FORMALIZED MATHEMATICS
Volume 12, Number 2, 2004
University of Białystok

Cauchy Sequence of Complex Unitary Space

Yasumasa Suzuki
Take, Yokosuka-shi
Japan

Noboru Endou
Gifu National College of Technology

Summary. As an extension of [13], we introduce the Cauchy sequence of complex unitary space and describe its properties.

MML Identifier: CLVECT_3.

The terminology and notation used in this paper are introduced in the following papers: [22], [3], [20], [9], [5], [12], [10], [11], [15], [2], [18], [4], [1], [21], [16], [17], [14], [13], [19], [6], [7], and [8].

For simplicity, we follow the rules: X denotes a complex unitary space, s_1, s_2, s_3 denote sequences of X, R_1 denotes a sequence of real numbers, C_1, C_2, C_3 denote complex sequences, z, z_1, z_2 denote Complexes, r denotes a real number, and k, n, m denote natural numbers.

The scheme *Rec Func Ex CUS* deals with a complex unitary space $\mathcal{A}$, a point $\mathcal{B}$ of $\mathcal{A}$, and a binary functor $\mathcal{F}$ yielding a point of $\mathcal{A}$, and states that:

> There exists a function f from $\mathbb{N}$ into the carrier of $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and for every element n of $\mathbb{N}$ and for every point x of $\mathcal{A}$ such that $x = f(n)$ holds $f(n+1) = \mathcal{F}(n, x)$

for all values of the parameters.

Let us consider X, s_1. The functor $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ yields a sequence of X and is defined as follows:

(Def. 1) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(0) = s_1(0)$ and for every n holds $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) + s_1(n+1)$.

One can prove the following propositions:

(1) $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} + (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2+s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(2) $(\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} - (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_2-s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

(3) $(\sum_{\alpha=0}^{\kappa}(z\cdot s_1)(\alpha))_{\kappa\in\mathbb{N}} = z\cdot(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

ISSN 1426–2630

(4) $(\sum_{\alpha=0}^{\kappa}(-s_1)(\alpha))_{\kappa\in\mathbb{N}} = -(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$.

(5) $z_1 \cdot (\sum_{\alpha=0}^{\kappa}(s_2)(\alpha))_{\kappa\in\mathbb{N}} + z_2 \cdot (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(z_1 \cdot s_2 + z_2 \cdot s_3)(\alpha))_{\kappa\in\mathbb{N}}$.

Let us consider X, s_1. We say that s_1 is summable if and only if:

(Def. 2) $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is convergent.

The functor $\sum s_1$ yields a point of X and is defined as follows:

(Def. 3) $\sum s_1 = \lim((\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})$.

Next we state several propositions:

(6) If s_2 is summable and s_3 is summable, then $s_2 + s_3$ is summable and $\sum(s_2 + s_3) = \sum s_2 + \sum s_3$.

(7) If s_2 is summable and s_3 is summable, then $s_2 - s_3$ is summable and $\sum(s_2 - s_3) = \sum s_2 - \sum s_3$.

(8) If s_1 is summable, then $z \cdot s_1$ is summable and $\sum(z \cdot s_1) = z \cdot \sum s_1$.

(9) If s_1 is summable, then s_1 is convergent and $\lim s_1 = 0_X$.

(10) Suppose X is Hilbert. Then s_1 is summable if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m)\| < r$.

(11) If s_1 is summable, then $(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}$ is bounded.

(12) If for every n holds $s_2(n) = s_1(0)$, then $(\sum_{\alpha=0}^{\kappa}(s_1 \uparrow 1)(\alpha))_{\kappa\in\mathbb{N}} = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}} \uparrow 1 - s_2$.

(13) If s_1 is summable, then for every k holds $s_1 \uparrow k$ is summable.

(14) If there exists k such that $s_1 \uparrow k$ is summable, then s_1 is summable.

Let us consider X, s_1, n. The functor $\sum_{\kappa=0}^{n} s_1(\kappa)$ yielding a point of X is defined by:

(Def. 4) $\sum_{\kappa=0}^{n} s_1(\kappa) = (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

One can prove the following propositions:

(15) $\sum_{\kappa=0}^{0} s_1(\kappa) = s_1(0)$.

(16) $\sum_{\kappa=0}^{1} s_1(\kappa) = \sum_{\kappa=0}^{0} s_1(\kappa) + s_1(1)$.

(17) $\sum_{\kappa=0}^{1} s_1(\kappa) = s_1(0) + s_1(1)$.

(18) $\sum_{\kappa=0}^{n+1} s_1(\kappa) = \sum_{\kappa=0}^{n} s_1(\kappa) + s_1(n+1)$.

