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Complex Linear Space of Complex Sequences

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Summary. In this article, we introduce a notion of complex linear space of complex sequence and complex unitary space.

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The notation and terminology used here are introduced in the following papers: [18], [21], [22], [17], [5], [6], [10], [3], [7], [16], [9], [12], [19], [4], [1], [11], [15], [14], [2], [20], [13], and [8].

1. Linear Space of Complex Sequence

The non empty set the set of complex sequences is defined by:

(Def. 1) For every set x holds $x \in$ the set of complex sequences iff x is a complex sequence.

Let z be a set. Let us assume that $z \in$ the set of complex sequences. The functor $\mathrm{id}_{\mathrm{seq}}(z)$ yields a complex sequence and is defined by:

(Def. 2) $\mathrm{id}_{\mathrm{seq}}(z) = z$.

Let z be a set. Let us assume that $z \in \mathbb{C}$. The functor $\mathrm{id}_{\mathbb{C}}(z)$ yielding a Complex is defined by:

(Def. 3) $\mathrm{id}_{\mathbb{C}}(z) = z$.

One can prove the following propositions:

(1) There exists a binary operation A_1 on the set of complex sequences such that

(i) for all elements a, b of the set of complex sequences holds $A_1(a, b) = \mathrm{id}_{\mathrm{seq}}(a) + \mathrm{id}_{\mathrm{seq}}(b)$, and

(ii) A_1 is commutative and associative.

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(2) There exists a function f from $[:\mathbb{C}, \text{the set of complex sequences}:]$ into the set of complex sequences such that for all sets r, x if $r \in \mathbb{C}$ and $x \in$ the set of complex sequences, then $f(\langle r, x\rangle) = \mathrm{id}_{\mathbb{C}}(r)\ \mathrm{id}_{\mathrm{seq}}(x)$.

The binary operation $\mathrm{add}_{\mathrm{seq}}$ on the set of complex sequences is defined as follows:

(Def. 4) For all elements a, b of the set of complex sequences holds $\mathrm{add}_{\mathrm{seq}}(a, b) = \mathrm{id}_{\mathrm{seq}}(a) + \mathrm{id}_{\mathrm{seq}}(b)$.

The function $\mathrm{mult}_{\mathrm{seq}}$ from $[:\mathbb{C}, \text{the set of complex sequences}:]$ into the set of complex sequences is defined as follows:

(Def. 5) For all sets z, x such that $z \in \mathbb{C}$ and $x \in$ the set of complex sequences holds $\mathrm{mult}_{\mathrm{seq}}(\langle z, x\rangle) = \mathrm{id}_{\mathbb{C}}(z)\ \mathrm{id}_{\mathrm{seq}}(x)$.

The element CZeroseq of the set of complex sequences is defined by:

(Def. 6) For every natural number n holds $(\mathrm{id}_{\mathrm{seq}}(\mathrm{CZeroseq}))(n) = 0_{\mathbb{C}}$.

One can prove the following propositions:

(3) For every complex sequence x holds $\mathrm{id}_{\mathrm{seq}}(x) = x$.

(4) For all vectors v, w of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $v + w = \mathrm{id}_{\mathrm{seq}}(v) + \mathrm{id}_{\mathrm{seq}}(w)$.

(5) For every Complex z and for every vector v of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $z \cdot v = z\ \mathrm{id}_{\mathrm{seq}}(v)$.

One can check that $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ is Abelian.

Next we state several propositions:

(6) For all vectors u, v, w of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(u + v) + w = u + (v + w)$.

(7) For every vector v of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $v + 0_{\langle \text{the set of complex sequences}, \mathrm{CZeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle} = v$.

(8) Let v be a vector of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$. Then there exists a vector w of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ such that $v + w = 0_{\langle \text{the set of complex sequences}, \mathrm{CZeroseq}, \mathrm{add}_{\mathrm{seq}}, \mathrm{mult}_{\mathrm{seq}}\rangle}$.

(9) For every Complex z and for all vectors v, w of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $z \cdot (v + w) = z \cdot v + z \cdot w$.

