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# On the Segmentation of a Simple Closed Curve[1]

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**Summary.** The main goal of the work was to introduce the concept of the segmentation of a simple closed curve into (arbitrary small) arcs. The existence of it has been proved by Yatsuka Nakamura [21]. The concept of the gap of a segmentation is also introduced. It is the smallest distance between disjoint segments in the segmentation. For this purpose, the relationship between segments of an arc [24] and segments on a simple closed curve [21] has been shown.

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The papers [30], [35], [10], [3], [2], [29], [1], [13], [8], [9], [7], [4], [34], [25], [33], [22], [20], [28], [15], [26], [27], [18], [6], [12], [31], [19], [14], [16], [17], [23], [5], [24], [21], [11], and [32] provide the notation and terminology for this paper.

## 1. Preliminaries

The scheme *AndScheme* deals with a non empty set $\mathcal{A}$ and two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

$\{a; a$ ranges over elements of $\mathcal{A} : \mathcal{P}[a] \ \wedge \ \mathcal{Q}[a]\} = \{a_1; a_1$ ranges over elements of $\mathcal{A} : \mathcal{P}[a_1]\} \cap \{a_2; a_2$ ranges over elements of $\mathcal{A} : \mathcal{Q}[a_2]\}$

for all values of the parameters.

For simplicity, we follow the rules: $C$ is a simple closed curve, $p$, $q$ are points of $\mathcal{E}^2_{\mathrm{T}}$, $i$, $j$, $k$, $n$ are natural numbers, and $e$ is a real number.

The following proposition is true

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(1) For all finite non empty subsets $A$, $B$ of $\mathbb{R}$ holds $\min(A \cup B) = \min(\min A, \min B)$.

Let $T$ be a non empty topological space. One can check that there exists a subset of $T$ which is compact and non empty.

Next we state several propositions:

(2) Let $T$ be a non empty topological space, $f$ be a continuous real map of $T$, and $A$ be a compact subset of $T$. Then $f^\circ A$ is compact.

(3) For every compact subset $A$ of $\mathbb{R}$ and for every non empty subset $B$ of $\mathbb{R}$ such that $B \subseteq A$ holds $\inf B \in A$.

(4) Let $A$, $B$ be compact non empty subsets of $\mathcal{E}^n_{\rm T}$, $f$ be a continuous real map of $[\!: \mathcal{E}^n_{\rm T}, \mathcal{E}^n_{\rm T} :\!]$, and $g$ be a real map of $\mathcal{E}^n_{\rm T}$. Suppose that for every point $p$ of $\mathcal{E}^n_{\rm T}$ there exists a subset $G$ of $\mathbb{R}$ such that $G = \{f(p, q); q$ ranges over points of $\mathcal{E}^n_{\rm T}$: $q \in B\}$ and $g(p) = \inf G$. Then $\inf(f^\circ[\!: A, B :\!]) = \inf(g^\circ A)$.

(5) Let $A$, $B$ be compact non empty subsets of $\mathcal{E}^n_{\rm T}$, $f$ be a continuous real map of $[\!: \mathcal{E}^n_{\rm T}, \mathcal{E}^n_{\rm T} :\!]$, and $g$ be a real map of $\mathcal{E}^n_{\rm T}$. Suppose that for every point $q$ of $\mathcal{E}^n_{\rm T}$ there exists a subset $G$ of $\mathbb{R}$ such that $G = \{f(p, q); p$ ranges over points of $\mathcal{E}^n_{\rm T}$: $p \in A\}$ and $g(q) = \inf G$. Then $\inf(f^\circ[\!: A, B :\!]) = \inf(g^\circ B)$.

(6) If $q \in \mathrm{LowerArc}(C)$ and $q \neq \mathrm{W}_{\min}(C)$, then $\mathrm{E}_{\max}(C) \leqslant_C q$.

(7) If $q \in \mathrm{UpperArc}(C)$, then $q \leqslant_C \mathrm{E}_{\max}(C)$.

## 2. The Euclidean Distance

Let us consider $n$. The functor $\mathrm{EuclDist}(n)$ yielding a real map of $[\!: \mathcal{E}^n_{\rm T}, \mathcal{E}^n_{\rm T} :\!]$ is defined as follows:

(Def. 1) For all points $p$, $q$ of $\mathcal{E}^n_{\rm T}$ holds $(\mathrm{EuclDist}(n))(p, q) = |p - q|$.

Let $T$ be a non empty topological space and let $f$ be a real map of $T$. Let us observe that $f$ is continuous if and only if:

(Def. 2) For every point $p$ of $T$ and for every neighbourhood $N$ of $f(p)$ there exists a neighbourhood $V$ of $p$ such that $f^\circ V \subseteq N$.

