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A Representation of Integers by Binary Arithmetics and Addition of Integers

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Summary. In this article, we introduce the new concept of 2's complement representation. Natural numbers that are congruent mod n can be represented by the same n bits binary. Using the concept introduced here, negative numbers that are congruent mod n also can be represented by the same n bit binary. We also show some properties of addition of integers using this concept.

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The articles [16], [20], [2], [3], [12], [11], [10], [9], [17], [13], [14], [6], [7], [1], [15], [18], [4], [21], [8], [5], and [19] provide the notation and terminology for this paper.

1. Preliminaries

We follow the rules: n denotes a non empty natural number, j, k, l, m denote natural numbers, and g, h, i denote integers.

We now state a number of propositions:

(1) If $m > 0$, then $m \cdot 2 \geqslant m + 1$.

(2) For every natural number m holds $2^m \geqslant m$.

(3) For every natural number m holds $\langle\underbrace{0, \ldots, 0}_{m}\rangle + \langle\underbrace{0, \ldots, 0}_{m}\rangle = \langle\underbrace{0, \ldots, 0}_{m}\rangle$.

(4) For every natural number k such that $k \leqslant l$ and $l \leqslant m$ holds $k = l$ or $k + 1 \leqslant l$ and $l \leqslant m$.

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(5) For every non empty natural number n and for all n-tuples x, y of *Boolean* such that $x = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ and $y = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ holds $\mathrm{carry}(x, y) = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$.

(6) For every non empty natural number n and for all n-tuples x, y of *Boolean* such that $x = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ and $y = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ holds $x + y = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$.

(7) For every non empty natural number n and for every n-tuple F of *Boolean* such that $F = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ holds $\mathrm{Intval}(F) = 0$.

(8) If $l + m \leqslant k - 1$, then $l < k$ and $m < k$.

(9) If $g \leqslant h + i$ and $h < 0$ and $i < 0$, then $g < h$ and $g < i$.

(10) If $l + m \leqslant 2^n - 1$, then $\mathrm{add\_ovfl}(n\text{-BinarySequence}(l), n\text{-BinarySequence}(m)) = \mathit{false}$.

(11) For every non empty natural number n and for all natural numbers l, m such that $l + m \leqslant 2^n - 1$ holds $\mathrm{Absval}((n\text{-BinarySequence}(l)) + (n\text{-BinarySequence}(m))) = l + m$.

(12) For every non empty natural number n and for every n-tuple z of *Boolean* such that $z_n = \mathit{true}$ holds $\mathrm{Absval}(z) \geqslant 2^{n -' 1}$.

(13) If $l + m \leqslant 2^{n -' 1} - 1$, then $(\mathrm{carry}(n\text{-BinarySequence}(l), n\text{-BinarySequence}(m)))_n = \mathit{false}$.

(14) For every non empty natural number n such that $l + m \leqslant 2^{n -' 1} - 1$ holds $\mathrm{Intval}((n\text{-BinarySequence}(l)) + (n\text{-BinarySequence}(m))) = l + m$.

(15) For every 1-tuple z of *Boolean* such that $z = \langle \mathit{true} \rangle$ holds $\mathrm{Intval}(z) = -1$.

(16) For every 1-tuple z of *Boolean* such that $z = \langle \mathit{false} \rangle$ holds $\mathrm{Intval}(z) = 0$.

(17) For every boolean set x holds $\mathit{true} \vee x = \mathit{true}$.

(18) For every non empty natural number n holds $0 \leqslant 2^{n -' 1} - 1$ and $-2^{n -' 1} \leqslant 0$.

(19) For all n-tuples x, y of *Boolean* such that $x = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ and $y = \underbrace{\langle 0, \ldots, 0 \rangle}_{n}$ holds x and y are summable.

(20) $i \cdot n \bmod n = 0$.

2. Majorant Power

Let m, j be natural numbers. The functor $\mathrm{MajP}(m, j)$ yielding a natural number is defined as follows:

(Def. 1) $2^{\mathrm{MajP}(m,j)} \geqslant j$ and $\mathrm{MajP}(m,j) \geqslant m$ and for every natural number k such that $2^k \geqslant j$ and $k \geqslant m$ holds $k \geqslant \mathrm{MajP}(m,j)$.

One can prove the following propositions:

(21) If $j \geqslant k$, then $\mathrm{MajP}(m,j) \geqslant \mathrm{MajP}(m,k)$.

(22) If $l \geqslant m$, then $\mathrm{MajP}(l,j) \geqslant \mathrm{MajP}(m,j)$.