(10) For all Complexes z_1, z_2 and for every vector v of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(z_1 + z_2) \cdot v = z_1 \cdot v + z_2 \cdot v$.

(11) For all Complexes z_1, z_2 and for every vector v of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $(z_1 \cdot z_2) \cdot v = z_1 \cdot (z_2 \cdot v)$.

(12) For every vector v of $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$ holds $1_{\mathbb{C}} \cdot v = v$.

The complex linear space the linear space of complex sequences is defined as follows:

(Def. 7) The linear space of complex sequences = $\langle$the set of complex sequences, CZeroseq, $\mathrm{add}_{\mathrm{seq}}$, $\mathrm{mult}_{\mathrm{seq}}\rangle$.

Let X be a complex linear space and let X_1 be a subset of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Add\_}(X_1, X)$ yields a binary operation on X_1 and is defined by:

(Def. 8) $\mathrm{Add\_}(X_1, X) = (\text{the addition of } X)\restriction[\!:X_1, X_1:\!]$.

Let X be a complex linear space and let X_1 be a subset of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Mult\_}(X_1, X)$ yields a function from $[\!:\mathbb{C}, X_1:\!]$ into X_1 and is defined as follows:

(Def. 9) $\mathrm{Mult\_}(X_1, X) = (\text{the external multiplication of } X)\restriction[\!:\mathbb{C}, X_1:\!]$.

Let X be a complex linear space and let X_1 be a subset of X. Let us assume that X_1 is linearly closed and non empty. The functor $\mathrm{Zero\_}(X_1, X)$ yielding an element of X_1 is defined by:

(Def. 10) $\mathrm{Zero\_}(X_1, X) = 0_X$.

One can prove the following proposition

(13) Let V be a complex linear space and V_1 be a subset of V. Suppose V_1 is linearly closed and non empty. Then $\langle V_1, \mathrm{Zero\_}(V_1, V), \mathrm{Add\_}(V_1, V), \mathrm{Mult\_}(V_1, V)\rangle$ is a subspace of V.

The subset the set of l2-complex sequences of the linear space of complex sequences is defined by the conditions (Def. 11).

(Def. 11)(i) The set of l2-complex sequences is non empty, and

(ii) for every set x holds $x \in$ the set of l2-complex sequences iff $x \in$ the set of complex sequences and $|\,\mathrm{id}_{\mathrm{seq}}(x)|\,|\,\mathrm{id}_{\mathrm{seq}}(x)|$ is summable.

One can prove the following propositions:

(14) The set of l2-complex sequences is linearly closed and the set of l2-complex sequences is non empty.

(15) $\langle$the set of l2-complex sequences, Zero_(the set of l2-complex sequences, the linear space of complex sequences), Add_(the set of l2-complex sequences, the linear space of complex sequences), Mult_(the set of l2-complex sequences, the linear space of complex sequences)$\rangle$ is a subspace of the linear space of complex sequences.

(16) $\langle$the set of l2-complex sequences, Zero_(the set of l2-complex sequences, the linear space of complex sequences), Add_(the set of l2-complex sequences, the linear space of complex sequences), Mult_(the set of l2-complex sequences, the linear space of complex sequences)$\rangle$ is a complex linear space.

(17)(i) The carrier of the linear space of complex sequences = the set of complex sequences,

(ii) for every set x holds x is an element of the linear space of complex sequences iff x is a complex sequence,

(iii) for every set x holds x is a vector of the linear space of complex sequences iff x is a complex sequence,

(iv) for every vector u of the linear space of complex sequences holds $u = \mathrm{id}_{\mathrm{seq}}(u)$,

(v) for all vectors u, v of the linear space of complex sequences holds $u + v = \mathrm{id}_{\mathrm{seq}}(u) + \mathrm{id}_{\mathrm{seq}}(v)$, and

(vi) for every Complex z and for every vector u of the linear space of complex sequences holds $z \cdot u = z\ \mathrm{id}_{\mathrm{seq}}(u)$.

