FORMALIZED MATHEMATICS
Volume 11, Number 3, 2003
University of Białystok

On the Two Short Axiomatizations of Ortholattices

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Summary. In the paper, two short axiom systems for Boolean algebras are introduced. In the first section we show that the single axiom (DN_1) proposed in [2] in terms of disjunction and negation characterizes Boolean algebras. To prove that (DN_1) is a single axiom for Robbins algebras (that is, Boolean algebras as well), we use the Otter theorem prover. The second section contains proof that the two classical axioms ($\mathrm{Meredith}_1$), ($\mathrm{Meredith}_2$) proposed by Meredith [3] may also serve as a basis for Boolean algebras. The results will be used to characterize ortholattices.

MML Identifier: `ROBBINS2`.

The terminology and notation used in this paper have been introduced in the following articles: [4], [5], and [1].

1. Single Axiom for Boolean Algebras

Let L be a non empty complemented lattice structure. We say that L satisfies (DN_1) if and only if:

(Def. 1) For all elements x, y, z, u of the carrier of L holds $(((x+y)^{\mathrm{c}}+z)^{\mathrm{c}}+(x+(z^{\mathrm{c}}+(z+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}}=z$.

Let us observe that TrivComplLat satisfies (DN_1) and TrivOrtLat satisfies (DN_1).

Let us observe that there exists a non empty complemented lattice structure which is join-commutative and join-associative and satisfies (DN_1).

Next we state a number of propositions:

[1]This work has been partially supported by CALCULEMUS grant HPRN-CT-2000-00102.

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ISSN 1426–2630

(1) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u, v be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + (((z+u)^{\mathrm{c}} + x)^{\mathrm{c}} + (y^{\mathrm{c}} + (y+v)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = y$.

(2) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + ((z+x)^{\mathrm{c}} + (y^{\mathrm{c}} + (y+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = y$.

(3) Let L be a non empty complemented lattice structure satisfying (DN_1) and x be an element of the carrier of L. Then $((x+x^{\mathrm{c}})^{\mathrm{c}} + x)^{\mathrm{c}} = x^{\mathrm{c}}$.

(4) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + ((z+x)^{\mathrm{c}} + (((y+y^{\mathrm{c}})^{\mathrm{c}} + y)^{\mathrm{c}} + (y+u)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = y$.

(5) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + ((z+x)^{\mathrm{c}} + y)^{\mathrm{c}})^{\mathrm{c}} = y$.

(6) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + (x^{\mathrm{c}} + y)^{\mathrm{c}})^{\mathrm{c}} = y$.

(7) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(((x+y)^{\mathrm{c}} + x)^{\mathrm{c}} + (x+y)^{\mathrm{c}})^{\mathrm{c}} = x$.

(8) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x + ((x+y)^{\mathrm{c}} + x)^{\mathrm{c}})^{\mathrm{c}} = (x+y)^{\mathrm{c}}$.

(9) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(((x+y)^{\mathrm{c}} + z)^{\mathrm{c}} + (x+z)^{\mathrm{c}})^{\mathrm{c}} = z$.

(10) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x + ((y+z)^{\mathrm{c}} + (y+x)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = (y+x)^{\mathrm{c}}$.

(11) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $((((x+y)^{\mathrm{c}} + z)^{\mathrm{c}} + (x^{\mathrm{c}} + y)^{\mathrm{c}})^{\mathrm{c}} + y)^{\mathrm{c}} = (x^{\mathrm{c}} + y)^{\mathrm{c}}$.

(12) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x + ((y+z)^{\mathrm{c}} + (z+x)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = (z+x)^{\mathrm{c}}$.

(13) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z, u be elements of the carrier of L. Then $((x+y)^{\mathrm{c}} + ((z+x)^{\mathrm{c}} + (y^{\mathrm{c}} + (u+y)^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}})^{\mathrm{c}} = y$.

(14) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x+y)^{\mathrm{c}} = (y+x)^{\mathrm{c}}$.

(15) Let L be a non empty complemented lattice structure satisfying (DN_1)

and x, y, z be elements of the carrier of L. Then $(((x+y)^{c}+(y+z)^{c})^{c}+z)^{c} = (y+z)^{c}$.

(16) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $((x+((x+y)^{c}+z)^{c})^{c}+z)^{c} = ((x+y)^{c}+z)^{c}$.

(17) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(((x+y)^{c}+x)^{c}+y)^{c} = (y+y)^{c}$.

(18) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x^{c}+(y+x)^{c})^{c} = x$.

(19) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{c}+y^{c})^{c} = y$.

(20) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x+(y+x^{c})^{c})^{c} = x^{c}$.

(21) Let L be a non empty complemented lattice structure satisfying (DN_1) and x be an element of the carrier of L. Then $(x+x)^{c} = x^{c}$.

(22) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(((x+y)^{c}+x)^{c}+y)^{c} = y^{c}$.

(23) Let L be a non empty complemented lattice structure satisfying (DN_1) and x be an element of the carrier of L. Then $(x^{c})^{c} = x$.

(24) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{c}+x)^{c}+y = (y^{c})^{c}$.

