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Angle and Triangle in Euclidean Topological Space

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Summary. Two transformations between the complex space and 2-dimensional Euclidean topological space are defined. By them, the concept of argument is induced to 2-dimensional vectors using argument of complex number. Similarly, the concept of an angle is introduced using the angle of two complex numbers. The concept of a triangle and related concepts are also defined in n-dimensional Euclidean topological spaces.

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The notation and terminology used in this paper have been introduced in the following articles: [17], [19], [18], [20], [4], [12], [21], [5], [16], [11], [3], [13], [15], [8], [2], [6], [7], [1], [10], [9], and [14].

We follow the rules: z, z_1, z_2 are elements of $\mathbb{C}$, r, r_1, r_2, x_1, x_2 are real numbers, and p, p_1, p_2, p_3, q are points of $\mathcal{E}^2_{\rm T}$.

Let z be an element of $\mathbb{C}$. The functor cpx2euc(z) yielding a point of $\mathcal{E}^2_{\rm T}$ is defined by:

(Def. 1) $\text{cpx2euc}(z) = [\Re(z), \Im(z)]$.

Let p be a point of $\mathcal{E}^2_{\rm T}$. The functor euc2cpx(p) yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 2) $\text{euc2cpx}(p) = p_{\mathbf{1}} + p_{\mathbf{2}} i$.

One can prove the following propositions:

(1) $\text{euc2cpx}(\text{cpx2euc}(z)) = z$.

(2) $\text{cpx2euc}(\text{euc2cpx}(p)) = p$.

(3) For every p there exists z such that $p = \text{cpx2euc}(z)$.

(4) For every z there exists p such that $z = \text{euc2cpx}(p)$.

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(5) For all z_1, z_2 such that $\mathrm{cpx2euc}(z_1) = \mathrm{cpx2euc}(z_2)$ holds $z_1 = z_2$.

(6) For all p_1, p_2 such that $\mathrm{euc2cpx}(p_1) = \mathrm{euc2cpx}(p_2)$ holds $p_1 = p_2$.

(7) $(\mathrm{cpx2euc}(z))_{\mathbf{1}} = \Re(z)$ and $(\mathrm{cpx2euc}(z))_{\mathbf{2}} = \Im(z)$.

(8) $\Re(\mathrm{euc2cpx}(p)) = p_{\mathbf{1}}$ and $\Im(\mathrm{euc2cpx}(p)) = p_{\mathbf{2}}$.

(9) $\mathrm{cpx2euc}(x_1 + x_2 i) = [x_1, x_2]$.

(10) $[\Re(z_1 + z_2), \Im(z_1 + z_2)] = [\Re(z_1) + \Re(z_2), \Im(z_1) + \Im(z_2)]$.

(11) $\mathrm{cpx2euc}(z_1 + z_2) = \mathrm{cpx2euc}(z_1) + \mathrm{cpx2euc}(z_2)$.

(12) $(p_1 + p_2)_{\mathbf{1}} + (p_1 + p_2)_{\mathbf{2}} i = ((p_1)_{\mathbf{1}} + (p_2)_{\mathbf{1}}) + ((p_1)_{\mathbf{2}} + (p_2)_{\mathbf{2}})i$.

(13) $\mathrm{euc2cpx}(p_1 + p_2) = \mathrm{euc2cpx}(p_1) + \mathrm{euc2cpx}(p_2)$.

(14) $[\Re(-z), \Im(-z)] = [-\Re(z), -\Im(z)]$.

(15) $\mathrm{cpx2euc}(-z) = -\mathrm{cpx2euc}(z)$.

(16) $(-p)_{\mathbf{1}} + (-p)_{\mathbf{2}} i = -p_{\mathbf{1}} + (-p_{\mathbf{2}})i$.

(17) $\mathrm{euc2cpx}(-p) = -\mathrm{euc2cpx}(p)$.

