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Subspaces and Cosets of Subspace of Real Unitary Space

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Summary. In this article, subspace and the coset of subspace of real unitary space are defined. And we discuss some of their fundamental properties.

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The articles [6], [3], [10], [7], [1], [11], [2], [5], [9], [8], and [4] provide the notation and terminology for this paper.

1. Definition and Axioms of the Subspace of Real Unitary Space

Let V be a real unitary space. A real unitary space is said to be a subspace of V if it satisfies the conditions (Def. 1).

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the zero of it $=$ the zero of V,

(iii) the addition of it $=$ (the addition of V)$\restriction$[: the carrier of it, the carrier of it :],

(iv) the external multiplication of it $=$ (the external multiplication of V)$\restriction$[: $\mathbb{R}$, the carrier of it :], and

(v) the scalar product of it $=$ (the scalar product of V)$\restriction$[: the carrier of it, the carrier of it :].

We now state a number of propositions:

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(1) Let V be a real unitary space, W_1, W_2 be subspaces of V, and x be a set. If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(2) For every real unitary space V and for every subspace W of V and for every set x such that $x \in W$ holds $x \in V$.

(3) For every real unitary space V and for every subspace W of V holds every vector of W is a vector of V.

(4) For every real unitary space V and for every subspace W of V holds $0_W = 0_V$.

(5) For every real unitary space V and for all subspaces W_1, W_2 of V holds $0_{(W_1)} = 0_{(W_2)}$.

(6) Let V be a real unitary space, W be a subspace of V, u, v be vectors of V, and w_1, w_2 be vectors of W. If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(7) Let V be a real unitary space, W be a subspace of V, v be a vector of V, w be a vector of W, and a be a real number. If $w = v$, then $a \cdot w = a \cdot v$.

(8) Let V be a real unitary space, W be a subspace of V, v_1, v_2 be vectors of V, and w_1, w_2 be vectors of W. If $w_1 = v_1$ and $w_2 = v_2$, then $(w_1|w_2) = (v_1|v_2)$.

(9) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and w be a vector of W. If $w = v$, then $-v = -w$.

(10) Let V be a real unitary space, W be a subspace of V, u, v be vectors of V, and w_1, w_2 be vectors of W. If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(11) For every real unitary space V and for every subspace W of V holds $0_V \in W$.

(12) For every real unitary space V and for all subspaces W_1, W_2 of V holds $0_{(W_1)} \in W_2$.

(13) For every real unitary space V and for every subspace W of V holds $0_W \in V$.

(14) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $u \in W$ and $v \in W$, then $u + v \in W$.

(15) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $v \in W$, then $a \cdot v \in W$.

(16) For every real unitary space V and for every subspace W of V and for every vector v of V such that $v \in W$ holds $-v \in W$.

(17) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $u \in W$ and $v \in W$, then $u - v \in W$.

(18) Let V be a real unitary space, V_1 be a subset of the carrier of V, D be a non empty set, d_1 be an element of D, A be a binary operation on D, M be a function from $[:\mathbb{R}, D:]$ into D, and S be a function from $[:D, D:]$ into $\mathbb{R}$. Suppose that

(i) $V_1 = D$,
(ii) $d_1 = 0_V$,
(iii) $A = (\text{the addition of } V) \restriction [\!: V_1, V_1 :\!]$,
(iv) $M = (\text{the external multiplication of } V) \restriction [\!: \mathbb{R}, V_1 :\!]$, and
(v) $S = (\text{the scalar product of } V) \restriction [\!: V_1, V_1 :\!]$.

Then $\langle D, d_1, A, M, S \rangle$ is a subspace of V.

(19) Every real unitary space V is a subspace of V.

(20) For all strict real unitary spaces V, X such that V is a subspace of X and X is a subspace of V holds $V = X$.

(21) Let V, X, Y be real unitary spaces. Suppose V is a subspace of X and X is a subspace of Y. Then V is a subspace of Y.

(22) Let V be a real unitary space and W_1, W_2 be subspaces of V. Suppose the carrier of $W_1 \subseteq$ the carrier of W_2. Then W_1 is a subspace of W_2.

(23) Let V be a real unitary space and W_1, W_2 be subspaces of V. Suppose that for every vector v of V such that $v \in W_1$ holds $v \in W_2$. Then W_1 is a subspace of W_2.

Let V be a real unitary space. Observe that there exists a subspace of V which is strict.

Next we state several propositions:

(24) Let V be a real unitary space and W_1, W_2 be strict subspaces of V. If the carrier of $W_1 =$ the carrier of W_2, then $W_1 = W_2$.

(25) Let V be a real unitary space and W_1, W_2 be strict subspaces of V. If for every vector v of V holds $v \in W_1$ iff $v \in W_2$, then $W_1 = W_2$.

(26) Let V be a strict real unitary space and W be a strict subspace of V. If the carrier of $W =$ the carrier of V, then $W = V$.

(27) Let V be a strict real unitary space and W be a strict subspace of V. If for every vector v of V holds $v \in W$ iff $v \in V$, then $W = V$.