Let us consider $n$. Note that $\mathrm{EuclDist}(n)$ is continuous.

## 3. On the Distance between Subsets of a Euclidean Space

The following proposition is true

(8) For all non empty compact subsets $A$, $B$ of $\mathcal{E}^n_{\rm T}$ such that $A$ misses $B$ holds $\mathrm{dist}_{\min}(A, B) > 0$.

## 4. On the Segments

The following propositions are true:

(9) If $p \leqslant_C q$ and $q \leqslant_C \mathrm{E}_{\mathrm{max}}(C)$ and $p \neq q$, then $\mathrm{Segment}(p, q, C) = \mathrm{Segment}(\mathrm{UpperArc}(C), \mathrm{W}_{\mathrm{min}}(C), \mathrm{E}_{\mathrm{max}}(C), p, q)$.

(10) If $\mathrm{E}_{\mathrm{max}}(C) \leqslant_C q$, then $\mathrm{Segment}(\mathrm{E}_{\mathrm{max}}(C), q, C) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E}_{\mathrm{max}}(C), \mathrm{W}_{\mathrm{min}}(C), \mathrm{E}_{\mathrm{max}}(C), q)$.

(11) If $\mathrm{E}_{\mathrm{max}}(C) \leqslant_C q$, then $\mathrm{Segment}(q, \mathrm{W}_{\mathrm{min}}(C), C) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E}_{\mathrm{max}}(C), \mathrm{W}_{\mathrm{min}}(C), q, \mathrm{W}_{\mathrm{min}}(C))$.

(12) If $p \leqslant_C q$ and $\mathrm{E}_{\mathrm{max}}(C) \leqslant_C p$, then $\mathrm{Segment}(p, q, C) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E}_{\mathrm{max}}(C), \mathrm{W}_{\mathrm{min}}(C), p, q)$.

(13) If $p \leqslant_C \mathrm{E}_{\mathrm{max}}(C)$ and $\mathrm{E}_{\mathrm{max}}(C) \leqslant_C q$, then $\mathrm{Segment}(p, q, C) = \mathrm{RSegment}(\mathrm{UpperArc}(C), \mathrm{W}_{\mathrm{min}}(C), \mathrm{E}_{\mathrm{max}}(C), p) \cup \mathrm{LSegment}(\mathrm{LowerArc}(C), \mathrm{E}_{\mathrm{max}}(C), \mathrm{W}_{\mathrm{min}}(C), q)$.

(14) If $p \leqslant_C \mathrm{E}_{\mathrm{max}}(C)$, then $\mathrm{Segment}(p, \mathrm{W}_{\mathrm{min}}(C), C) = \mathrm{RSegment}(\mathrm{UpperArc}(C), \mathrm{W}_{\mathrm{min}}(C), \mathrm{E}_{\mathrm{max}}(C), p) \cup \mathrm{LSegment}(\mathrm{LowerArc}(C), \mathrm{E}_{\mathrm{max}}(C), \mathrm{W}_{\mathrm{min}}(C), \mathrm{W}_{\mathrm{min}}(C))$.

(15) $\mathrm{RSegment}(\mathrm{UpperArc}(C), \mathrm{W}_{\mathrm{min}}(C), \mathrm{E}_{\mathrm{max}}(C), p) = \mathrm{Segment}(\mathrm{UpperArc}(C), \mathrm{W}_{\mathrm{min}}(C), \mathrm{E}_{\mathrm{max}}(C), p, \mathrm{E}_{\mathrm{max}}(C))$.

(16) $\mathrm{LSegment}(\mathrm{LowerArc}(C), \mathrm{E}_{\mathrm{max}}(C), \mathrm{W}_{\mathrm{min}}(C), p) = \mathrm{Segment}(\mathrm{LowerArc}(C), \mathrm{E}_{\mathrm{max}}(C), \mathrm{W}_{\mathrm{min}}(C), \mathrm{E}_{\mathrm{max}}(C), p)$.

(17) For every point $p$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in C$ and $p \neq \mathrm{W}_{\mathrm{min}}(C)$ holds $\mathrm{Segment}(p, \mathrm{W}_{\mathrm{min}}(C), C)$ is an arc from $p$ to $\mathrm{W}_{\mathrm{min}}(C)$.

(18) For all points $p$, $q$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \neq q$ and $p \leqslant_C q$ holds $\mathrm{Segment}(p, q, C)$ is an arc from $p$ to $q$.

(19) $C = \mathrm{Segment}(\mathrm{W}_{\mathrm{min}}(C), \mathrm{W}_{\mathrm{min}}(C), C)$.

(20) For every point $q$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $q \in C$ holds $\mathrm{Segment}(q, \mathrm{W}_{\mathrm{min}}(C), C)$ is compact.