(25) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{c})^{c} = y+x$.

(26) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+((y+z)^{c}+(y+x)^{c})^{c} = ((y+x)^{c})^{c}$.

(27) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $x+y = y+x$.

One can verify that every non empty complemented lattice structure which satisfies (DN_1) is also join-commutative.

Next we state a number of propositions:

(28) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{c}+x)^{c}+y = y$.

(29) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $((x+y)^{c}+y)^{c}+x = x$.

(30) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $x+((y+x)^{c}+y)^{c} = x$.

(31) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x+y^{c})^{c}+(y^{c}+y)^{c} = (x+y^{c})^{c}$.

(32) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x+y)^c+(y+y^c)^c = (x+y)^c$.

(33) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x+y)^c+(y^c+y)^c = (x+y)^c$.

(34) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(((x+y^c)^c)^c+y)^c = (y^c+y)^c$.

(35) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x + y^c + y)^c = (y^c + y)^c$.

(36) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(((x + y^c + z)^c + y)^c + (y^c + y)^c)^c = y$.

(37) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+((y+z)^c+(y+x)^c)^c = y + x$.

(38) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+(y+((z+y)^c+x)^c)^c = (z + y)^c + x$.

(39) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+((y+x)^c+(y+z)^c)^c = y + x$.

(40) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $((x + y)^c + ((x + y)^c + (x + z)^c)^c)^c + y = y$.

(41) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(((x+y^c+z)^c+y)^c)^c = y$.

(42) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x + (y + x^c + z)^c = x$.

(43) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x^c + (y + x + z)^c = x^c$.

(44) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x + y)^c + x = x + y^c$.

(45) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y be elements of the carrier of L. Then $(x + (x + y^c)^c)^c = (x + y)^c$.

(46) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $((x+y)^c+(x+z))^c+y = y$.

(47) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(((x + y)^c + z)^c + (x^c + y)^c)^c + y = ((x^c + y)^c)^c$.

(48) Let L be a non empty complemented lattice structure satisfying (DN_1)

and x, y, z be elements of the carrier of L. Then $(((x+y)^c+z)^c+(x^c+y)^c)^c+y=x^c+y$.

(49) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x^c+(((y+x)^c)^c+(y+z))^c)^c+(y+z)=((y+x)^c)^c+(y+z)$.

(50) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x^c+(y+x+(y+z))^c)^c+(y+z)=((y+x)^c)^c+(y+z)$.

(51) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x^c+(y+x+(y+z))^c)^c+(y+z)=(y+x)+(y+z)$.

(52) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x^c)^c+(y+z)=(y+x)+(y+z)$.

(53) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x+y)+(x+z)=y+(x+z)$.

(54) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x+y)+(x+z)=z+(x+y)$.

(55) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+(y+z)=z+(y+x)$.

(56) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $x+(y+z)=y+(z+x)$.

(57) Let L be a non empty complemented lattice structure satisfying (DN_1) and x, y, z be elements of the carrier of L. Then $(x+y)+z=x+(y+z)$.

Let us observe that every non empty complemented lattice structure which satisfies (DN_1) is also join-associative and every non empty complemented lattice structure which satisfies (DN_1) is also Robbins.

One can prove the following propositions:

(58) Let L be a non empty complemented lattice structure and x, z be elements of the carrier of L. Suppose L is join-commutative, join-associative, and Huntington. Then $(z+x)*(z+x^c)=z$.

(59) Let L be a non empty complemented lattice structure such that L is join-commutative, join-associative, and Robbins. Then L satisfies (DN_1).

Let us mention that every non empty complemented lattice structure which is join-commutative, join-associative, and Robbins satisfies also (DN_1).

Let us observe that there exists a pre-ortholattice which is de Morgan and satisfies (DN_1).

One can verify that every pre-ortholattice which is de Morgan satisfies (DN_1) is also Boolean and every well-complemented pre-ortholattice which is Boolean satisfies also (DN_1).

2. Meredith Two Axioms for Boolean Algebras

Let L be a non empty complemented lattice structure. We say that L satisfies (Meredith$_1$) if and only if:

(Def. 2) For all elements x, y of the carrier of L holds $(x^{\rm c} + y)^{\rm c} + x = x$.

We say that L satisfies (Meredith$_2$) if and only if:

(Def. 3) For all elements x, y, z of the carrier of L holds $(x^{\rm c} + y)^{\rm c} + (z + y) = y + (z + x)$.

Let us note that every non empty complemented lattice structure which satisfies (Meredith$_1$) and (Meredith$_2$) is also join-commutative, join-associative, and Huntington and every non empty complemented lattice structure which is join-commutative, join-associative, and Huntington satisfies also (Meredith$_1$) and (Meredith$_2$).

Let us note that there exists a pre-ortholattice which is de Morgan and satisfies (Meredith$_1$), (Meredith$_2$), and (DN$_1$).

Let us observe that every pre-ortholattice which is de Morgan satisfies (Meredith$_1$) and (Meredith$_2$) is also Boolean and every well-complemented pre-ortholattice which is Boolean satisfies also (Meredith$_1$) and (Meredith$_2$).

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Received June 28, 2003