FORMALIZED MATHEMATICS
Volume 12, Number **2**, 2004
University of Białystok

Solving Roots of Polynomial Equation of Degree 2 and 3 with Complex Coefficients

Yuzhong Ding
QingDao University of Science and Technology

Xiquan Liang
QingDao University of Science and Technology

Summary. In the article, solving complex roots of polynomial equation of degree 2 and 3 with real coefficients and complex roots of polynomial equation of degree 2 and 3 with complex coefficients is discussed.

MML Identifier: POLYEQ_3.

The terminology and notation used here are introduced in the following articles: [20], [15], [2], [5], [3], [8], [17], [16], [14], [10], [12], [7], [18], [1], [13], [21], [9], [19], [11], [6], and [4].

1. Solving Complex Roots of Polynomial Equation of Degree 2 and 3 with Real Coefficients

We follow the rules: a, b, c, d, a', b', c', d', x, y, x_1, u, v are real numbers and s, t, h, z, z_1, z_2, z_3, z_4, s_1, s_2, s_3, p, q are elements of $\mathbb{C}$.

Let a be a real number and let us consider z. Then $a \cdot z$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 1) $a \cdot z = (a + 0i) \cdot z$.

Then $a + z$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 2) $a + z = z + (a + 0i)$.

Let us consider z. Then $z^{\mathbf{2}}$ is an element of $\mathbb{C}$ and it can be characterized by the condition:

(Def. 3) $z^{\mathbf{2}} = (\Re(z)^{\mathbf{2}} - \Im(z)^{\mathbf{2}}) + (2 \cdot (\Re(z) \cdot \Im(z)))i$.

Let us consider a, b, c, z. Then $\mathrm{Poly2}(a, b, c, z)$ is an element of $\mathbb{C}$.

The following propositions are true:

ISSN 1426–2630

(1) $(a+ci)\cdot(b+di)=(a\cdot b-c\cdot d)+(a\cdot d+b\cdot c)i$.

(2) If $z=x+yi$, then $z^{\mathbf{2}}=(x^{\mathbf{2}}-y^{\mathbf{2}})+(2\cdot x\cdot y)i$.

(3) For all a, b holds $(a+0i)\cdot(b+0i)=a\cdot b+0i$.

(4) If $a\neq 0$ and $\Delta(a,b,c)\geqslant 0$ and $\mathrm{Poly2}(a,b,c,z)=0$, then $z=\frac{-b+\sqrt{\Delta(a,b,c)}}{2\cdot a}$ or $z=\frac{-b-\sqrt{\Delta(a,b,c)}}{2\cdot a}$ or $z=-\frac{b}{2\cdot a}$.

(5) If $a\neq 0$ and $\Delta(a,b,c)<0$ and $\mathrm{Poly2}(a,b,c,z)=0_{\mathbb{C}}$, then $z=-\frac{b}{2\cdot a}+\frac{\sqrt{-\Delta(a,b,c)}}{2\cdot a}i$ or $z=-\frac{b}{2\cdot a}+(-\frac{\sqrt{-\Delta(a,b,c)}}{2\cdot a})i$.

(6) If $b\neq 0$ and for every z holds $\mathrm{Poly2}(0,b,c,z)=0_{\mathbb{C}}$, then $z=-\frac{c}{b}$.

(7) Let a, b, c be real numbers and z, z_1, z_2 be elements of $\mathbb{C}$. Suppose $a\neq 0$. Suppose that for every element z of $\mathbb{C}$ holds $\mathrm{Poly2}(a,b,c,z)=\mathrm{Quard}(a,z_1,z_2,z)$. Then $\frac{b}{a}+0i=-(z_1+z_2)$ and $\frac{c}{a}+0i=z_1\cdot z_2$.

Let z be an element of $\mathbb{C}$. The functor $z^{\mathbf{3}}$ yielding an element of $\mathbb{C}$ is defined by:

(Def. 4) $z^{\mathbf{3}}=z^{\mathbf{2}}\cdot z$.

Let a, b, c, d be real numbers and let z be an element of $\mathbb{C}$. The functor $\mathrm{Poly}_3(a,b,c,d,z)$ yielding an element of $\mathbb{C}$ is defined as follows:

(Def. 5) $\mathrm{Poly}_3(a,b,c,d,z)=a\cdot z^{\mathbf{3}}+b\cdot z^{\mathbf{2}}+c\cdot z+d$.