(21) For all points $q_1$, $q_2$ of $\mathcal{E}^2_{\mathrm{T}}$ such that $q_1 \leqslant_C q_2$ holds $\mathrm{Segment}(q_1, q_2, C)$ is compact.

## 5. The Concept of a Segmentation

Let us consider $C$. A finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ is said to be a segmentation of $C$ if it satisfies the conditions (Def. 3).

(Def. 3) $\mathrm{It}_1 = \mathrm{W}_{\mathrm{min}}(C)$ and it is one-to-one and $8 \leqslant \mathrm{len}\,\mathrm{it}$ and $\mathrm{rng}\,\mathrm{it} \subseteq C$ and for every natural number $i$ such that $1 \leqslant i$ and $i < \mathrm{len}\,\mathrm{it}$ holds $\mathrm{it}_i \leqslant_C \mathrm{it}_{i+1}$ and for every natural number $i$ such that $1 \leqslant i$ and $i + 1 < \mathrm{len}\,\mathrm{it}$ holds $\mathrm{Segment}(\mathrm{it}_i, \mathrm{it}_{i+1}, C) \cap \mathrm{Segment}(\mathrm{it}_{i+1}, \mathrm{it}_{i+2}, C) =$

$\{\mathrm{it}_{i+1}\}$ and $\mathrm{Segment}(\mathrm{it}_{\mathrm{len}\,\mathrm{it}}, \mathrm{it}_1, C) \cap \mathrm{Segment}(\mathrm{it}_1, \mathrm{it}_2, C) = \{\mathrm{it}_1\}$ and $\mathrm{Segment}(\mathrm{it}_{\mathrm{len}\,\mathrm{it} -' 1}, \mathrm{it}_{\mathrm{len}\,\mathrm{it}}, C) \cap \mathrm{Segment}(\mathrm{it}_{\mathrm{len}\,\mathrm{it}}, \mathrm{it}_1, C) = \{\mathrm{it}_{\mathrm{len}\,\mathrm{it}}\}$ and $\mathrm{Segment}(\mathrm{it}_{\mathrm{len}\,\mathrm{it} -' 1}, \mathrm{it}_{\mathrm{len}\,\mathrm{it}}, C)$ misses $\mathrm{Segment}(\mathrm{it}_1, \mathrm{it}_2, C)$ and for all natural numbers $i$, $j$ such that $1 \leqslant i$ and $i < j$ and $j < \mathrm{len}\,\mathrm{it}$ and $i$ and $j$ are not adjacent holds $\mathrm{Segment}(\mathrm{it}_i, \mathrm{it}_{i+1}, C)$ misses $\mathrm{Segment}(\mathrm{it}_j, \mathrm{it}_{j+1}, C)$ and for every natural number $i$ such that $1 < i$ and $i + 1 < \mathrm{len}\,\mathrm{it}$ holds $\mathrm{Segment}(\mathrm{it}_{\mathrm{len}\,\mathrm{it}}, \mathrm{it}_1, C)$ misses $\mathrm{Segment}(\mathrm{it}_i, \mathrm{it}_{i+1}, C)$.

Let us consider $C$. One can verify that every segmentation of $C$ is non trivial.

One can prove the following proposition

(22) For every segmentation $S$ of $C$ and for every $i$ such that $1 \leqslant i$ and $i \leqslant \mathrm{len}\,S$ holds $S_i \in C$.

## 6. The Segments of a Segmentation

Let us consider $C$, let $i$ be a natural number, and let $S$ be a segmentation of $C$. The functor $\mathrm{Segm}(S, i)$ yields a subset of $\mathcal{E}^2_{\mathrm{T}}$ and is defined by:

(Def. 4) $\mathrm{Segm}(S, i) = \begin{cases} \mathrm{Segment}(S_i, S_{i+1}, C), & \text{if } 1 \leqslant i \text{ and } i < \mathrm{len}\,S, \\ \mathrm{Segment}(S_{\mathrm{len}\,S}, S_1, C), & \text{otherwise.} \end{cases}$

The following proposition is true

(23) For every segmentation $S$ of $C$ such that $i \in \mathrm{dom}\,S$ holds $\mathrm{Segm}(S, i) \subseteq C$.

Let us consider $C$, let $S$ be a segmentation of $C$, and let us consider $i$. Note that $\mathrm{Segm}(S, i)$ is non empty and compact.

We now state several propositions:

(24) For every segmentation $S$ of $C$ and for every $p$ such that $p \in C$ there exists a natural number $i$ such that $i \in \mathrm{dom}\,S$ and $p \in \mathrm{Segm}(S, i)$.

