FORMALIZED MATHEMATICS
Volume 12, Number 1, 2004
University of Białystok

Transitive Closure of Fuzzy Relations[1]

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MML Identifier: LFUZZY_1.

The papers [22], [11], [25], [8], [9], [2], [3], [20], [21], [10], [1], [27], [7], [24], [23], [15], [19], [26], [4], [5], [6], [14], [12], [17], [18], [13], and [16] provide the terminology and notation for this paper.

1. Inclusion of Fuzzy Sets

In this paper X, Y denote non empty sets.

Let X be a non empty set. Observe that every membership function of X is real-yielding.

Let f, g be real-yielding functions. The predicate $f \sqsubseteq g$ is defined by:

(Def. 1) $\operatorname{dom} f \subseteq \operatorname{dom} g$ and for every set x such that $x \in \operatorname{dom} f$ holds $f(x) \leqslant g(x)$.

Let X be a non empty set and let f, g be membership functions of X. Let us observe that $f \sqsubseteq g$ if and only if:

(Def. 2) For every element x of X holds $f(x) \leqslant g(x)$.

We introduce $f \subseteq g$ as a synonym of $f \sqsubseteq g$.

Let X, Y be non empty sets and let f, g be membership functions of X, Y. Let us observe that $f \sqsubseteq g$ if and only if:

(Def. 3) For every element x of X and for every element y of Y holds $f(\langle x, y\rangle) \leqslant g(\langle x, y\rangle)$.

One can prove the following propositions:

[1]This work has been partially supported by the Polish Academy of Sciences and the Japan Society for the Promotion of Science (JSPS Grant 0324101 and JSPS Grant-in-aid 15700195) when the first author was visiting Białystok Technical University as postdoctoral fellow.

ISSN 1426–2630

(1) For all membership functions R, S of X such that for every element x of X holds $R(x) = S(x)$ holds $R = S$.

(2) Let R, S be membership functions of X, Y. Suppose that for every element x of X and for every element y of Y holds $R(\langle x, y \rangle) = S(\langle x, y \rangle)$. Then $R = S$.

(3) For all membership functions R, S of X holds $R = S$ iff $R \subseteq S$ and $S \subseteq R$.

(4) For every membership function R of X holds $R \subseteq R$.

(5) For all membership functions R, S, T of X such that $R \subseteq S$ and $S \subseteq T$ holds $R \subseteq T$.

(6) Let X, Y, Z be non empty sets, R, S be membership functions of X, Y, and T, U be membership functions of Y, Z. If $R \subseteq S$ and $T \subseteq U$, then $R\,T \subseteq S\,U$.

Let X be a non empty set and let f, g be membership functions of X. Let us note that the functor $\min(f, g)$ is commutative. Let us note that the functor $\max(f, g)$ is commutative.

We now state two propositions:

(7) For all membership functions f, g of X holds $\min(f, g) \subseteq f$.

(8) For all membership functions f, g of X holds $f \subseteq \max(f, g)$.

2. Properties of Fuzzy Relations

Let X be a non empty set and let R be a membership function of X, X. We say that R is reflexive if and only if:

(Def. 4) $\mathrm{Imf}(X, X) \subseteq R$.

Let X be a non empty set and let R be a membership function of X, X. Let us observe that R is reflexive if and only if:

(Def. 5) For every element x of X holds $R(\langle x, x \rangle) = 1$.

Let X be a non empty set and let R be a membership function of X, X. We say that R is symmetric if and only if:

(Def. 6) $\mathrm{converse}\, R = R$.

Let X be a non empty set and let R be a membership function of X, X. Let us observe that R is symmetric if and only if:

(Def. 7) For all elements x, y of X holds $R(\langle x, y \rangle) = R(\langle y, x \rangle)$.

Let X be a non empty set and let R be a membership function of X, X. We say that R is transitive if and only if:

(Def. 8) $R\,R \subseteq R$.

Let X be a non empty set and let R be a membership function of X, X. Let us observe that R is transitive if and only if:

(Def. 9) For all elements x, y, z of X holds $R(\langle x, y\rangle) \sqcap R(\langle y, z\rangle) \preceq R(\langle x, z\rangle)$.

Let X be a non empty set and let R be a membership function of X, X. We say that R is antisymmetric if and only if:

(Def. 10) For all elements x, y of X such that $R(\langle x, y\rangle) \neq 0$ and $R(\langle y, x\rangle) \neq 0$ holds $x = y$.

Let X be a non empty set and let R be a membership function of X, X. Let us observe that R is antisymmetric if and only if:

(Def. 11) For all elements x, y of X such that $R(\langle x, y\rangle) \neq 0$ and $x \neq y$ holds $R(\langle y, x\rangle) = 0$.

Let us consider X. Note that $\mathrm{Imf}(X, X)$ is symmetric, transitive, reflexive, and antisymmetric.

Let us consider X. Observe that there exists a membership function of X, X which is reflexive, transitive, symmetric, and antisymmetric.

Next we state two propositions:

(9) For all membership functions R, S of X, X such that R is symmetric and S is symmetric holds converse $\min(R, S) = \min(R, S)$.

(10) For all membership functions R, S of X, X such that R is symmetric and S is symmetric holds converse $\max(R, S) = \max(R, S)$.

Let us consider X and let R, S be symmetric membership functions of X, X. Note that $\min(R, S)$ is symmetric and $\max(R, S)$ is symmetric.

One can prove the following proposition

(11) For all membership functions R, S of X, X such that R is transitive and S is transitive holds $\min(R, S)\ \min(R, S) \subseteq \min(R, S)$.

Let us consider X and let R, S be transitive membership functions of X, X. Observe that $\min(R, S)$ is transitive.