We now state a number of propositions:

(8) $(0_{\mathbb{C}})^{\mathbf{3}}=0_{\mathbb{C}}$.

(9) $(1_{\mathbb{C}})^{\mathbf{3}}=1_{\mathbb{C}}$.

(10) $(-1_{\mathbb{C}})^{\mathbf{3}}=-1_{\mathbb{C}}$.

(11) $\Re(z^{\mathbf{3}})=\Re(z)^3-3\cdot\Re(z)\cdot\Im(z)^{\mathbf{2}}$ and $\Im(z^{\mathbf{3}})=-\Im(z)^3+3\cdot\Re(z)^{\mathbf{2}}\cdot\Im(z)$.

(12) If for every z holds $\mathrm{Poly}_3(a,b,c,d,z)=\mathrm{Poly}_3(a',b',c',d',z)$, then $a=a'$ and $b=b'$ and $c=c'$ and $d=d'$.

(13) $(z+h)^{\mathbf{3}}=z^{\mathbf{3}}+3\cdot h\cdot z^{\mathbf{2}}+3\cdot h^{\mathbf{2}}\cdot z+h^{\mathbf{3}}$.

(14) $(z\cdot h)^{\mathbf{3}}=z^{\mathbf{3}}\cdot h^{\mathbf{3}}$.

(15) If $b\neq 0$ and $\mathrm{Poly}_3(0,b,c,d,z)=0_{\mathbb{C}}$ and $\Delta(b,c,d)\geqslant 0$, then $z=\frac{-c+\sqrt{\Delta(b,c,d)}}{2\cdot b}$ or $z=\frac{-c-\sqrt{\Delta(b,c,d)}}{2\cdot b}$ or $z=-\frac{c}{2\cdot b}$.

(16) If $b\neq 0$ and $\mathrm{Poly}_3(0,b,c,d,z)=0_{\mathbb{C}}$ and $\Delta(b,c,d)<0$, then $z=-\frac{c}{2\cdot b}+\frac{\sqrt{-\Delta(b,c,d)}}{2\cdot b}i$ or $z=-\frac{c}{2\cdot b}+(-\frac{\sqrt{-\Delta(b,c,d)}}{2\cdot b})i$.

(17) If $a\neq 0$ and $\mathrm{Poly}_3(a,0,c,0,z)=0$ and $4\cdot a\cdot c\leqslant 0$, then $z=\frac{\sqrt{-4\cdot a\cdot c}}{2\cdot a}$ or $z=\frac{-\sqrt{-4\cdot a\cdot c}}{2\cdot a}$ or $z=0$.

(18) If $a\neq 0$ and $\mathrm{Poly}_3(a,b,c,0,z)=0$ and $\Delta(a,b,c)\geqslant 0$, then $z=\frac{-b+\sqrt{\Delta(a,b,c)}}{2\cdot a}$ or $z=\frac{-b-\sqrt{\Delta(a,b,c)}}{2\cdot a}$ or $z=-\frac{b}{2\cdot a}$ or $z=0$.

(19) If $a\neq 0$ and $\mathrm{Poly}_3(a,b,c,0,z)=0_{\mathbb{C}}$ and $\Delta(a,b,c)<0$, then $z=-\frac{b}{2\cdot a}+\frac{\sqrt{-\Delta(a,b,c)}}{2\cdot a}i$ or $z=-\frac{b}{2\cdot a}+(-\frac{\sqrt{-\Delta(a,b,c)}}{2\cdot a})i$ or $z=0_{\mathbb{C}}$.

(20) If $a \geqslant 0$ and $y^{\mathbf{2}} = a$, then $y = \sqrt{a}$ or $y = -\sqrt{a}$.