(19) $s_1(n+1) = \sum_{\kappa=0}^{n+1} s_1(\kappa) - \sum_{\kappa=0}^{n} s_1(\kappa)$.

(20) $s_1(1) = \sum_{\kappa=0}^{1} s_1(\kappa) - \sum_{\kappa=0}^{0} s_1(\kappa)$.

Let us consider X, s_1, n, m. The functor $\sum_{\kappa=n+1}^{m} s_1(\kappa)$ yielding a point of X is defined by:

(Def. 5) $\sum_{\kappa=n+1}^{m} s_1(\kappa) = \sum_{\kappa=0}^{n} s_1(\kappa) - \sum_{\kappa=0}^{m} s_1(\kappa)$.

One can prove the following four propositions:

(21) $\sum_{\kappa=1+1}^{0} s_1(\kappa) = s_1(1)$.

(22) $\sum_{\kappa=n+1+1}^{n} s_1(\kappa) = s_1(n+1)$.

(23) Suppose X is Hilbert. Then s_1 is summable if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|\sum_{\kappa=0}^{n} s_1(\kappa) - \sum_{\kappa=0}^{m} s_1(\kappa)\| < r$.

(24) Suppose X is Hilbert. Then s_1 is summable if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $\|\sum_{\kappa=n+1}^{m} s_1(\kappa)\| < r$.

Let us consider C_1, n. The functor $\sum_{\kappa=0}^{n} C_1(\kappa)$ yielding a Complex is defined as follows:

(Def. 6) $\sum_{\kappa=0}^{n} C_1(\kappa) = (\sum_{\alpha=0}^{\kappa}(C_1)(\alpha))_{\kappa\in\mathbb{N}}(n)$.

Let us consider C_1, n, m. The functor $\sum_{\kappa=n+1}^{m} C_1(\kappa)$ yielding a Complex is defined by:

(Def. 7) $\sum_{\kappa=n+1}^{m} C_1(\kappa) = \sum_{\kappa=0}^{n} C_1(\kappa) - \sum_{\kappa=0}^{m} C_1(\kappa)$.

Let us consider X, s_1. We say that s_1 is absolutely summable if and only if:

(Def. 8) $\|s_1\|$ is summable.

The following propositions are true:

(25) If s_2 is absolutely summable and s_3 is absolutely summable, then $s_2 + s_3$ is absolutely summable.

(26) If s_1 is absolutely summable, then $z \cdot s_1$ is absolutely summable.

(27) If for every n holds $\|s_1\|(n) \leqslant R_1(n)$ and R_1 is summable, then s_1 is absolutely summable.

(28) If for every n holds $s_1(n) \neq 0_X$ and $R_1(n) = \frac{\|s_1(n+1)\|}{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 < 1$, then s_1 is absolutely summable.

(29) If $r > 0$ and there exists m such that for every n such that $n \geqslant m$ holds $\|s_1(n)\| \geqslant r$, then s_1 is not convergent or $\lim s_1 \neq 0_X$.

(30) If for every n holds $s_1(n) \neq 0_X$ and there exists m such that for every n such that $n \geqslant m$ holds $\frac{\|s_1(n+1)\|}{\|s_1(n)\|} \geqslant 1$, then s_1 is not summable.

(31) If for every n holds $s_1(n) \neq 0_X$ and for every n holds $R_1(n) = \frac{\|s_1(n+1)\|}{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 > 1$, then s_1 is not summable.

(32) If for every n holds $R_1(n) = \sqrt[n]{\|s_1(n)\|}$ and R_1 is convergent and $\lim R_1 < 1$, then s_1 is absolutely summable.

(33) If for every n holds $R_1(n) = \sqrt[n]{\|s_1\|(n)}$ and there exists m such that for every n such that $n \geqslant m$ holds $R_1(n) \geqslant 1$, then s_1 is not summable.

(34) If for every n holds $R_1(n) = \sqrt[n]{\|s_1\|(n)}$ and R_1 is convergent and $\lim R_1 > 1$, then s_1 is not summable.

(35) $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}$ is non-decreasing.

(36) For every n holds $(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(n) \geqslant 0$.

(37) For every n holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)\| \leqslant (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(38) For every n holds $\|\sum_{\kappa=0}^{n} s_1(\kappa)\| \leqslant \sum_{\kappa=0}^{n}\|s_1\|(\kappa)$.

(39) For all n, m holds $\|(\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}}(n)\| \leqslant |(\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(m) - (\sum_{\alpha=0}^{\kappa}\|s_1\|(\alpha))_{\kappa\in\mathbb{N}}(n)|$.