(18) $\mathrm{cpx2euc}(z_1 - z_2) = \mathrm{cpx2euc}(z_1) - \mathrm{cpx2euc}(z_2)$.

(19) $\mathrm{euc2cpx}(p_1 - p_2) = \mathrm{euc2cpx}(p_1) - \mathrm{euc2cpx}(p_2)$.

(20) $\mathrm{cpx2euc}(0_{\mathbb{C}}) = 0_{\mathcal{E}^2_{\mathrm{T}}}$.

(21) $\mathrm{euc2cpx}(0_{\mathcal{E}^2_{\mathrm{T}}}) = 0_{\mathbb{C}}$.

(22) If $\mathrm{euc2cpx}(p) = 0_{\mathbb{C}}$, then $p = 0_{\mathcal{E}^2_{\mathrm{T}}}$.

(23) $\mathrm{cpx2euc}((r + 0i) \cdot z) = r \cdot \mathrm{cpx2euc}(z)$.

(24) $(r + 0i) \cdot (r_1 + r_2 i) = r \cdot r_1 + (r \cdot r_2)i$.

(25) $\mathrm{euc2cpx}(r \cdot p) = (r + 0i) \cdot \mathrm{euc2cpx}(p)$.

(26) $|\mathrm{euc2cpx}(p)| = \sqrt{(p_{\mathbf{1}})^{\mathbf{2}} + (p_{\mathbf{2}})^{\mathbf{2}}}$.

(27) For every finite sequence f of elements of $\mathbb{R}$ such that $\mathrm{len}\, f = 2$ holds $|f| = \sqrt{f(1)^{\mathbf{2}} + f(2)^{\mathbf{2}}}$.

(28) For every finite sequence f of elements of $\mathbb{R}$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathrm{len}\, f = 2$ and $p = f$ holds $|p| = |f|$.

(29) $|\mathrm{cpx2euc}(z)| = \sqrt{\Re(z)^{\mathbf{2}} + \Im(z)^{\mathbf{2}}}$.

(30) $|\mathrm{cpx2euc}(z)| = |z|$.

(31) $|\mathrm{euc2cpx}(p)| = |p|$.

Let us consider p. The functor $\mathrm{Arg}\, p$ yields a real number and is defined as follows:

(Def. 3) $\mathrm{Arg}\, p = \mathrm{Arg}\, \mathrm{euc2cpx}(p)$.

We now state a number of propositions:

(32) For every element z of $\mathbb{C}$ and for every p such that $z = \mathrm{euc2cpx}(p)$ or $p = \mathrm{cpx2euc}(z)$ holds $\mathrm{Arg}\, z = \mathrm{Arg}\, p$.

(33) For every p holds $0 \leqslant \mathrm{Arg}\, p$ and $\mathrm{Arg}\, p < 2 \cdot \pi$.

(34) For all real numbers x_1, x_2 and for every p such that $x_1 = |p| \cdot \cos \operatorname{Arg} p$ and $x_2 = |p| \cdot \sin \operatorname{Arg} p$ holds $p = [x_1, x_2]$.

(35) $\operatorname{Arg}(0_{\mathcal{E}^2_{\mathrm{T}}}) = 0$.

(36) For every p such that $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ holds if $\operatorname{Arg} p < \pi$, then $\operatorname{Arg}(-p) = \operatorname{Arg} p + \pi$ and if $\operatorname{Arg} p \geqslant \pi$, then $\operatorname{Arg}(-p) = \operatorname{Arg} p - \pi$.

(37) For every p such that $\operatorname{Arg} p = 0$ holds $p = [|p|, 0]$ and $p_{\mathbf{2}} = 0$.

(38) For every p such that $p \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ holds $\operatorname{Arg} p < \pi$ iff $\operatorname{Arg}(-p) \geqslant \pi$.

(39) For all p_1, p_2 such that $p_1 \neq p_2$ or $p_1 - p_2 \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ holds $\operatorname{Arg}(p_1 - p_2) < \pi$ iff $\operatorname{Arg}(p_2 - p_1) \geqslant \pi$.