(25) Let $S$ be a segmentation of $C$ and given $i$, $j$. Suppose $1 \leqslant i$ and $i < j$ and $j < \mathrm{len}\,S$ and $i$ and $j$ are not adjacent. Then $\mathrm{Segm}(S, i)$ misses $\mathrm{Segm}(S, j)$.

(26) For every segmentation $S$ of $C$ and for every $j$ such that $1 < j$ and $j < \mathrm{len}\,S -' 1$ holds $\mathrm{Segm}(S, \mathrm{len}\,S)$ misses $\mathrm{Segm}(S, j)$.

(27) Let $S$ be a segmentation of $C$ and given $i$, $j$. Suppose $1 \leqslant i$ and $i < j$ and $j < \mathrm{len}\,S$ and $i$ and $j$ are adjacent. Then $\mathrm{Segm}(S, i) \cap \mathrm{Segm}(S, j) = \{S_{i+1}\}$.

(28) Let $S$ be a segmentation of $C$ and given $i$, $j$. Suppose $1 \leqslant i$ and $i < j$ and $j < \mathrm{len}\,S$ and $i$ and $j$ are adjacent. Then $\mathrm{Segm}(S, i)$ meets $\mathrm{Segm}(S, j)$.

(29) For every segmentation $S$ of $C$ holds $\mathrm{Segm}(S, \mathrm{len}\,S) \cap \mathrm{Segm}(S, 1) = \{S_1\}$.

(30) For every segmentation $S$ of $C$ holds $\mathrm{Segm}(S, \mathrm{len}\,S)$ meets $\mathrm{Segm}(S, 1)$.

(31) For every segmentation $S$ of $C$ holds $\mathrm{Segm}(S, \mathrm{len}\,S) \cap \mathrm{Segm}(S, \mathrm{len}\,S -' 1) = \{S_{\mathrm{len}\,S}\}$.

(32) For every segmentation $S$ of $C$ holds $\mathrm{Segm}(S, \mathrm{len}\,S)$ meets $\mathrm{Segm}(S, \mathrm{len}\,S -' 1)$.

## 7. The Diameter of a Segmentation

Let us consider $n$ and let $C$ be a subset of $\mathcal{E}^n_{\rm T}$. The functor $\emptyset C$ yielding a real number is defined by:

(Def. 5) There exists a subset $W$ of $\mathcal{E}^n$ such that $W = C$ and $\emptyset C = \emptyset W$.

Let us consider $C$ and let $S$ be a segmentation of $C$. The functor $\emptyset S$ yielding a real number is defined as follows:

(Def. 6) There exists a non empty finite subset $S_1$ of $\mathbb{R}$ such that $S_1 = \{\emptyset \operatorname{Segm}(S, i) : i \in \operatorname{dom} S\}$ and $\emptyset S = \max S_1$.

We now state three propositions:

(33) For every segmentation $S$ of $C$ and for every $i$ holds $\emptyset \operatorname{Segm}(S, i) \leqslant \emptyset S$.

(34) For every segmentation $S$ of $C$ and for every real number $e$ such that for every $i$ holds $\emptyset \operatorname{Segm}(S, i) < e$ holds $\emptyset S < e$.

(35) For every real number $e$ such that $e > 0$ there exists a segmentation $S$ of $C$ such that $\emptyset S < e$.

## 8. The Concept of the Gap of a Segmentation

Let us consider $C$ and let $S$ be a segmentation of $C$. The functor $\operatorname{Gap}(S)$ yields a real number and is defined by the condition (Def. 7).

(Def. 7) There exist non empty finite subsets $S_1$, $S_2$ of $\mathbb{R}$ such that $S_1 = \{\operatorname{dist}_{\min}(\operatorname{Segm}(S, i), \operatorname{Segm}(S, j)) : 1 \leqslant i \wedge i < j \wedge j < \operatorname{len} S \wedge i$ and $j$ are not adjacent$\}$ and $S_2 = \{\operatorname{dist}_{\min}(\operatorname{Segm}(S, \operatorname{len} S), \operatorname{Segm}(S, k)) : 1 < k \wedge k < \operatorname{len} S -' 1\}$ and $\operatorname{Gap}(S) = \min(\min S_1, \min S_2)$.

Next we state two propositions:

(36) Let $S$ be a segmentation of $C$. Then there exists a finite non empty subset $F$ of $\mathbb{R}$ such that $F = \{\operatorname{dist}_{\min}(\operatorname{Segm}(S, i), \operatorname{Segm}(S, j)) : 1 \leqslant i \wedge i < j \wedge j \leqslant \operatorname{len} S \wedge \operatorname{Segm}(S, i)$ misses $\operatorname{Segm}(S, j)\}$ and $\operatorname{Gap}(S) = \min F$.

(37) For every segmentation $S$ of $C$ holds $\operatorname{Gap}(S) > 0$.

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