2. Unitary Space with Complex Coefficient

We introduce complex unitary space structures which are extensions of CLS structure and are systems

⟨ a carrier, a zero, an addition, an external multiplication, a scalar product ⟩,

where the carrier is a set, the zero is an element of the carrier, the addition is a binary operation on the carrier, the external multiplication is a function from [: $\mathbb{C}$, the carrier :] into the carrier, and the scalar product is a function from [: the carrier, the carrier :] into $\mathbb{C}$.

Let us note that there exists a complex unitary space structure which is non empty and strict.

Let D be a non empty set, let Z be an element of D, let a be a binary operation on D, let m be a function from [: $\mathbb{C}$, D :] into D, and let s be a function from [: D, D :] into $\mathbb{C}$. Note that $\langle D, Z, a, m, s \rangle$ is non empty.

We adopt the following rules: X is a non empty complex unitary space structure, a, b are Complexes, and x, y are points of X.

Let us consider X and let us consider x, y. The functor $(x|y)$ yields a Complex and is defined by:

(Def. 12) $(x|y) = (\text{the scalar product of } X)(\langle x, y \rangle)$.

Let I_1 be a non empty complex unitary space structure. We say that I_1 is complex unitary space-like if and only if the condition (Def. 13) is satisfied.

(Def. 13) Let x, y, w be points of I_1 and given a. Then $(x|x) = 0$ iff $x = 0_{(I_1)}$ and $0 \leqslant \Re((x|x))$ and $0 = \Im((x|x))$ and $(x|y) = \overline{(y|x)}$ and $((x+y)|w) = (x|w) + (y|w)$ and $((a \cdot x)|y) = a \cdot (x|y)$.

Let us note that there exists a non empty complex unitary space structure which is complex unitary space-like, complex linear space-like, Abelian, add-associative, right zeroed, right complementable, and strict.

A complex unitary space is a complex unitary space-like complex linear space-like Abelian add-associative right zeroed right complementable non empty complex unitary space structure.

We use the following convention: X is a complex unitary space and x, y, z, u, v are points of X.

Next we state a number of propositions:

(18) $(0_X|0_X) = 0$.

(19) $(x|(y+z)) = (x|y) + (x|z)$.

(20) $(x|(a \cdot y)) = \overline{a} \cdot (x|y)$.

(21) $((a \cdot x)|y) = (x|(\overline{a} \cdot y))$.

(22) $((a \cdot x + b \cdot y)|z) = a \cdot (x|z) + b \cdot (y|z)$.

(23) $(x|(a \cdot y + b \cdot z)) = \overline{a} \cdot (x|y) + \overline{b} \cdot (x|z)$.

(24) $((-x)|y) = (x|-y)$.

(25) $((-x)|y) = -(x|y)$.

(26) $(x|-y) = -(x|y)$.

(27) $((-x)|-y) = (x|y)$.

(28) $((x-y)|z) = (x|z) - (y|z)$.

(29) $(x|(y-z)) = (x|y) - (x|z)$.

(30) $((x-y)|(u-v)) = ((x|u) - (x|v) - (y|u)) + (y|v)$.

(31) $(0_X|x) = 0$.

(32) $(x|0_X) = 0$.

(33) $((x+y)|(x+y)) = (x|x) + (x|y) + (y|x) + (y|y)$.

(34) $((x+y)|(x-y)) = (((x|x) - (x|y)) + (y|x)) - (y|y)$.

(35) $((x-y)|(x-y)) = ((x|x) - (x|y) - (y|x)) + (y|y)$.

(36) $|(x|x)| = \Re((x|x))$.

(37) $|(x|y)| \leqslant \sqrt{|(x|x)|} \cdot \sqrt{|(y|y)|}$.

Let us consider X and let us consider x, y. We say that x, y are orthogonal if and only if:

(Def. 14) $(x|y) = 0$.

Let us note that the predicate x, y are orthogonal is symmetric.

We now state several propositions:

(38) If x, y are orthogonal, then x, $-y$ are orthogonal.

(39) If x, y are orthogonal, then $-x$, y are orthogonal.

(40) If x, y are orthogonal, then $-x$, $-y$ are orthogonal.

(41) x, 0_X are orthogonal.

