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Complex Banach Space of Bounded Complex Sequences

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Summary. An extension of [18]. In this article, we introduce two complex Banach spaces. One of them is the space of bounded complex sequences. The other one is the space of complex bounded functions, which is defined by the set of all complex bounded functions.

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The articles [21], [6], [23], [24], [17], [20], [2], [19], [12], [4], [5], [7], [22], [3], [1], [16], [15], [14], [10], [13], [11], [8], and [9] provide the terminology and notation for this paper.

1. Complex Banach Space of Bounded Complex Sequences

The subset the set of bounded complex sequences of the linear space of complex sequences is defined by the condition (Def. 1).

(Def. 1) Let x be a set. Then $x \in$ the set of bounded complex sequences if and only if $x \in$ the set of complex sequences and $\mathrm{id}_{\mathrm{seq}}(x)$ is bounded.

Let us note that the set of bounded complex sequences is non empty and the set of bounded complex sequences is linearly closed.

One can prove the following proposition

(1) $\langle$the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences), Mult_(the set of bounded complex sequences, the linear space of complex sequences)$\rangle$ is a subspace of the linear space of complex sequences.

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Let us mention that ⟨the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences),

Mult_(the set of bounded complex sequences, the linear space of complex sequences)⟩ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

The function Clinfty-norm from the set of bounded complex sequences into $\mathbb{R}$ is defined by:

(Def. 2) For every set x such that $x \in$ the set of bounded complex sequences holds Clinfty-norm$(x) = \sup \operatorname{rng} |\operatorname{id}_{\rm seq}(x)|$.

Next we state the proposition

(2) For every complex sequence s_1 holds s_1 is bounded and $\sup \operatorname{rng} |s_1| = 0$ iff for every natural number n holds $s_1(n) = 0_{\mathbb{C}}$.

One can check that ⟨the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences),

Mult_(the set of bounded complex sequences, the linear space of complex sequences), Clinfty-norm⟩ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

The non empty complex normed space structure Clinfty-Space is defined by the condition (Def. 3).

(Def. 3) Clinfty-Space = ⟨the set of bounded complex sequences, Zero_(the set of bounded complex sequences, the linear space of complex sequences), Add_(the set of bounded complex sequences, the linear space of complex sequences), Mult_(the set of bounded complex sequences, the linear space of complex sequences), Clinfty-norm⟩.

Next we state two propositions:

(3) The carrier of Clinfty-Space = the set of bounded complex sequences and for every set x holds x is a vector of Clinfty-Space iff x is a complex sequence and $\operatorname{id}_{\rm seq}(x)$ is bounded and $0_{\text{Clinfty-Space}} =$ CZeroseq and for every vector u of Clinfty-Space holds $u = \operatorname{id}_{\rm seq}(u)$ and for all vectors u, v of Clinfty-Space holds $u + v = \operatorname{id}_{\rm seq}(u) + \operatorname{id}_{\rm seq}(v)$ and for every Complex c and for every vector u of Clinfty-Space holds $c \cdot u = c\ \operatorname{id}_{\rm seq}(u)$ and for every vector u of Clinfty-Space holds $-u = -\operatorname{id}_{\rm seq}(u)$ and $\operatorname{id}_{\rm seq}(-u) = -\operatorname{id}_{\rm seq}(u)$ and for all vectors u, v of Clinfty-Space holds $u - v = \operatorname{id}_{\rm seq}(u) - \operatorname{id}_{\rm seq}(v)$ and for every vector v of Clinfty-Space holds $\operatorname{id}_{\rm seq}(v)$ is bounded and for every vector v of Clinfty-Space holds $\|v\| = \sup \operatorname{rng} |\operatorname{id}_{\rm seq}(v)|$.

(4) Let x, y be points of Clinfty-Space and c be a Complex. Then $\|x\| = 0$ iff $x = 0_{\text{Clinfty-Space}}$ and $0 \leqslant \|x\|$ and $\|x+y\| \leqslant \|x\|+\|y\|$ and $\|c \cdot x\| = |c| \cdot \|x\|$.

Let us note that Clinfty-Space is complex normed space-like, complex linear space-like, Abelian, add-associative, right zeroed, and right complementable.

We now state two propositions:

(5) For every sequence v_1 of Clinfty-Space such that v_1 is Cauchy sequence by norm holds v_1 is convergent.

(6) Clinfty-Space is a complex Banach space.

2. Another Example of Complex Banach Space

Let X be a non empty set, let Y be a complex normed space, and let I_1 be a function from X into the carrier of Y. We say that I_1 is bounded if and only if:

(Def. 4) There exists a real number K such that $0 \leqslant K$ and for every element x of X holds $\|I_1(x)\| \leqslant K$.

