b_1, and C be a path from b_1 to c_1. Then $A + -B + B + C$, $A + C$ are homotopic.

(40) Suppose a, b are connected and a, c are connected and c, d are connected. Let A be a path from a to b, B be a path from c to d, and C be a path from a to c. Then $A + -A + C + B + -B$, C are homotopic.

(41) Let A be a path from a_1 to b_1, B be a path from c_1 to d_1, and C be a path from a_1 to c_1. Then $A + -A + C + B + -B$, C are homotopic.

(42) Suppose a, b are connected and a, c are connected and d, c are connected. Let A be a path from a to b, B be a path from c to d, and C be a path from a to c. Then $A + (-A + C + B) + -B$, C are homotopic.

(43) Let A be a path from a_1 to b_1, B be a path from c_1 to d_1, and C be a path from a_1 to c_1. Then $A + (-A + C + B) + -B$, C are homotopic.

(44) Suppose that
- (i) a, b are connected,
- (ii) b, c are connected,
- (iii) c, d are connected,
- (iv) d, e are connected, and
- (v) a, f are connected.

Let A be a path from a to b, B be a path from b to c, C be a path from c to d, D be a path from d to e, and E be a path from f to c. Then $(A + (B + C)) + D$, $A + B + -E + (E + C + D)$ are homotopic.

(45) Let A be a path from a_1 to b_1, B be a path from b_1 to c_1, C be a path from c_1 to d_1, D be a path from d_1 to e_1, and E be a path from f_1 to c_1. Then $(A + (B + C)) + D$, $A + B + -E + (E + C + D)$ are homotopic.

3. The Fundamental Group

Let T be a topological structure and let t be a point of T. A loop of t is a path from t to t.

Let T be a non empty topological structure and let t be a point of T. The functor Loops(t) is defined by:

(Def. 1) For every set x holds $x \in$ Loops(t) iff x is a loop of t.

Let T be a non empty topological structure and let t be a point of T. Observe that Loops(t) is non empty.

Let X be a non empty topological space and let a be a point of X. The functor EqRel(X, a) yielding a binary relation on Loops(a) is defined by:

(Def. 2) For all loops P, Q of a holds $\langle P, Q\rangle \in \mathrm{EqRel}(X, a)$ iff P, Q are homotopic.

Let X be a non empty topological space and let a be a point of X. One can check that EqRel(X, a) is non empty, total, symmetric, and transitive.

We now state two propositions:

(46) For all loops P, Q of a holds $Q \in [P]_{\mathrm{EqRel}(X,a)}$ iff P, Q are homotopic.

(47) For all loops P, Q of a holds $[P]_{\mathrm{EqRel}(X,a)} = [Q]_{\mathrm{EqRel}(X,a)}$ iff P, Q are homotopic.

Let X be a non empty topological space and let a be a point of X. The functor FundamentalGroup(X, a) yielding a strict groupoid is defined by the conditions (Def. 3).

(Def. 3)(i) The carrier of FundamentalGroup(X, a) = Classes EqRel(X, a), and

(ii) for all elements x, y of FundamentalGroup(X, a) there exist loops P, Q of a such that $x = [P]_{\mathrm{EqRel}(X,a)}$ and $y = [Q]_{\mathrm{EqRel}(X,a)}$ and (the multiplication of FundamentalGroup(X, a))(x, y) $= [P + Q]_{\mathrm{EqRel}(X,a)}$.

We introduce $\pi_1(X, a)$ as a synonym of FundamentalGroup(X, a).

Let X be a non empty topological space and let a be a point of X. One can verify that $\pi_1(X, a)$ is non empty.

Next we state the proposition

(48) For every set x holds $x \in$ the carrier of $\pi_1(X, a)$ iff there exists a loop P of a such that $x = [P]_{\mathrm{EqRel}(X,a)}$.

Let X be a non empty topological space and let a be a point of X. Note that $\pi_1(X, a)$ is associative and group-like.

Let T be a non empty topological space, let x_0, x_1 be points of T, and let P be a path from x_0 to x_1. Let us assume that x_0, x_1 are connected. The functor π_1-iso(P) yielding a map from $\pi_1(T, x_1)$ into $\pi_1(T, x_0)$ is defined by:

(Def. 4) For every loop Q of x_1 holds $(\pi_1\text{-iso}(P))([Q]_{\mathrm{EqRel}(T,x_1)}) = [P + Q + -P]_{\mathrm{EqRel}(T,x_0)}$.

For simplicity, we follow the rules: x_0, x_1 denote points of X, P, Q denote paths from x_0 to x_1, y_0, y_1 denote points of T, and R, V denote paths from y_0 to y_1.