(21) Suppose $a \neq 0$ and $\mathrm{Poly}_3(a, 0, c, d, z) = 0_{\mathbb{C}}$ and $\Im(z) = 0$. Let given u, v. Suppose $\Re(z) = u + v$ and $3 \cdot v \cdot u + \frac{c}{a} = 0$. Then

(i) $z = \sqrt[3]{-\frac{d}{2 \cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}} + \sqrt[3]{-\frac{d}{2 \cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}}$, or

(ii) $z = \sqrt[3]{-\frac{d}{2 \cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}} + \sqrt[3]{-\frac{d}{2 \cdot a} + \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}}$, or

(iii) $z = \sqrt[3]{-\frac{d}{2 \cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}} + \sqrt[3]{-\frac{d}{2 \cdot a} - \sqrt{\frac{d^{\mathbf{2}}}{4 \cdot a^{\mathbf{2}}} + (\frac{c}{3 \cdot a})^3}}$.

(22) Suppose $a \neq 0$ and $\mathrm{Poly}_3(a, 0, c, d, z) = 0_{\mathbb{C}}$ and $\Im(z) \neq 0$. Let given u, v. Suppose $\Re(z) = u + v$ and $3 \cdot v \cdot u + \frac{c}{4 \cdot a} = 0$ and $\frac{c}{a} \geqslant 0$. Then

(i) $z = (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})} +$
$\sqrt{3 \cdot (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}} + \frac{c}{a}}\, i$, or

(ii) $z = (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})} +$
$(-\sqrt{3 \cdot (\sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} + \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}} + \frac{c}{a}})\, i$,
or

(iii) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} +$
$\sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}} + \frac{c}{a}}\, i$, or

(iv) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} +$
$(-\sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} + \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}} + \frac{c}{a}})\, i$, or

(v) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} +$
$\sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}} + \frac{c}{a}}\, i$, or

(vi) $z = 2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3}} +$
$(-\sqrt{3 \cdot (2 \cdot \sqrt[3]{\frac{d}{16 \cdot a} - \sqrt{(\frac{d}{16 \cdot a})^{\mathbf{2}} + (\frac{c}{12 \cdot a})^3})^{\mathbf{2}} + \frac{c}{a}})\, i$.

(23) Suppose $a \neq 0$ and $\mathrm{Poly}_3(a, b, c, d, z) = 0_{\mathbb{C}}$ and $\Im(z) = 0$. Let given u, v, x_1. Suppose $x_1 = \Re(z) + \frac{b}{3 \cdot a}$ and $\Re(z) = (u + v) - \frac{b}{3 \cdot a}$ and $3 \cdot u \cdot v + \frac{3 \cdot a \cdot c - b^{\mathbf{2}}}{3 \cdot a^{\mathbf{2}}} = 0$. Then

(i) $z = ((\sqrt[3]{(-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}}) + \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3}} + \sqrt[3]{-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}} - \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3)}} - \frac{b}{3\cdot a}) + 0i$, or

(ii) $z = ((\sqrt[3]{(-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}}) + \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3}} + \sqrt[3]{(-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}}) + \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3)}} - \frac{b}{3\cdot a}) + 0i$, or

(iii) $z = ((\sqrt[3]{-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}} - \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3}} + \sqrt[3]{-(\frac{b}{3\cdot a})^3 - \frac{3\cdot a\cdot d-b\cdot c}{6\cdot a^{\mathbf{2}}} - \sqrt{\frac{(2\cdot(\frac{b}{3\cdot a})^3+\frac{3\cdot a\cdot d-b\cdot c}{3\cdot a^{\mathbf{2}}})^{\mathbf{2}}}{4} + (\frac{3\cdot a\cdot c-b^{\mathbf{2}}}{9\cdot a^{\mathbf{2}}})^3)}} - \frac{b}{3\cdot a}) + 0i$.

(24) If $z_1 \neq 0$ and $\mathrm{Poly1}(z_1, z_2, z) = 0$, then $z = -\frac{z_2}{z_1}$.

(25) If $z_2 \neq 0$, then it is not true that there exists z such that $\mathrm{Poly1}(0, z_2, z) = 0$.

2. Complex Roots of Polynomial Equation of Degree 2 and 3 with Complex Coefficients

Let us consider z_1, z_2, z_3, z. The functor $\mathrm{CPoly2}(z_1, z_2, z_3, z)$ yields an element of $\mathbb{C}$ and is defined by:

(Def. 6) $\mathrm{CPoly2}(z_1, z_2, z_3, z) = z_1 \cdot z^{\mathbf{2}} + z_2 \cdot z + z_3$.

