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The Series on Banach Algebra

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Summary. In this article, the basic properties of the series on Banach algebra are described. The Neumann series is introduced in the last section.

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The notation and terminology used in this paper are introduced in the following articles: [19], [21], [22], [4], [5], [3], [2], [18], [6], [1], [20], [10], [11], [12], [17], [9], [7], [8], [14], [13], [15], and [16].

1. Basic Properties of Sequences of Norm Space

Let X be a non empty normed structure and let s_1 be a sequence of X. The functor $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ yielding a sequence of X is defined as follows:

(Def. 1) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(0) = s_1(0)$ and for every natural number n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) + s_1(n+1)$.

One can prove the following proposition

(1) Let X be an add-associative right zeroed right complementable non empty normed structure and s_1 be a sequence of X. Suppose that for every natural number n holds $s_1(n) = 0_X$. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) = 0_X$.

Let X be a real normed space and let s_1 be a sequence of X. We say that s_1 is summable if and only if:

(Def. 2) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent.

Let X be a real normed space. One can verify that there exists a sequence of X which is summable.

Let X be a real normed space and let s_1 be a sequence of X. The functor $\sum s_1$ yields an element of X and is defined by:

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(Def. 3) $\sum s_1 = \lim((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})$.

Let X be a real normed space and let s_1 be a sequence of X. We say that s_1 is norm-summable if and only if:

(Def. 4) $\|s_1\|$ is summable.

Next we state several propositions:

(2) For every real normed space X and for every sequence s_1 of X and for every natural number m holds $0 \leqslant \|s_1\|(m)$.

(3) For every real normed space X and for all elements x, y, z of X holds $\|x-y\| = \|(x-z)+(z-y)\|$.

(4) Let X be a real normed space and s_1 be a sequence of X. Suppose s_1 is convergent. Let s be a real number. Suppose $0 < s$. Then there exists a natural number n such that for every natural number m if $n \leqslant m$, then $\|s_1(m) - s_1(n)\| < s$.

(5) Let X be a real normed space and s_1 be a sequence of X. Then s_1 is Cauchy sequence by norm if and only if for every real number p such that $p > 0$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $\|s_1(m) - s_1(n)\| < p$.

(6) Let X be a real normed space and s_1 be a sequence of X. Suppose that for every natural number n holds $s_1(n) = 0_X$. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(m) = 0$.

Let X be a real normed space and let s_1 be a sequence of X. Let us observe that s_1 is constant if and only if:

(Def. 5) There exists an element r of X such that for every natural number n holds $s_1(n) = r$.

Let X be a real normed space, let s_1 be a sequence of X, and let k be a natural number. The functor $s_1 \uparrow k$ yielding a sequence of X is defined as follows:

(Def. 6) For every natural number n holds $(s_1 \uparrow k)(n) = s_1(n+k)$.

Let X be a non empty 1-sorted structure, let N_1 be an increasing sequence of naturals, and let s_1 be a sequence of X. Then $s_1 \cdot N_1$ is a function from $\mathbb{N}$ into the carrier of X.

Let X be a non empty 1-sorted structure, let N_1 be an increasing sequence of naturals, and let s_1 be a sequence of X. Then $s_1 \cdot N_1$ is a sequence of X.

Let X be a real normed space and let s_1, s_2 be sequences of X. We say that s_1 is a subsequence of s_2 if and only if:

(Def. 7) There exists an increasing sequence N_1 of naturals such that $s_1 = s_2 \cdot N_1$.

Next we state a number of propositions:

(7) Let X be a non empty 1-sorted structure, s_1 be a sequence of X, N_1 be an increasing sequence of naturals, and n be a natural number. Then $(s_1 \cdot N_1)(n) = s_1(N_1(n))$.

(8) For every real normed space X and for every sequence s_1 of X holds $s_1 \uparrow 0 = s_1$.

(9) For every real normed space X and for every sequence s_1 of X and for all natural numbers k, m holds $s_1 \uparrow k \uparrow m = s_1 \uparrow m \uparrow k$.

(10) For every real normed space X and for every sequence s_1 of X and for all natural numbers k, m holds $s_1 \uparrow k \uparrow m = s_1 \uparrow (k + m)$.

(11) Let X be a real normed space and s_1, s_2 be sequences of X. If s_2 is a subsequence of s_1 and s_1 is convergent, then s_2 is convergent.