(40) For every p holds $\operatorname{Arg} p \in]0, \pi[$ iff $p_{\mathbf{2}} > 0$.

(41) For every p such that $\operatorname{Arg} p \neq 0$ holds $\operatorname{Arg} p < \pi$ iff $\sin \operatorname{Arg} p > 0$.

(42) For all p_1, p_2 such that $\operatorname{Arg} p_1 < \pi$ and $\operatorname{Arg} p_2 < \pi$ holds $\operatorname{Arg}(p_1 + p_2) < \pi$.

Let us consider p_1, p_2, p_3. The functor $\measuredangle(p_1, p_2, p_3)$ yielding a real number is defined as follows:

(Def. 4) $\measuredangle(p_1, p_2, p_3) = \measuredangle(\operatorname{euc2cpx}(p_1), \operatorname{euc2cpx}(p_2), \operatorname{euc2cpx}(p_3))$.

The following propositions are true:

(43) For all p_1, p_2, p_3 holds $0 \leqslant \measuredangle(p_1, p_2, p_3)$ and $\measuredangle(p_1, p_2, p_3) < 2 \cdot \pi$.

(44) For all p_1, p_2, p_3 holds $\measuredangle(p_1, p_2, p_3) = \measuredangle(p_1 - p_2, 0_{\mathcal{E}^2_{\mathrm{T}}}, p_3 - p_2)$.

(45) For all p_1, p_2, p_3 such that $\measuredangle(p_1, p_2, p_3) = 0$ holds $\operatorname{Arg}(p_1 - p_2) = \operatorname{Arg}(p_3 - p_2)$ and $\measuredangle(p_3, p_2, p_1) = 0$.

(46) For all p_1, p_2, p_3 such that $\measuredangle(p_1, p_2, p_3) \neq 0$ holds $\measuredangle(p_3, p_2, p_1) = 2 \cdot \pi - \measuredangle(p_1, p_2, p_3)$.

(47) For all p_1, p_2, p_3 such that $\measuredangle(p_3, p_2, p_1) \neq 0$ holds $\measuredangle(p_3, p_2, p_1) = 2 \cdot \pi - \measuredangle(p_1, p_2, p_3)$.

(48) For all elements x, y of $\mathbb{C}$ holds $\Re((x|y)) = \Re(x) \cdot \Re(y) + \Im(x) \cdot \Im(y)$.

(49) For all elements x, y of $\mathbb{C}$ holds $\Im((x|y)) = -\Re(x) \cdot \Im(y) + \Im(x) \cdot \Re(y)$.

(50) For all p, q holds $|(p, q)| = p_{\mathbf{1}} \cdot q_{\mathbf{1}} + p_{\mathbf{2}} \cdot q_{\mathbf{2}}$.

(51) For all p_1, p_2 holds $|(p_1, p_2)| = \Re((\operatorname{euc2cpx}(p_1) | \operatorname{euc2cpx}(p_2)))$.

(52) For all p_1, p_2, p_3 such that $p_1 \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ and $p_2 \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ holds $|(p_1, p_2)| = 0$ iff $\measuredangle(p_1, 0_{\mathcal{E}^2_{\mathrm{T}}}, p_2) = \frac{\pi}{2}$ or $\measuredangle(p_1, 0_{\mathcal{E}^2_{\mathrm{T}}}, p_2) = \frac{3}{2} \cdot \pi$.