We now state a number of propositions:

(26) If for every z holds $\mathrm{CPoly2}(z_1, z_2, z_3, z) = \mathrm{CPoly2}(s_1, s_2, s_3, z)$, then $z_1 = s_1$ and $z_2 = s_2$ and $z_3 = s_3$.

(27) $\frac{-a+\sqrt{a^{\mathbf{2}}+b^{\mathbf{2}}}}{2} \geqslant 0$ and $\frac{a+\sqrt{a^{\mathbf{2}}+b^{\mathbf{2}}}}{2} \geqslant 0$.

(28) If $z^{\mathbf{2}} = s$ and $\Im(s) \geqslant 0$, then $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$ or $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$.

(29) If $z^{\mathbf{2}} = s$ and $\Im(s) = 0$ and $\Re(s) > 0$, then $z = \sqrt{\Re(s)}$ or $z = -\sqrt{\Re(s)}$.

(30) If $z^{\mathbf{2}} = s$ and $\Im(s) = 0$ and $\Re(s) < 0$, then $z = 0 + \sqrt{-\Re(s)}i$ or $z = 0 + (-\sqrt{-\Re(s)})i$.

(31) If $z^{\mathbf{2}} = s$ and $\Im(s) < 0$, then $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$ or $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$.

(32) Suppose $z^{\mathbf{2}} = s$. Then

(i) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$, or

(ii) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iii) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iv) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$.

(33) $\mathrm{CPoly2}(0_{\mathbb{C}}, 0_{\mathbb{C}}, 0_{\mathbb{C}}, z) = 0$.

(34) If $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, 0_{\mathbb{C}}, 0_{\mathbb{C}}, z) = 0$, then $z = 0$.

(35) If $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, 0_{\mathbb{C}}, z) = 0$, then $z = -\frac{z_2}{z_1}$ or $z = 0$.

(36) Suppose $z_1 \neq 0_{\mathbb{C}}$ and $\mathrm{CPoly2}(z_1, 0_{\mathbb{C}}, z_3, z) = 0_{\mathbb{C}}$. Let given s. Suppose $s = -\frac{z_3}{z_1}$. Then

(i) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$, or

(ii) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iii) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iv) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$.

(37) Suppose $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, z_3, z) = 0_{\mathbb{C}}$. Let given h, t. Suppose $h = (\frac{z_2}{2 \cdot z_1})^{\mathbf{2}} - \frac{z_3}{z_1}$ and $t = \frac{z_2}{2 \cdot z_1}$. Then

(i) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}}i) - t$, or

(ii) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}})i) - t$, or

(iii) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}})i) - t$, or

(iv) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}}i) - t$.

Let us consider z_1, z_2, z_3, z_4, z. The functor $\mathrm{CPoly2}(z_1, z_2, z_3, z_4, z)$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 7) $\mathrm{CPoly2}(z_1, z_2, z_3, z_4, z) = z_1 \cdot z^{\mathbf{3}} + z_2 \cdot z^{\mathbf{2}} + z_3 \cdot z + z_4$.

One can prove the following propositions:

(38) If $z^{\mathbf{2}} = 1$, then $z = 1$ or $z = -1$.

(39) $z_{\mathbb{N}}^{3} = z \cdot z \cdot z$ and $z_{\mathbb{N}}^{3} = z^{\mathbf{2}} \cdot z$ and $z_{\mathbb{N}}^{3} = z^{\mathbf{3}}$.

(40) If $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, 0_{\mathbb{C}}, 0_{\mathbb{C}}, z) = 0_{\mathbb{C}}$, then $z = -\frac{z_2}{z_1}$ or $z = 0$.

(41) Suppose $z_1 \neq 0_{\mathbb{C}}$ and $\mathrm{CPoly2}(z_1, 0_{\mathbb{C}}, z_3, 0_{\mathbb{C}}, z) = 0_{\mathbb{C}}$. Let given s. Suppose $s = -\frac{z_3}{z_1}$. Then

(i) $z = 0_{\mathbb{C}}$, or

(ii) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$, or

(iii) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(iv) $z = \sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}})i$, or

(v) $z = -\sqrt{\frac{\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(s)+\sqrt{\Re(s)^{\mathbf{2}}+\Im(s)^{\mathbf{2}}}}{2}}i$.