Let A be a set and let X be a non empty set. Then $\chi_{A,X}$ is a membership function of X.

One can prove the following propositions:

(12) For every binary relation r on X such that r is reflexive in X holds $\chi_{r,[\![X, X]\!]}$ is reflexive.

(13) For every binary relation r on X such that r is antisymmetric holds $\chi_{r,[\![X, X]\!]}$ is antisymmetric.

(14) For every binary relation r on X such that r is symmetric holds $\chi_{r,[\![X, X]\!]}$ is symmetric.

(15) For every binary relation r on X such that r is transitive holds $\chi_{r,[\![X, X]\!]}$ is transitive.

(16) $\mathrm{Zmf}(X, X)$ is symmetric, antisymmetric, and transitive.

(17) $\mathrm{Umf}(X, X)$ is symmetric, transitive, and reflexive.

(18) For every membership function R of X, X holds $\max(R, \text{converse}\, R)$ is symmetric.

(19) For every membership function R of X, X holds $\min(R, \text{converse}\, R)$ is symmetric.

(20) Let R be a membership function of X, X and R' be a membership function of X, X. If R' is symmetric and $R \subseteq R'$, then $\max(R, \text{converse}\, R) \subseteq R'$.

(21) Let R be a membership function of X, X and R' be a membership function of X, X. If R' is symmetric and $R' \subseteq R$, then $R' \subseteq \min(R, \text{converse}\, R)$.

3. Transitive Closure

Let X be a non empty set, let R be a membership function of X, X, and let n be a natural number. The functor R^n yielding a membership function of X, X is defined by the condition (Def. 12).

(Def. 12) There exists a function F from $\mathbb{N}$ into $[0,1]^{[: X, X :]}$ such that

(i) $R^n = F(n)$,

(ii) $F(0) = \text{Imf}(X, X)$, and

(iii) for every natural number k there exists a membership function Q of X, X such that $F(k) = Q$ and $F(k+1) = Q\, R$.

In the sequel X denotes a non empty set and R denotes a membership function of X, X.

Next we state several propositions:

(22) $\text{Imf}(X, X)\, R = R$.

(23) $R\, \text{Imf}(X, X) = R$.

(24) $R^0 = \text{Imf}(X, X)$.

(25) $R^1 = R$.

(26) For every natural number n holds $R^{(n+1)} = R^n\, R$.

(27) For all natural numbers m, n holds $R^{(m+n)} = R^m\, R^n$.

(28) For all natural numbers m, n holds $R^{(m \cdot n)} = (R^n)^m$.

Let X be a non empty set and let R be a membership function of X, X. The functor $\text{TrCl}\, R$ yields a membership function of X, X and is defined as follows:

(Def. 13) $\text{TrCl}\, R = \bigsqcup_{\text{FuzzyLattice}[: X, X :]} \{R^n; n \text{ ranges over natural numbers: } n > 0\}$.

Next we state several propositions:

(29) For all elements x, y of X holds $(\text{TrCl}\, R)(\langle x, y \rangle) = \bigsqcup_{\text{RealPoset}[0,1]} \pi_{\langle x, y \rangle} \{R^n; n \text{ ranges over natural numbers: } n > 0\}$.

(30) $R \subseteq \text{TrCl}\, R$.

(31) For every natural number n such that $n > 0$ holds $R^n \subseteq \mathrm{TrCl}\, R$.

(32) For every subset Q of FuzzyLattice X and for every element x of X holds $(\bigsqcup_{\mathrm{FuzzyLattice}\, X} Q)(x) = \bigsqcup_{\mathrm{RealPoset}[0,1]} \pi_x Q$.

(33) Let R be a complete Heyting lattice, X be a subset of R, and y be an element of R. Then $y \sqcap \bigsqcup_R X = \bigsqcup_R \{y \sqcap x; x$ ranges over elements of R: $x \in X\}$.

(34) Let R be a membership function of X, X and Q be a subset of FuzzyLattice$[: X, X :]$. Then $R\,({}^{@}\bigsqcup_{\mathrm{FuzzyLattice}[: X, X :]} Q) = \bigsqcup_{\mathrm{FuzzyLattice}[: X, X :]} \{R\,({}^{@}r); r$ ranges over elements of FuzzyLattice$[: X, X :]$: $r \in Q\}$.

(35) Let R be a membership function of X, X and Q be a subset of FuzzyLattice$[: X, X :]$. Then $({}^{@}\bigsqcup_{\mathrm{FuzzyLattice}[: X, X :]} Q)\, R = \bigsqcup_{\mathrm{FuzzyLattice}[: X, X :]} \{({}^{@}r)\, R; r$ ranges over elements of FuzzyLattice$[: X, X :]$: $r \in Q\}$.

(36) Let R be a membership function of X, X. Then $\mathrm{TrCl}\, R\ \mathrm{TrCl}\, R = \bigsqcup_{\mathrm{FuzzyLattice}[: X, X :]} \{R^i\, R^j; i$ ranges over natural numbers, j ranges over natural numbers: $i > 0 \ \wedge\ j > 0\}$.

Let X be a non empty set and let R be a membership function of X, X. Note that $\mathrm{TrCl}\, R$ is transitive.

We now state four propositions:

(37) Let R be a membership function of X, X and n be a natural number. If R is transitive and $n > 0$, then $R^n \subseteq R$.

(38) For every membership function R of X, X such that R is transitive holds $R = \mathrm{TrCl}\, R$.

(39) For all membership functions R, S of X, X and for every natural number n such that $R \subseteq S$ holds $R^n \subseteq S^n$.

(40) For all membership functions R, S of X, X such that S is transitive and $R \subseteq S$ holds $\mathrm{TrCl}\, R \subseteq S$.

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Received November 23, 2003