(40) For all n, m holds $\|\sum_{\kappa=0}^{m} s_1(\kappa) - \sum_{\kappa=0}^{n} s_1(\kappa)\| \leqslant |\sum_{\kappa=0}^{m}\|s_1\|(\kappa) - \sum_{\kappa=0}^{n}\|s_1\|(\kappa)|$.

(41) For all n, m holds $\|\sum_{\kappa=m+1}^{n} s_1(\kappa)\| \leqslant |\sum_{\kappa=m+1}^{n}\|s_1\|(\kappa)|$.

(42) If X is Hilbert, then if s_1 is absolutely summable, then s_1 is summable.

Let us consider X, s_1, C_1. The functor $C_1 \cdot s_1$ yields a sequence of X and is defined by:

(Def. 9) For every n holds $(C_1 \cdot s_1)(n) = C_1(n) \cdot s_1(n)$.

Next we state several propositions:

(43) $C_1 \cdot (s_2 + s_3) = C_1 \cdot s_2 + C_1 \cdot s_3$.

(44) $(C_2 + C_3) \cdot s_1 = C_2 \cdot s_1 + C_3 \cdot s_1$.

(45) $(C_2\, C_3) \cdot s_1 = C_2 \cdot (C_3 \cdot s_1)$.

(46) $(z\, C_1) \cdot s_1 = z \cdot (C_1 \cdot s_1)$.

(47) $C_1 \cdot -s_1 = (-C_1) \cdot s_1$.

(48) If C_1 is convergent and s_1 is convergent, then $C_1 \cdot s_1$ is convergent.

(49) If C_1 is bounded and s_1 is bounded, then $C_1 \cdot s_1$ is bounded.

(50) If C_1 is convergent and s_1 is convergent, then $C_1 \cdot s_1$ is convergent and $\lim(C_1 \cdot s_1) = \lim C_1 \cdot \lim s_1$.

Let us consider C_1. We say that C_1 is Cauchy if and only if:

(Def. 10) For every r such that $r > 0$ there exists k such that for all n, m such that $n \geqslant k$ and $m \geqslant k$ holds $|C_1(n) - C_1(m)| < r$.

We introduce C_1 is a Cauchy sequence as a synonym of C_1 is Cauchy.

Next we state four propositions:

(51) If X is Hilbert, then if s_1 is Cauchy and C_1 is Cauchy, then $C_1 \cdot s_1$ is Cauchy.

(52) For every n holds $(\sum_{\alpha=0}^{\kappa}((C_1 - C_1 \uparrow 1) \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n) = (\sum_{\alpha=0}^{\kappa}(C_1 \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) - (C_1 \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(n+1)$.

(53) For every n holds $(\sum_{\alpha=0}^{\kappa}(C_1 \cdot s_1)(\alpha))_{\kappa\in\mathbb{N}}(n+1) = (C_1 \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(n+1) - (\sum_{\alpha=0}^{\kappa}((C_1 \uparrow 1 - C_1) \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(54) For every n holds $\sum_{\kappa=0}^{n+1}(C_1 \cdot s_1)(\kappa) = (C_1 \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(n+1) - \sum_{\kappa=0}^{n}((C_1 \uparrow 1 - C_1) \cdot (\sum_{\alpha=0}^{\kappa}(s_1)(\alpha))_{\kappa\in\mathbb{N}})(\kappa)$.

References

[1] Agnieszka Banachowicz and Anna Winnicka. Complex sequences. *Formalized Mathematics*, 4(**1**):121–124, 1993.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[4] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Noboru Endou. Complex linear space and complex normed space. *Formalized Mathematics*, 12(**2**):93–102, 2004.
[7] Noboru Endou. Complex linear space of complex sequences. *Formalized Mathematics*, 12(**2**):109–117, 2004.
[8] Noboru Endou. Convergent sequences in complex unitary space. *Formalized Mathematics*, 12(**2**):159–165, 2004.
[9] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[10] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[11] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.
[12] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[13] Elżbieta Kraszewska and Jan Popiołek. Series in Banach and Hilbert Spaces. *Formalized Mathematics*, 2(**5**):695–699, 1991.
[14] Adam Naumowicz. Conjugate sequences, bounded complex sequences and convergent complex sequences. *Formalized Mathematics*, 6(**2**):265–268, 1997.
[15] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.
[16] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[17] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.
[18] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Formalized Mathematics*, 2(**4**):449–452, 1991.
[19] Yasunari Shidama and Artur Korniłowicz. Convergence and the limit of complex sequences. Series. *Formalized Mathematics*, 6(**3**):403–410, 1997.
[20] Andrzej Trybulec. Subsets of complex numbers. *To appear in Formalized Mathematics.*
[21] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

Received March 18, 2004