(42) If x, y are orthogonal, then $((x+y)|(x+y)) = (x|x) + (y|y)$.

(43) If x, y are orthogonal, then $((x-y)|(x-y)) = (x|x) + (y|y)$.

Let us consider X, x. The functor $\|x\|$ yields a real number and is defined as follows:

(Def. 15) $\|x\| = \sqrt{|(x|x)|}$.

We now state several propositions:

(44) $\|x\| = 0$ iff $x = 0_X$.

(45) $\|a \cdot x\| = |a| \cdot \|x\|$.

(46) $0 \leqslant \|x\|$.

(47) $|(x|y)| \leqslant \|x\| \cdot \|y\|$.

(48) $\|x + y\| \leqslant \|x\| + \|y\|$.

(49) $\|-x\| = \|x\|$.

(50) $\|x\| - \|y\| \leqslant \|x - y\|$.

(51) $|\|x\| - \|y\|| \leqslant \|x - y\|$.

Let us consider X, x, y. The functor $\rho(x, y)$ yielding a real number is defined as follows:

(Def. 16) $\rho(x, y) = \|x - y\|$.

One can prove the following proposition

(52) $\rho(x, y) = \rho(y, x)$.

Let us consider X, x, y. Let us observe that the functor $\rho(x, y)$ is commutative.

We now state a number of propositions:

(53) $\rho(x, x) = 0$.

(54) $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$.

(55) $x \neq y$ iff $\rho(x, y) \neq 0$.

(56) $\rho(x, y) \geqslant 0$.

(57) $x \neq y$ iff $\rho(x, y) > 0$.

(58) $\rho(x, y) = \sqrt{|((x - y)|(x - y))|}$.

(59) $\rho(x + y, u + v) \leqslant \rho(x, u) + \rho(y, v)$.

(60) $\rho(x - y, u - v) \leqslant \rho(x, u) + \rho(y, v)$.

(61) $\rho(x - z, y - z) = \rho(x, y)$.

(62) $\rho(x - z, y - z) \leqslant \rho(z, x) + \rho(z, y)$.

We follow the rules: s_1, s_2, s_3, s_4 are sequences of X and k, n, m are natural numbers.

The scheme *Ex Seq in CUS* deals with a non empty complex unitary space structure $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

> There exists a sequence s_1 of $\mathcal{A}$ such that for every n holds $s_1(n) = \mathcal{F}(n)$

for all values of the parameters.

Let us consider X and let us consider s_1. The functor $-s_1$ yielding a sequence of X is defined by:

(Def. 17) For every n holds $(-s_1)(n) = -s_1(n)$.

Let us consider X, let us consider s_1, and let us consider x. The functor $s_1 + x$ yielding a sequence of X is defined by:

(Def. 18) For every n holds $(s_1 + x)(n) = s_1(n) + x$.

One can prove the following proposition

(63) $s_2 + s_3 = s_3 + s_2$.

Let us consider X, s_2, s_3. Let us observe that the functor $s_2 + s_3$ is commutative.

One can prove the following propositions:

(64) $s_2 + (s_3 + s_4) = (s_2 + s_3) + s_4$.

(65) If s_2 is constant and s_3 is constant and $s_1 = s_2 + s_3$, then s_1 is constant.

(66) If s_2 is constant and s_3 is constant and $s_1 = s_2 - s_3$, then s_1 is constant.

(67) If s_2 is constant and $s_1 = a \cdot s_2$, then s_1 is constant.

(68) s_1 is constant iff for every n holds $s_1(n) = s_1(n + 1)$.

(69) s_1 is constant iff for all n, k holds $s_1(n) = s_1(n + k)$.

(70) s_1 is constant iff for all n, m holds $s_1(n) = s_1(m)$.

(71) $s_2 - s_3 = s_2 + -s_3$.

(72) $s_1 = s_1 + 0_X$.

(73) $a \cdot (s_2 + s_3) = a \cdot s_2 + a \cdot s_3$.

(74) $(a + b) \cdot s_1 = a \cdot s_1 + b \cdot s_1$.

(75) $(a \cdot b) \cdot s_1 = a \cdot (b \cdot s_1)$.