The following proposition is true

(7) Let X be a non empty set, Y be a complex normed space, and f be a function from X into the carrier of Y. If for every element x of X holds $f(x) = 0_Y$, then f is bounded.

Let X be a non empty set and let Y be a complex normed space. One can check that there exists a function from X into the carrier of Y which is bounded.

Let X be a non empty set and let Y be a complex normed space. The functor CBdFuncs(X, Y) yields a subset of ComplexVectSpace(X, Y) and is defined by:

(Def. 5) For every set x holds $x \in$ CBdFuncs(X, Y) iff x is a bounded function from X into the carrier of Y.

Let X be a non empty set and let Y be a complex normed space. Note that CBdFuncs(X, Y) is non empty.

One can prove the following propositions:

(8) For every non empty set X and for every complex normed space Y holds CBdFuncs(X, Y) is linearly closed.

(9) Let X be a non empty set and Y be a complex normed space. Then $\langle$CBdFuncs(X, Y), Zero_(CBdFuncs(X, Y), ComplexVectSpace(X, Y)), Add_(CBdFuncs(X, Y), ComplexVectSpace(X, Y)), Mult_(CBdFuncs(X, Y), ComplexVectSpace(X, Y))$\rangle$ is a subspace of ComplexVectSpace(X, Y).

Let X be a non empty set and let Y be a complex normed space. Note that $\langle$CBdFuncs(X, Y), Zero_(CBdFuncs(X, Y), ComplexVectSpace(X, Y)), Add_(CBdFuncs(X, Y), ComplexVectSpace(X, Y)), Mult_(CBdFuncs(X, Y), ComplexVectSpace(X, Y))$\rangle$ is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

We now state the proposition

(10) Let X be a non empty set and Y be a complex normed space. Then $\langle \mathrm{CBdFuncs}(X,Y), \mathrm{Zero\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Add\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Mult\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y))\rangle$ is a complex linear space.

Let X be a non empty set and let Y be a complex normed space. The set of bounded complex sequences from X into Y yielding a complex linear space is defined by:

(Def. 6) The set of bounded complex sequences from X into Y $= \langle \mathrm{CBdFuncs}(X,Y), \mathrm{Zero\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Add\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y)), \mathrm{Mult\_}(\mathrm{CBdFuncs}(X,Y), \mathrm{ComplexVectSpace}(X,Y))\rangle$.

Let X be a non empty set and let Y be a complex normed space. One can verify that the set of bounded complex sequences from X into Y is strict.

The following three propositions are true:

(11) Let X be a non empty set, Y be a complex normed space, f, g, h be vectors of the set of bounded complex sequences from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element x of X holds $h'(x) = f'(x) + g'(x)$.

(12) Let X be a non empty set, Y be a complex normed space, f, h be vectors of the set of bounded complex sequences from X into Y, and f', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $h' = h$. Let c be a Complex. Then $h = c \cdot f$ if and only if for every element x of X holds $h'(x) = c \cdot f'(x)$.

(13) Let X be a non empty set and Y be a complex normed space. Then $0_{\text{the set of bounded complex sequences from } X \text{ into } Y} = X \longmapsto 0_Y$.

Let X be a non empty set, let Y be a complex normed space, and let f be a set. Let us assume that $f \in \mathrm{CBdFuncs}(X,Y)$. The functor $\mathrm{modetrans}(f,X,Y)$ yields a bounded function from X into the carrier of Y and is defined by:

(Def. 7) $\mathrm{modetrans}(f,X,Y) = f$.

Let X be a non empty set, let Y be a complex normed space, and let u be a function from X into the carrier of Y. The functor $\mathrm{PreNorms}(u)$ yielding a non empty subset of $\mathbb{R}$ is defined by:

(Def. 8) $\mathrm{PreNorms}(u) = \{\|u(t)\| : t$ ranges over elements of $X\}$.

We now state three propositions:

(14) Let X be a non empty set, Y be a complex normed space, and g be a bounded function from X into the carrier of Y. Then $\mathrm{PreNorms}(g)$ is non empty and upper bounded.

(15) Let X be a non empty set, Y be a complex normed space, and g be a function from X into the carrier of Y. Then g is bounded if and only if

PreNorms(g) is upper bounded.

(16) Let X be a non empty set and Y be a complex normed space. Then there exists a function N_1 from CBdFuncs(X, Y) into $\mathbb{R}$ such that for every set f if $f \in$ CBdFuncs(X, Y), then $N_1(f) =$ sup PreNorms(modetrans(f, X, Y)).

Let X be a non empty set and let Y be a complex normed space. The functor CBdFuncsNorm(X, Y) yielding a function from CBdFuncs(X, Y) into $\mathbb{R}$ is defined by:

(Def. 9) For every set x such that $x \in$ CBdFuncs(X, Y) holds CBdFuncsNorm(X, Y)(x) = sup PreNorms(modetrans(x, X, Y)).