(42) Suppose $z_1 \neq 0$ and $\mathrm{CPoly2}(z_1, z_2, z_3, 0_{\mathbb{C}}, z) = 0_{\mathbb{C}}$. Let given s, h, t. Suppose $s = -\frac{z_3}{z_1}$ and $h = (\frac{z_2}{2 \cdot z_1})^{\mathbf{2}} - \frac{z_3}{z_1}$ and $t = \frac{z_2}{2 \cdot z_1}$. Then

(i) $z = 0$, or

(ii) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}}i) - t$, or

(iii) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}})i) - t$, or

(iv) $z = (\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + (-\sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}})i) - t$, or

(v) $z = (-\sqrt{\frac{\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}} + \sqrt{\frac{-\Re(h)+\sqrt{\Re(h)^{\mathbf{2}}+\Im(h)^{\mathbf{2}}}}{2}}i) - t$.

(43) If $z = s - (\frac{1}{3} + 0i) \cdot z_1$, then $z^{\mathbf{2}} = s^{\mathbf{2}} + (-(\frac{2}{3} + 0i)) \cdot z_1 \cdot s + (\frac{1}{9} + 0i) \cdot {z_1}^{\mathbf{2}}$.

(44) If $z = s - (\frac{1}{3}+0i) \cdot z_1$, then $z^{\mathbf{3}} = ((s^{\mathbf{3}} - z_1 \cdot s^{\mathbf{2}}) + (\frac{1}{3}+0i) \cdot {z_1}^{\mathbf{2}} \cdot s) - (\frac{1}{27}+0i) \cdot {z_1}^{\mathbf{3}}$.

(45) Suppose $\mathrm{CPoly2}(1_{\mathbb{C}}, z_1, z_2, z_3, z) = 0_{\mathbb{C}}$. Let given p, q, s. Suppose $z = s - (\frac{1}{3} + 0i) \cdot z_1$ and $p = -(\frac{1}{3} + 0i) \cdot {z_1}^{\mathbf{2}} + z_2$ and $q = ((\frac{2}{27} + 0i) \cdot {z_1}^{\mathbf{3}} - (\frac{1}{3} + 0i) \cdot z_1 \cdot z_2) + z_3$. Then $\mathrm{CPoly2}(1_{\mathbb{C}}, 0_{\mathbb{C}}, p, q, s) = 0_{\mathbb{C}}$.

(46) For every element z of $\mathbb{C}$ holds $|z| \cdot \cos \operatorname{Arg} z + (|z| \cdot \sin \operatorname{Arg} z)i = (|z| + 0i) \cdot (\cos \operatorname{Arg} z + \sin \operatorname{Arg} z i)$.

(47) For every element z of $\mathbb{C}$ and for every natural number n holds $z_{\mathbb{N}}^{n+1} = (z_{\mathbb{N}}^{n}) \cdot z$.

(48) For every element z of $\mathbb{C}$ holds $z_{\mathbb{N}}^{1} = z$.

(49) For every element z of $\mathbb{C}$ holds $z_{\mathbb{N}}^{2} = z \cdot z$.

(50) For every natural number n such that $n > 0$ holds $0_{\mathbb{N}}^{n} = 0$.

(51) For all elements x, y of $\mathbb{C}$ and for every natural number n holds $(x \cdot y)_{\mathbb{N}}^{n} = (x_{\mathbb{N}}^{n}) \cdot y_{\mathbb{N}}^{n}$.

(52) For every real number x such that $x > 0$ and for every natural number n holds $(x + 0i)_{\mathbb{N}}^{n} = x^{n} + 0i$.

(53) For every real number x and for every natural number n holds $(\cos x + \sin x i)_{\mathbb{N}}^{n} = \cos(n \cdot x) + \sin(n \cdot x)i$.

(54) For every element z of $\mathbb{C}$ and for every natural number n such that $z \neq 0_{\mathbb{C}}$ or $n > 0$ holds $z_{\mathbb{N}}^{n} = |z|^{n} \cdot \cos(n \cdot \operatorname{Arg} z) + (|z|^{n} \cdot \sin(n \cdot \operatorname{Arg} z))i$.