(76) $1_{\mathbb{C}} \cdot s_1 = s_1$.

(77) $(-1_{\mathbb{C}}) \cdot s_1 = -s_1$.

(78) $s_1 - x = s_1 + -x$.

(79) $s_2 - s_3 = -(s_3 - s_2)$.

(80) $s_1 = s_1 - 0_X$.

(81) $s_1 = --s_1$.

(82) $s_2 - (s_3 + s_4) = s_2 - s_3 - s_4$.

(83) $(s_2 + s_3) - s_4 = s_2 + (s_3 - s_4)$.

(84) $s_2 - (s_3 - s_4) = (s_2 - s_3) + s_4$.

(85) $a \cdot (s_2 - s_3) = a \cdot s_2 - a \cdot s_3$.

3. Complex Unitary Space of Complex Sequence

Next we state the proposition

(86) There exists a function f from $[\!:$ the set of l2-complex sequences, the set of l2-complex sequences $:\!]$ into $\mathbb{C}$ such that for all sets x, y if $x \in$ the set of l2-complex sequences and $y \in$ the set of l2-complex sequences, then $f(\langle x, y \rangle) = \sum(\mathrm{id}_{\mathrm{seq}}(x)\,\overline{\mathrm{id}_{\mathrm{seq}}(y)})$.

The function $\mathrm{scalar}_{\mathrm{cl}}$ from $[\!:$ the set of l2-complex sequences, the set of l2-complex sequences $:\!]$ into $\mathbb{C}$ is defined by the condition (Def. 19).

(Def. 19) Let x, y be sets. Suppose $x \in$ the set of l2-complex sequences and $y \in$ the set of l2-complex sequences. Then $\mathrm{scalar}_{\mathrm{cl}}(\langle x, y \rangle) = \sum(\mathrm{id}_{\mathrm{seq}}(x)\,\overline{\mathrm{id}_{\mathrm{seq}}(y)})$.

Let us observe that $\langle$the set of l2-complex sequences, Zero_(the set of l2-complex sequences, the linear space of complex sequences), Add_(the set of l2-complex sequences, the linear space of complex sequences), Mult_(the set of l2-complex sequences, the linear space of complex sequences), $\mathrm{scalar}_{\mathrm{cl}}\rangle$ is non empty.

The non empty complex unitary space structure Complexl2-Space is defined by the condition (Def. 20).

(Def. 20) Complexl2-Space = $\langle$the set of l2-complex sequences, Zero_(the set of l2-complex sequences, the linear space of complex sequences), Add_(the set of l2-complex sequences, the linear space of complex sequences), Mult_(the set of l2-complex sequences, the linear space of complex sequences), $\mathrm{scalar}_{\mathrm{cl}}\rangle$.

The following propositions are true:

(87) Let l be a complex unitary space structure. Suppose $\langle$the carrier of l, the zero of l, the addition of l, the external multiplication of $l\rangle$ is a complex linear space. Then l is a complex linear space.

(88) For every complex sequence s_1 such that for every natural number n holds $s_1(n) = 0_{\mathbb{C}}$ holds s_1 is summable and $\sum s_1 = 0_{\mathbb{C}}$.

Let us observe that Complexl2-Space is Abelian, add-associative, right zeroed, right complementable, and complex linear space-like.

References

[1] Agnieszka Banachowicz and Anna Winnicka. Complex sequences. *Formalized Mathematics*, 4(**1**):121–124, 1993.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[4] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[8] Noboru Endou. Complex linear space and complex normed space. *Formalized Mathematics*, 12(**2**):93–102, 2004.
[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[10] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[11] Adam Naumowicz. Conjugate sequences, bounded complex sequences and convergent complex sequences. *Formalized Mathematics*, 6(**2**):265–268, 1997.
[12] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[13] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[14] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Formalized Mathematics*, 2(**4**):449–452, 1991.
[15] Yasunari Shidama and Artur Korniłowicz. Convergence and the limit of complex sequences. Series. *Formalized Mathematics*, 6(**3**):403–410, 1997.
[16] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[17] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[19] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.
[20] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[21] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[22] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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