(23) If $m \geqslant 1$, then $\mathrm{MajP}(m,1) = m$.

(24) If $j \leqslant 2^m$, then $\mathrm{MajP}(m,j) = m$.

(25) If $j > 2^m$, then $\mathrm{MajP}(m,j) > m$.

3. 2's Complement

Let m be a natural number and let i be an integer.

The functor $\mathrm{2sComplement}(m,i)$ yields a m-tuple of *Boolean* and is defined by:

(Def. 2) $$\mathrm{2sComplement}(m,i) = \begin{cases} m\text{-BinarySequence}(|2^{\mathrm{MajP}(m,|i|)} + i|), & \text{if } i < 0, \\ m\text{-BinarySequence}(|i|), & \text{otherwise.} \end{cases}$$

The following propositions are true:

(26) For every natural number m holds $\mathrm{2sComplement}(m,0) = \langle \underbrace{0,\ldots,0}_{m} \rangle$.

(27) For every integer i such that $i \leqslant 2^{n-'1} - 1$ and $-2^{n-'1} \leqslant i$ holds $\mathrm{Intval}(\mathrm{2sComplement}(n,i)) = i$.

(28) For all integers h, i such that $h \geqslant 0$ and $i \geqslant 0$ or $h < 0$ and $i < 0$ but $h \bmod 2^n = i \bmod 2^n$ holds $\mathrm{2sComplement}(n,h) = \mathrm{2sComplement}(n,i)$.

(29) For all integers h, i such that $h \geqslant 0$ and $i \geqslant 0$ or $h < 0$ and $i < 0$ but $h \equiv i (\mathrm{mod}\, 2^n)$ holds $\mathrm{2sComplement}(n,h) = \mathrm{2sComplement}(n,i)$.

(30) For all natural numbers l, m such that $l \bmod 2^n = m \bmod 2^n$ holds $n\text{-BinarySequence}(l) = n\text{-BinarySequence}(m)$.

(31) For all natural numbers l, m such that $l \equiv m (\mathrm{mod}\, 2^n)$ holds $n\text{-BinarySequence}(l) = n\text{-BinarySequence}(m)$.

(32) For every natural number j such that $1 \leqslant j$ and $j \leqslant n$ holds $(\mathrm{2sComplement}(n+1,i))_j = (\mathrm{2sComplement}(n,i))_j$.

(33) There exists an element x of *Boolean* such that $\mathrm{2sComplement}(m+1,i) = (\mathrm{2sComplement}(m,i)) ^\frown \langle x \rangle$.

(34) There exists an element x of *Boolean* such that $(m+1)\text{-BinarySequence}(l) = (m\text{-BinarySequence}(l)) ^\frown \langle x \rangle$.

(35) Let n be a non empty natural number. Suppose $-2^n \leqslant h + i$ and $h < 0$ and $i < 0$ and $-2^{n-'1} \leqslant h$ and $-2^{n-'1} \leqslant i$. Then $(\mathrm{carry}(\mathrm{2sComplement}(n+1,h), \mathrm{2sComplement}(n+1,i)))_{n+1} = \mathit{true}$.

(36) For every non empty natural number n such that $-2^{n-'1} \leqslant h+i$ and $h+i \leqslant 2^{n-'1}-1$ and $h \geqslant 0$ and $i \geqslant 0$ holds $\mathrm{Intval}(\mathrm{2sComplement}(n,h)+\mathrm{2sComplement}(n,i)) = h+i$.

(37) Let n be a non empty natural number. Suppose $-2^{(n+1)-'1} \leqslant h+i$ and $h+i \leqslant 2^{(n+1)-'1}-1$ and $h<0$ and $i<0$ and $-2^{n-'1} \leqslant h$ and $-2^{n-'1} \leqslant i$. Then $\mathrm{Intval}(\mathrm{2sComplement}(n+1,h)+\mathrm{2sComplement}(n+1,i)) = h+i$.

(38) Let n be a non empty natural number. Suppose that $-2^{n-'1} \leqslant h$ and $h \leqslant 2^{n-'1}-1$ and $-2^{n-'1} \leqslant i$ and $i \leqslant 2^{n-'1}-1$ and $-2^{n-'1} \leqslant h+i$ and $h+i \leqslant 2^{n-'1}-1$ and $h \geqslant 0$ and $i<0$ or $h<0$ and $i \geqslant 0$. Then $\mathrm{Intval}(\mathrm{2sComplement}(n,h)+\mathrm{2sComplement}(n,i)) = h+i$.

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