(12) Let X be a real normed space and s_1, s_2 be sequences of X. If s_2 is a subsequence of s_1 and s_1 is convergent, then $\lim s_2 = \lim s_1$.

(13) Let X be a real normed space, s_1 be a sequence of X, and k be a natural number. Then $s_1 \uparrow k$ is a subsequence of s_1.

(14) Let X be a real normed space, s_1, s_2 be sequences of X, and k be a natural number. If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim(s_1 \uparrow k) = \lim s_1$.

(15) Let X be a real normed space and s_1, s_2 be sequences of X. Suppose s_1 is convergent and there exists a natural number k such that $s_1 = s_2 \uparrow k$. Then s_2 is convergent.

(16) Let X be a real normed space and s_1, s_2 be sequences of X. Suppose s_1 is convergent and there exists a natural number k such that $s_1 = s_2 \uparrow k$. Then $\lim s_2 = \lim s_1$.

(17) For every real normed space X and for every sequence s_1 of X such that s_1 is constant holds s_1 is convergent.

(18) Let X be a real normed space and s_1 be a sequence of X. If for every natural number n holds $s_1(n) = 0_X$, then s_1 is norm-summable.

Let X be a real normed space. Observe that there exists a sequence of X which is norm-summable.

Next we state three propositions:

(19) Let X be a real normed space and s be a sequence of X. If s is summable, then s is convergent and $\lim s = 0_X$.

(20) For every real normed space X and for all sequences s_3, s_4 of X holds $(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa}(s_4)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_3 + s_4)(\alpha))_{\kappa\in\mathbb{N}}$.

(21) For every real normed space X and for all sequences s_3, s_4 of X holds $(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa}(s_4)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_3 - s_4)(\alpha))_{\kappa\in\mathbb{N}}$.

Let X be a real normed space and let s_1 be a norm-summable sequence of X. Observe that $\|s_1\|$ is summable.

Let X be a real normed space. One can check that every sequence of X which is summable is also convergent.

The following propositions are true:

(22) Let X be a real normed space and s_2, s_5 be sequences of X. If s_2 is summable and s_5 is summable, then s_2+s_5 is summable and $\sum(s_2+s_5) = \sum s_2 + \sum s_5$.

(23) Let X be a real normed space and s_2, s_5 be sequences of X. If s_2 is summable and s_5 is summable, then s_2-s_5 is summable and $\sum(s_2-s_5) = \sum s_2 - \sum s_5$.

Let X be a real normed space and let s_2, s_5 be summable sequences of X. One can verify that $s_2 + s_5$ is summable and $s_2 - s_5$ is summable.

We now state two propositions:

(24) For every real normed space X and for every sequence s_1 of X and for every real number z holds $(\sum_{\alpha=0}^{\kappa}(z \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}} = z \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

(25) Let X be a real normed space, s_1 be a summable sequence of X, and z be a real number. Then $z \cdot s_1$ is summable and $\sum(z \cdot s_1) = z \cdot \sum s_1$.

Let X be a real normed space, let z be a real number, and let s_1 be a summable sequence of X. Observe that $z \cdot s_1$ is summable.

One can prove the following two propositions:

(26) Let X be a real normed space and s, s_3 be sequences of X. If for every natural number n holds $s_3(n) = s(0)$, then $(\sum_{\alpha=0}^{\kappa}(s \uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}} \uparrow 1 - s_3$.

(27) Let X be a real normed space and s be a sequence of X. If s is summable, then for every natural number n holds $s \uparrow n$ is summable.

Let X be a real normed space, let s_1 be a summable sequence of X, and let n be a natural number. Observe that $s_1 \uparrow n$ is summable.

Next we state the proposition

(28) Let X be a real normed space and s_1 be a sequence of X. Then $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded if and only if s_1 is norm-summable.

Let X be a real normed space and let s_1 be a norm-summable sequence of X. One can check that $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}$ is upper bounded.

One can prove the following propositions:

(29) Let X be a real Banach space and s_1 be a sequence of X. Then s_1 is summable if and only if for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)\| < p$.

(30) Let X be a real normed space, s be a sequence of X, and n, m be natural numbers. If $n \leqslant m$, then $\|(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(n)\| \leqslant |(\sum_{\alpha=0}^{\kappa}\|s\|(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|s\|(\alpha))_{\kappa\in\mathbb{N}}(n)|$.

(31) For every real Banach space X and for every sequence s_1 of X such that s_1 is norm-summable holds s_1 is summable.