Next we state three propositions:

(49) If x_0, x_1 are connected and P, Q are homotopic, then π_1-iso$(P) = \pi_1$-iso(Q).

(50) If R, V are homotopic, then π_1-iso$(R) = \pi_1$-iso(V).

(51) If x_0, x_1 are connected, then π_1-iso(P) is a homomorphism from $\pi_1(X, x_1)$ to $\pi_1(X, x_0)$.

Let T be a non empty arcwise connected topological space, let x_0, x_1 be points of T, and let P be a path from x_0 to x_1. Then π_1-iso(P) is a homomorphism from $\pi_1(T, x_1)$ to $\pi_1(T, x_0)$.

The following propositions are true:

(52) If x_0, x_1 are connected, then π_1-iso(P) is one-to-one.

(53) If x_0, x_1 are connected, then π_1-iso(P) is onto.

Let T be a non empty arcwise connected topological space, let x_0, x_1 be points of T, and let P be a path from x_0 to x_1. One can verify that π_1-iso(P) is one-to-one and onto.

One can prove the following propositions:

(54) If x_0, x_1 are connected, then $(\pi_1\text{-iso}(P))^{-1} = \pi_1$-iso$(-P)$.

(55) $(\pi_1\text{-iso}(R))^{-1} = \pi_1$-iso$(-R)$.

(56) If x_0, x_1 are connected, then for every homomorphism h from $\pi_1(X, x_1)$ to $\pi_1(X, x_0)$ such that $h = \pi_1$-iso(P) holds h is an isomorphism.

(57) π_1-iso(R) is an isomorphism.

(58) If x_0, x_1 are connected, then $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.

(59) $\pi_1(T, y_0)$ and $\pi_1(T, y_1)$ are isomorphic.

4. Euclidean Topological Space

Let n be a natural number, let a, b be points of $\mathcal{E}^n_{\mathrm{T}}$, and let P, Q be paths from a to b. The functor RealHomotopy(P, Q) yields a map from $[\!:\mathbb{I}, \mathbb{I}:\!]$ into $\mathcal{E}^n_{\mathrm{T}}$ and is defined by:

(Def. 5) For all elements s, t of $\mathbb{I}$ holds $(\text{RealHomotopy}(P, Q))(s, t) = (1 - t) \cdot P(s) + t \cdot Q(s)$.

The following proposition is true

(60) For all points a, b of $\mathcal{E}^n_{\mathrm{T}}$ and for all paths P, Q from a to b holds P, Q are homotopic.

Let n be a natural number, let a, b be points of $\mathcal{E}^n_{\mathrm{T}}$, and let P, Q be paths from a to b. Then RealHomotopy(P, Q) is a homotopy between P and Q.

Let n be a natural number, let a, b be points of $\mathcal{E}^n_{\mathrm{T}}$, and let P, Q be paths from a to b. One can check that every homotopy between P and Q is continuous.

Next we state the proposition

(61) For every point a of $\mathcal{E}^n_{\mathrm{T}}$ and for every loop C of a holds the carrier of $\pi_1(\mathcal{E}^n_{\mathrm{T}}, a) = \{[C]_{\mathrm{EqRel}(\mathcal{E}^n_{\mathrm{T}},a)}\}$.

Let n be a natural number and let a be a point of $\mathcal{E}^n_{\mathrm{T}}$. Note that $\pi_1(\mathcal{E}^n_{\mathrm{T}}, a)$ is trivial.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[3] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[9] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[10] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[11] Adam Grabowski. Introduction to the homotopy theory. *Formalized Mathematics*, 6(**4**):449–454, 1997.
[12] Adam Grabowski and Artur Korniłowicz. Algebraic properties of homotopies. *Formalized Mathematics*, 12(**3**):251–260, 2004.
[13] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[14] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[15] Michał Muzalewski. Categories of groups. *Formalized Mathematics*, 2(**4**):563–571, 1991.
[16] Yatsuka Nakamura. Half open intervals in real numbers. *Formalized Mathematics*, 10(**1**):21–22, 2002.
[17] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[18] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[19] Konrad Raczkowski and Paweł Sadowski. Equivalence relations and classes of abstraction. *Formalized Mathematics*, 1(**3**):441–444, 1990.
[20] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[21] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[22] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[23] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[24] Wojciech A. Trybulec. Groups. *Formalized Mathematics*, 1(**5**):821–827, 1990.
[25] Wojciech A. Trybulec and Michał J. Trybulec. Homomorphisms and isomorphisms of groups. Quotient group. *Formalized Mathematics*, 2(**4**):573–578, 1991.
[26] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[27] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[28] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.
[29] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Formalized Mathematics*, 6(**1**):123–130, 1997.

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