(55) For all natural numbers n, k and for every real number x such that $n \neq 0$ holds $(\cos(\frac{x+2 \cdot \pi \cdot k}{n}) + \sin(\frac{x+2 \cdot \pi \cdot k}{n})i)_{\mathbb{N}}^{n} = \cos x + \sin x i$.

(56) Let z be an element of $\mathbb{C}$ and n, k be natural numbers. If $n \neq 0$, then $z = (\sqrt[n]{|z|} \cdot \cos(\frac{\operatorname{Arg} z+2 \cdot \pi \cdot k}{n}) + (\sqrt[n]{|z|} \cdot \sin(\frac{\operatorname{Arg} z+2 \cdot \pi \cdot k}{n}))i)_{\mathbb{N}}^{n}$.

Let z be an element of $\mathbb{C}$ and let n be a non empty natural number. An element of $\mathbb{C}$ is called a complex root of n, z if:

(Def. 8) $\mathrm{It}_{\mathbb{N}}^{n} = z$.

Next we state several propositions:

(57) Let z be an element of $\mathbb{C}$, n be a non empty natural number, and k be a natural number. Then $\sqrt[n]{|z|} \cdot \cos(\frac{\operatorname{Arg} z+2\cdot\pi\cdot k}{n}) + (\sqrt[n]{|z|} \cdot \sin(\frac{\operatorname{Arg} z+2\cdot\pi\cdot k}{n}))i$ is a complex root of n, z.

(58) For every element z of $\mathbb{C}$ and for every complex root v of 1, z holds $v = z$.

(59) For every non empty natural number n and for every complex root v of n, $0_{\mathbb{C}}$ holds $v = 0_{\mathbb{C}}$.

(60) Let n be a non empty natural number, z be an element of $\mathbb{C}$, and v be a complex root of n, z. If $v = 0_{\mathbb{C}}$, then $z = 0_{\mathbb{C}}$.

(61) Let n be a non empty natural number and k be a natural number. Then $\cos(\frac{2\cdot\pi\cdot k}{n}) + \sin(\frac{2\cdot\pi\cdot k}{n})i$ is a complex root of n, $1_{\mathbb{C}}$.

(62) For every natural number k holds $\cos(\frac{2\cdot\pi\cdot k}{3}) + \sin(\frac{2\cdot\pi\cdot k}{3})i$ is a complex root of 3, $1_{\mathbb{C}}$.

(63) For all elements z, s of $\mathbb{C}$ and for every natural number n such that $s \neq 0$ and $z \neq 0$ and $n \geqslant 1$ and $s_{\mathbb{N}}^{n} = z_{\mathbb{N}}^{n}$ holds $|s| = |z|$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[4] Wenpai Chang, Yatsuka Nakamura, and Piotr Rudnicki. Inner products and angles of complex numbers. *Formalized Mathematics*, 11(**3**):275–280, 2003.

[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[6] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[7] Rafał Kwiatek. Factorial and Newton coefficients. *Formalized Mathematics*, 1(**5**):887–890, 1990.

[8] Xiquan Liang. Solving roots of polynomial equations of degree 2 and 3 with real coefficients. *Formalized Mathematics*, 9(**2**):347–350, 2001.

[9] Anna Justyna Milewska. The field of complex numbers. *Formalized Mathematics*, 9(**2**):265–269, 2001.

[10] Jan Popiołek. Quadratic inequalities. *Formalized Mathematics*, 2(**4**):507–509, 1991.

[11] Konrad Raczkowski. Integer and rational exponents. *Formalized Mathematics*, 2(**1**):125–130, 1991.

[12] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.

[13] Yasunari Shidama and Artur Korniłowicz. Convergence and the limit of complex sequences. Series. *Formalized Mathematics*, 6(**3**):403–410, 1997.

[14] Andrzej Trybulec. Introduction to arithmetics. *To appear in Formalized Mathematics.*

[15] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*

[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[17] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[18] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

[19] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[21] Yuguang Yang and Yasunari Shidama. Trigonometric functions and existence of circle ratio. *Formalized Mathematics*, 7(**2**):255–263, 1998.

Received January 26, 2004