(32) Let X be a real normed space, r_1 be a sequence of real numbers, and s_5 be a sequence of X. Suppose r_1 is summable and there exists a natural

number m such that for every natural number n such that $m \leqslant n$ holds $\|s_5(n)\| \leqslant r_1(n)$. Then s_5 is norm-summable.

(33) Let X be a real normed space and s_2, s_5 be sequences of X. Suppose for every natural number n holds $0 \leqslant \|s_2\|(n)$ and $\|s_2\|(n) \leqslant \|s_5\|(n)$ and s_5 is norm-summable. Then s_2 is norm-summable and $\sum\|s_2\| \leqslant \sum\|s_5\|$.

(34) Let X be a real normed space and s_1 be a sequence of X. Suppose that

(i) for every natural number n holds $\|s_1\|(n) > 0$, and

(ii) there exists a natural number m such that for every natural number n such that $n \geqslant m$ holds $\frac{\|s_1\|(n+1)}{\|s_1\|(n)} \geqslant 1$.

Then s_1 is not norm-summable.

(35) Let X be a real normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose for every natural number n holds $r_1(n) = \sqrt[n]{\|s_1\|(n)}$ and r_1 is convergent and $\lim r_1 < 1$. Then s_1 is norm-summable.

(36) Let X be a real normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose that

(i) for every natural number n holds $r_1(n) = \sqrt[n]{\|s_1\|(n)}$, and

(ii) there exists a natural number m such that for every natural number n such that $m \leqslant n$ holds $r_1(n) \geqslant 1$.

Then $\|s_1\|$ is not summable.

(37) Let X be a real normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose for every natural number n holds $r_1(n) = \sqrt[n]{\|s_1\|(n)}$ and r_1 is convergent and $\lim r_1 > 1$. Then s_1 is not norm-summable.

(38) Let X be a real normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose $\|s_1\|$ is non-increasing and for every natural number n holds $r_1(n) = 2^n \cdot \|s_1\|(2^n)$. Then s_1 is norm-summable if and only if r_1 is summable.

(39) Let X be a real normed space, s_1 be a sequence of X, and p be a real number. Suppose $p > 1$ and for every natural number n such that $n \geqslant 1$ holds $\|s_1\|(n) = \frac{1}{n^p}$. Then s_1 is norm-summable.

(40) Let X be a real normed space, s_1 be a sequence of X, and p be a real number. Suppose $p \leqslant 1$ and for every natural number n such that $n \geqslant 1$ holds $\|s_1\|(n) = \frac{1}{n^p}$. Then s_1 is not norm-summable.

(41) Let X be a real normed space, s_1 be a sequence of X, and r_1 be a sequence of real numbers. Suppose for every natural number n holds $s_1(n) \neq 0_X$ and $r_1(n) = \frac{\|s_1\|(n+1)}{\|s_1\|(n)}$ and r_1 is convergent and $\lim r_1 < 1$. Then s_1 is norm-summable.

(42) Let X be a real normed space and s_1 be a sequence of X. Suppose that

(i) for every natural number n holds $s_1(n) \neq 0_X$, and

(ii) there exists a natural number m such that for every natural number n such that $n \geqslant m$ holds $\frac{\|s_1\|(n+1)}{\|s_1\|(n)} \geqslant 1$.

Then s_1 is not norm-summable.

Let X be a real Banach space. Observe that every sequence of X which is norm-summable is also summable.

2. Basic Properties of Sequences of Banach Algebra

The scheme *ExNCBASeq* deals with a non empty normed algebra structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

There exists a sequence S of $\mathcal{A}$ such that for every natural number n holds $S(n) = \mathcal{F}(n)$

for all values of the parameters.