One can prove the following propositions:

(17) Let X be a non empty set, Y be a complex normed space, and f be a bounded function from X into the carrier of Y. Then modetrans(f, X, Y) = f.

(18) Let X be a non empty set, Y be a complex normed space, and f be a bounded function from X into the carrier of Y. Then CBdFuncsNorm(X, Y)(f) = sup PreNorms(f).

Let X be a non empty set and let Y be a complex normed space. The complex normed space of bounded functions from X into Y yields a non empty complex normed space structure and is defined by:

(Def. 10) The complex normed space of bounded functions from X into Y = ⟨CBdFuncs(X, Y), Zero_(CBdFuncs(X, Y), ComplexVectSpace(X, Y)), Add_(CBdFuncs(X, Y), ComplexVectSpace(X, Y)), Mult_(CBdFuncs(X, Y), ComplexVectSpace(X, Y)), CBdFuncsNorm(X, Y)⟩.

The following propositions are true:

(19) Let X be a non empty set and Y be a complex normed space. Then $X \longmapsto 0_Y = 0_{\text{the complex normed space of bounded functions from } X \text{ into } Y}$.

(20) Let X be a non empty set, Y be a complex normed space, f be a point of the complex normed space of bounded functions from X into Y, and g be a bounded function from X into the carrier of Y. If $g = f$, then for every element t of X holds $\|g(t)\| \leqslant \|f\|$.

(21) Let X be a non empty set, Y be a complex normed space, and f be a point of the complex normed space of bounded functions from X into Y. Then $0 \leqslant \|f\|$.

(22) Let X be a non empty set, Y be a complex normed space, and f be a point of the complex normed space of bounded functions from X into Y. Suppose $f = 0_{\text{the complex normed space of bounded functions from } X \text{ into } Y}$. Then $0 = \|f\|$.

(23) Let X be a non empty set, Y be a complex normed space, f, g, h be points of the complex normed space of bounded functions from X into Y,

and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f + g$ if and only if for every element x of X holds $h'(x) = f'(x) + g'(x)$.

(24) Let X be a non empty set, Y be a complex normed space, f, h be points of the complex normed space of bounded functions from X into Y, and f', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $h' = h$. Let c be a Complex. Then $h = c \cdot f$ if and only if for every element x of X holds $h'(x) = c \cdot f'(x)$.

(25) Let X be a non empty set, Y be a complex normed space, f, g be points of the complex normed space of bounded functions from X into Y, and c be a Complex. Then

(i) $\|f\| = 0$ iff $f = 0_{\text{the complex normed space of bounded functions from } X \text{ into } Y}$,

(ii) $\|c \cdot f\| = |c| \cdot \|f\|$, and

(iii) $\|f + g\| \leqslant \|f\| + \|g\|$.

(26) Let X be a non empty set and Y be a complex normed space. Then the complex normed space of bounded functions from X into Y is complex normed space-like.

(27) Let X be a non empty set and Y be a complex normed space. Then the complex normed space of bounded functions from X into Y is a complex normed space.

Let X be a non empty set and let Y be a complex normed space. One can check that the complex normed space of bounded functions from X into Y is complex normed space-like, complex linear space-like, Abelian, add-associative, right zeroed, and right complementable.

One can prove the following three propositions:

(28) Let X be a non empty set, Y be a complex normed space, f, g, h be points of the complex normed space of bounded functions from X into Y, and f', g', h' be bounded functions from X into the carrier of Y. Suppose $f' = f$ and $g' = g$ and $h' = h$. Then $h = f - g$ if and only if for every element x of X holds $h'(x) = f'(x) - g'(x)$.

(29) Let X be a non empty set and Y be a complex normed space. Suppose Y is complete. Let s_1 be a sequence of the complex normed space of bounded functions from X into Y. If s_1 is Cauchy sequence by norm, then s_1 is convergent.

(30) Let X be a non empty set and Y be a complex Banach space. Then the complex normed space of bounded functions from X into Y is a complex Banach space.

Let X be a non empty set and let Y be a complex Banach space. Note that the complex normed space of bounded functions from X into Y is complete.

3. Some Properties of Complex Sequences

We now state four propositions:

(31) For all complex sequences s_2, s_3 such that s_2 is bounded and s_3 is bounded holds $s_2 + s_3$ is bounded.

(32) For every Complex c and for every complex sequence s_1 such that s_1 is bounded holds $c\, s_1$ is bounded.

(33) For every complex sequence s_1 holds s_1 is bounded iff $|s_1|$ is bounded.

(34) For all complex sequences s_2, s_3, s_4 holds $s_2 = s_3 - s_4$ iff for every natural number n holds $s_2(n) = s_3(n) - s_4(n)$.

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