(53) Let given p_1, p_2. Suppose $p_1 \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$ and $p_2 \neq 0_{\mathcal{E}^2_{\mathrm{T}}}$. Then $-(p_1)_{\mathbf{1}} \cdot (p_2)_{\mathbf{2}} + (p_1)_{\mathbf{2}} \cdot (p_2)_{\mathbf{1}} = |p_1| \cdot |p_2|$ or $-(p_1)_{\mathbf{1}} \cdot (p_2)_{\mathbf{2}} + (p_1)_{\mathbf{2}} \cdot (p_2)_{\mathbf{1}} = -|p_1| \cdot |p_2|$ if and only if $\measuredangle(p_1, 0_{\mathcal{E}^2_{\mathrm{T}}}, p_2) = \frac{\pi}{2}$ or $\measuredangle(p_1, 0_{\mathcal{E}^2_{\mathrm{T}}}, p_2) = \frac{3}{2} \cdot \pi$.

(54) For all p_1, p_2, p_3 such that $p_1 \neq p_2$ and $p_3 \neq p_2$ holds $|(p_1 - p_2, p_3 - p_2)| = 0$ iff $\measuredangle(p_1, p_2, p_3) = \frac{\pi}{2}$ or $\measuredangle(p_1, p_2, p_3) = \frac{3}{2} \cdot \pi$.

(55) For all p_1, p_2, p_3 such that $p_1 \neq p_2$ but $p_3 \neq p_2$ but $\measuredangle(p_1, p_2, p_3) = \frac{\pi}{2}$ or $\measuredangle(p_1, p_2, p_3) = \frac{3}{2} \cdot \pi$ holds $|p_1 - p_2|^{\mathbf{2}} + |p_3 - p_2|^{\mathbf{2}} = |p_1 - p_3|^{\mathbf{2}}$.

(56) For all p_1, p_2, p_3 such that $p_2 \neq p_1$ and $p_1 \neq p_3$ and $p_3 \neq p_2$ and $\measuredangle(p_2, p_1, p_3) < \pi$ and $\measuredangle(p_1, p_3, p_2) < \pi$ and $\measuredangle(p_3, p_2, p_1) < \pi$ holds $\measuredangle(p_2, p_1, p_3) + \measuredangle(p_1, p_3, p_2) + \measuredangle(p_3, p_2, p_1) = \pi$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\rm T}$. The functor Triangle(p_1, p_2, p_3) yields a subset of $\mathcal{E}^n_{\rm T}$ and is defined as follows:

(Def. 5) Triangle$(p_1, p_2, p_3) = \mathcal{L}(p_1, p_2) \cup \mathcal{L}(p_2, p_3) \cup \mathcal{L}(p_3, p_1)$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\rm T}$. The functor ClInsideOfTriangle(p_1, p_2, p_3) yields a subset of $\mathcal{E}^n_{\rm T}$ and is defined as follows:

(Def. 6) ClInsideOfTriangle$(p_1, p_2, p_3) = \{p; p$ ranges over points of $\mathcal{E}^n_{\rm T}$: $\bigvee_{a_1, a_2, a_3 : \text{real number}} (0 \leqslant a_1 \wedge 0 \leqslant a_2 \wedge 0 \leqslant a_3 \wedge a_1 + a_2 + a_3 = 1 \wedge p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3)\}$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\rm T}$. The functor InsideOfTriangle(p_1, p_2, p_3) yielding a subset of $\mathcal{E}^n_{\rm T}$ is defined by:

(Def. 7) InsideOfTriangle$(p_1, p_2, p_3) =$ ClInsideOfTriangle$(p_1, p_2, p_3) \setminus$Triangle$(p_1, p_2, p_3)$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\rm T}$. The functor OutsideOfTriangle(p_1, p_2, p_3) yielding a subset of $\mathcal{E}^n_{\rm T}$ is defined by the condition (Def. 8).

(Def. 8) OutsideOfTriangle$(p_1, p_2, p_3) = \{p; p$ ranges over points of $\mathcal{E}^n_{\rm T}$: $\bigvee_{a_1, a_2, a_3 : \text{real number}} ((0 > a_1 \vee 0 > a_2 \vee 0 > a_3) \wedge a_1 + a_2 + a_3 = 1 \wedge p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3)\}$.