The following proposition is true

(43) Let X be a Banach algebra, x, y, z be elements of X, and a, b be real numbers. Then $x+y = y+x$ and $(x+y)+z = x+(y+z)$ and $x+0_X = x$ and there exists an element t of X such that $x+t = 0_X$ and $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ and $1\cdot x = x$ and $0\cdot x = 0_X$ and $a\cdot 0_X = 0_X$ and $(-1)\cdot x = -x$ and $x\cdot \mathbf{1}_X = x$ and $\mathbf{1}_X \cdot x = x$ and $x\cdot(y+z) = x\cdot y + x\cdot z$ and $(y+z)\cdot x = y\cdot x + z\cdot x$ and $a\cdot(x\cdot y) = (a\cdot x)\cdot y$ and $a\cdot(x+y) = a\cdot x + a\cdot y$ and $(a+b)\cdot x = a\cdot x + b\cdot x$ and $(a\cdot b)\cdot x = a\cdot(b\cdot x)$ and $(a\cdot b)\cdot(x\cdot y) = a\cdot x\cdot(b\cdot y)$ and $a\cdot(x\cdot y) = x\cdot(a\cdot y)$ and $0_X\cdot x = 0_X$ and $x\cdot 0_X = 0_X$ and $x\cdot(y-z) = x\cdot y - x\cdot z$ and $(y-z)\cdot x = y\cdot x - z\cdot x$ and $(x+y)-z = x+(y-z)$ and $(x-y)+z = x-(y-z)$ and $x-y-z = x-(y+z)$ and $x+y = (x-z)+(z+y)$ and $x-y = (x-z)+(z-y)$ and $x = (x-y)+y$ and $x = y-(y-x)$ and $\|x\| = 0$ iff $x = 0_X$ and $\|a\cdot x\| = |a|\cdot\|x\|$ and $\|x+y\| \leqslant \|x\|+\|y\|$ and $\|x\cdot y\| \leqslant \|x\|\cdot\|y\|$ and $\|\mathbf{1}_X\| = 1$ and X is complete.

Let X be a non empty multiplicative loop structure and let v be an element of X. We say that v is invertible if and only if:

(Def. 8) There exists an element w of X such that $v\cdot w = \mathbf{1}_X$ and $w\cdot v = \mathbf{1}_X$.

Let X be a non empty normed algebra structure, let S be a sequence of X, and let a be an element of X. The functor $a\cdot S$ yielding a sequence of X is defined by:

(Def. 9) For every natural number n holds $(a\cdot S)(n) = a\cdot S(n)$.

Let X be a non empty normed algebra structure, let S be a sequence of X, and let a be an element of X. The functor $S\cdot a$ yields a sequence of X and is defined by:

(Def. 10) For every natural number n holds $(S\cdot a)(n) = S(n)\cdot a$.

Let X be a non empty normed algebra structure and let s_2, s_5 be sequences of X. The functor $s_2\cdot s_5$ yielding a sequence of X is defined as follows:

(Def. 11) For every natural number n holds $(s_2 \cdot s_5)(n) = s_2(n) \cdot s_5(n)$.

Let X be a Banach algebra and let x be an element of X. Let us assume that x is invertible. The functor x^{-1} yielding an element of X is defined as follows:

(Def. 12) $x \cdot x^{-1} = \mathbf{1}_X$ and $x^{-1} \cdot x = \mathbf{1}_X$.

Let X be a Banach algebra and let z be an element of X. The functor $(z^\kappa)_{\kappa \in \mathbb{N}}$ yielding a sequence of X is defined as follows:

(Def. 13) $(z^\kappa)_{\kappa \in \mathbb{N}}(0) = \mathbf{1}_X$ and for every natural number n holds $(z^\kappa)_{\kappa \in \mathbb{N}}(n+1) = (z^\kappa)_{\kappa \in \mathbb{N}}(n) \cdot z$.

Let X be a Banach algebra, let z be an element of X, and let n be a natural number. The functor $z_{\mathbb{N}}^n$ yields an element of X and is defined by:

(Def. 14) $z_{\mathbb{N}}^n = (z^\kappa)_{\kappa \in \mathbb{N}}(n)$.

One can prove the following four propositions:

(44) For every Banach algebra X and for every element z of X holds $z_{\mathbb{N}}^0 = \mathbf{1}_X$.

(45) For every Banach algebra X and for every element z of X such that $\|z\| < 1$ holds $(z^\kappa)_{\kappa \in \mathbb{N}}$ is summable and norm-summable.

(46) Let X be a Banach algebra and x be a point of X. If $\|\mathbf{1}_X - x\| < 1$, then $((\mathbf{1}_X - x)^\kappa)_{\kappa \in \mathbb{N}}$ is summable and $((\mathbf{1}_X - x)^\kappa)_{\kappa \in \mathbb{N}}$ is norm-summable.

(47) For every Banach algebra X and for every point x of X such that $\|\mathbf{1}_X - x\| < 1$ holds x is invertible and $x^{-1} = \sum(((\mathbf{1}_X - x)^\kappa)_{\kappa \in \mathbb{N}})$.

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