Let n be a natural number and let p_1, p_2, p_3 be points of $\mathcal{E}^n_{\rm T}$. The functor plane(p_1, p_2, p_3) yielding a subset of $\mathcal{E}^n_{\rm T}$ is defined as follows:

(Def. 9) plane$(p_1, p_2, p_3) =$ OutsideOfTriangle$(p_1, p_2, p_3) \cup$ClInsideOfTriangle(p_1, p_2, p_3).

One can prove the following propositions:

(57) Let n be a natural number and p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\rm T}$. Suppose $p \in$ plane(p_1, p_2, p_3). Then there exist real numbers a_1, a_2, a_3 such that $a_1 + a_2 + a_3 = 1$ and $p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$.

(58) For every natural number n and for all points p_1, p_2, p_3 of $\mathcal{E}^n_{\rm T}$ holds Triangle$(p_1, p_2, p_3) \subseteq$ ClInsideOfTriangle(p_1, p_2, p_3).

Let n be a natural number and let q_1, q_2 be points of $\mathcal{E}^n_{\rm T}$. We say that q_1, q_2 are lindependent2 if and only if:

(Def. 10) For all real numbers a_1, a_2 such that $a_1 \cdot q_1 + a_2 \cdot q_2 = 0_{\mathcal{E}^n_{\rm T}}$ holds $a_1 = 0$ and $a_2 = 0$.

We introduce q_1, q_2 are ldependent2 as an antonym of q_1, q_2 are lindependent2.

One can prove the following propositions:

(59) Let n be a natural number and q_1, q_2 be points of $\mathcal{E}^n_{\rm T}$. If q_1, q_2 are lindependent2, then $q_1 \neq q_2$ and $q_1 \neq 0_{\mathcal{E}^n_{\rm T}}$ and $q_2 \neq 0_{\mathcal{E}^n_{\rm T}}$.

(60) Let n be a natural number and p_1, p_2, p_3, p_0 be points of $\mathcal{E}^n_{\rm T}$. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p_0 \in$ plane(p_1, p_2, p_3). Then there exist real numbers a_1, a_2, a_3 such that

(i) $p_0 = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$,

(ii) $a_1 + a_2 + a_3 = 1$, and

(iii) for all real numbers b_1, b_2, b_3 such that $p_0 = b_1 \cdot p_1 + b_2 \cdot p_2 + b_3 \cdot p_3$ and $b_1 + b_2 + b_3 = 1$ holds $b_1 = a_1$ and $b_2 = a_2$ and $b_3 = a_3$.

(61) Let n be a natural number and p_1, p_2, p_3, p_0 be points of $\mathcal{E}^n_{\rm T}$. Given real numbers a_1, a_2, a_3 such that $p_0 = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$ and $a_1 + a_2 + a_3 = 1$. Then $p_0 \in$ plane(p_1, p_2, p_3).

(62) Let n be a natural number and p_1, p_2, p_3 be points of $\mathcal{E}^n_{\rm T}$. Then plane$(p_1, p_2, p_3) = \{p; p$ ranges over points of $\mathcal{E}^n_{\rm T}$: $\bigvee_{a_1, a_2, a_3 : \text{real number}} (a_1 + a_2 + a_3 = 1 \ \wedge \ p = a_1 \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3)\}$.

(63) For all p_1, p_2, p_3 such that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 holds plane$(p_1, p_2, p_3) = \mathcal{R}^2$.

Let n be a natural number and let p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\rm T}$. Let us assume that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p \in$ plane(p_1, p_2, p_3). The functor tricord1(p_1, p_2, p_3, p) yields a real number and is defined as follows:

(Def. 11) There exist real numbers a_2, a_3 such that tricord1$(p_1, p_2, p_3, p) + a_2 + a_3 = 1$ and $p =$ tricord1$(p_1, p_2, p_3, p) \cdot p_1 + a_2 \cdot p_2 + a_3 \cdot p_3$.

Let n be a natural number and let p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\rm T}$. Let us assume that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p \in$ plane(p_1, p_2, p_3). The functor tricord2(p_1, p_2, p_3, p) yielding a real number is defined as follows:

(Def. 12) There exist real numbers a_1, a_3 such that $a_1 +$ tricord2$(p_1, p_2, p_3, p) + a_3 = 1$ and $p = a_1 \cdot p_1 +$ tricord2$(p_1, p_2, p_3, p) \cdot p_2 + a_3 \cdot p_3$.

Let n be a natural number and let p_1, p_2, p_3, p be points of $\mathcal{E}^n_{\rm T}$. Let us assume that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 and $p \in$ plane(p_1, p_2, p_3). The functor tricord2(p_1, p_2, p_3, p) yielding a real number is defined as follows:

(Def. 13) There exist real numbers a_1, a_2 such that $a_1 + a_2 +$ tricord2$(p_1, p_2, p_3, p) = 1$ and $p = a_1 \cdot p_1 + a_2 \cdot p_2 +$ tricord2$(p_1, p_2, p_3, p) \cdot p_3$.

Let us consider p_1, p_2, p_3. The functor trcmap1(p_1, p_2, p_3) yielding a map from $\mathcal{E}^2_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ is defined as follows:

(Def. 14) For every p holds (trcmap1$(p_1, p_2, p_3))(p) =$ tricord1(p_1, p_2, p_3, p).

Let us consider p_1, p_2, p_3. The functor trcmap2(p_1, p_2, p_3) yields a map from $\mathcal{E}^2_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ and is defined as follows:

(Def. 15) For every p holds (trcmap2$(p_1, p_2, p_3))(p) =$ tricord2(p_1, p_2, p_3, p).

Let us consider p_1, p_2, p_3. The functor trcmap3(p_1, p_2, p_3) yielding a map from $\mathcal{E}^2_{\rm T}$ into $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def. 16) For every p holds (trcmap3$(p_1, p_2, p_3))(p) =$ tricord2(p_1, p_2, p_3, p).

Next we state several propositions:

(64) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in \mathrm{OutsideOfTriangle}(p_1, p_2, p_3)$ if and only if one of the following conditions is satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) < 0$, or

(ii) $\mathrm{tricord2}(p_1, p_2, p_3, p) < 0$, or

(iii) $\mathrm{tricord2}(p_1, p_2, p_3, p) < 0$.

(65) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in \mathrm{Triangle}(p_1, p_2, p_3)$ if and only if the following conditions are satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) \geqslant 0$,

(ii) $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$,

(iii) $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$, and

(iv) $\mathrm{tricord1}(p_1, p_2, p_3, p) = 0$ or $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$ or $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$.

(66) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in \mathrm{Triangle}(p_1, p_2, p_3)$ if and only if one of the following conditions is satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) = 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$, or

(ii) $\mathrm{tricord1}(p_1, p_2, p_3, p) \geqslant 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$, or

(iii) $\mathrm{tricord1}(p_1, p_2, p_3, p) \geqslant 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) \geqslant 0$ and $\mathrm{tricord2}(p_1, p_2, p_3, p) = 0$.

(67) Let given p_1, p_2, p_3, p. Suppose $p_2 - p_1$, $p_3 - p_1$ are lindependent2. Then $p \in \mathrm{InsideOfTriangle}(p_1, p_2, p_3)$ if and only if the following conditions are satisfied:

(i) $\mathrm{tricord1}(p_1, p_2, p_3, p) > 0$,

(ii) $\mathrm{tricord2}(p_1, p_2, p_3, p) > 0$, and

(iii) $\mathrm{tricord2}(p_1, p_2, p_3, p) > 0$.

(68) For all p_1, p_2, p_3 such that $p_2 - p_1$, $p_3 - p_1$ are lindependent2 holds $\mathrm{InsideOfTriangle}(p_1, p_2, p_3)$ is non empty.

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