(28) Let V be a real unitary space, W be a subspace of V, and V_1 be a subset of the carrier of V. If the carrier of $W = V_1$, then V_1 is linearly closed.

(29) Let V be a real unitary space, W be a subspace of V, and V_1 be a subset of the carrier of V. Suppose $V_1 \neq \emptyset$ and V_1 is linearly closed. Then there exists a strict subspace W of V such that $V_1 =$ the carrier of W.

2. Definition of Zero Subspace and Improper Subspace of Real Unitary Space

Let V be a real unitary space. The functor $\mathbf{0}_V$ yields a strict subspace of V and is defined by:

(Def. 2) The carrier of $\mathbf{0}_V = \{0_V\}$.

Let V be a real unitary space. The functor Ω_V yielding a strict subspace of V is defined by:

(Def. 3) $\Omega_V =$ the unitary space structure of V.

3. Theorems of Zero Subspace and Improper Subspace

Next we state several propositions:

(30) For every real unitary space V and for every subspace W of V holds $\mathbf{0}_W = \mathbf{0}_V$.

(31) For every real unitary space V and for all subspaces W_1, W_2 of V holds $\mathbf{0}_{(W_1)} = \mathbf{0}_{(W_2)}$.

(32) For every real unitary space V and for every subspace W of V holds $\mathbf{0}_W$ is a subspace of V.

(33) For every real unitary space V and for every subspace W of V holds $\mathbf{0}_V$ is a subspace of W.

(34) For every real unitary space V and for all subspaces W_1, W_2 of V holds $\mathbf{0}_{(W_1)}$ is a subspace of W_2.

(35) Every strict real unitary space V is a subspace of Ω_V.

4. The Cosets of Subspace of Real Unitary Space

Let V be a real unitary space, let v be a vector of V, and let W be a subspace of V. The functor $v + W$ yields a subset of the carrier of V and is defined as follows:

(Def. 4) $v + W = \{v + u; u$ ranges over vectors of V: $u \in W\}$.

Let V be a real unitary space and let W be a subspace of V. A subset of the carrier of V is said to be a coset of W if:

(Def. 5) There exists a vector v of V such that it $= v + W$.

5. Theorems of the Cosets

We now state a number of propositions:

(36) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $0_V \in v + W$ if and only if $v \in W$.

(37) For every real unitary space V and for every subspace W of V and for every vector v of V holds $v \in v + W$.

(38) For every real unitary space V and for every subspace W of V holds $0_V + W =$ the carrier of W.

(39) For every real unitary space V and for every vector v of V holds $v+\mathbf{0}_V = \{v\}$.

(40) For every real unitary space V and for every vector v of V holds $v+\Omega_V =$ the carrier of V.

(41) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $0_V \in v + W$ if and only if $v + W =$ the carrier of W.

(42) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $v \in W$ if and only if $v + W =$ the carrier of W.

(43) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $v \in W$, then $a \cdot v + W =$ the carrier of W.

(44) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $a \neq 0$ and $a \cdot v + W =$ the carrier of W, then $v \in W$.

(45) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $v \in W$ if and only if $-v + W =$ the carrier of W.

(46) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in W$ if and only if $v + W = v + u + W$.

(47) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in W$ if and only if $v + W = (v - u) + W$.

(48) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $v \in u + W$ if and only if $u + W = v + W$.

(49) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $v + W = -v + W$ if and only if $v \in W$.

(50) Let V be a real unitary space, W be a subspace of V, and u, v_1, v_2 be vectors of V. If $u \in v_1 + W$ and $u \in v_2 + W$, then $v_1 + W = v_2 + W$.

(51) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $u \in v + W$ and $u \in -v + W$, then $v \in W$.

(52) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $a \neq 1$ and $a \cdot v \in v + W$, then $v \in W$.

(53) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and a be a real number. If $v \in W$, then $a \cdot v \in v + W$.

(54) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then $-v \in v + W$ if and only if $v \in W$.

(55) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u + v \in v + W$ if and only if $u \in W$.

(56) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $v - u \in v + W$ if and only if $u \in W$.

(57) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in v + W$ if and only if there exists a vector v_1 of V such that

$v_1 \in W$ and $u = v + v_1$.

(58) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. Then $u \in v + W$ if and only if there exists a vector v_1 of V such that $v_1 \in W$ and $u = v - v_1$.

(59) Let V be a real unitary space, W be a subspace of V, and v_1, v_2 be vectors of V. Then there exists a vector v of V such that $v_1 \in v + W$ and $v_2 \in v + W$ if and only if $v_1 - v_2 \in W$.

(60) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $v + W = u + W$, then there exists a vector v_1 of V such that $v_1 \in W$ and $v + v_1 = u$.

(61) Let V be a real unitary space, W be a subspace of V, and u, v be vectors of V. If $v + W = u + W$, then there exists a vector v_1 of V such that $v_1 \in W$ and $v - v_1 = u$.

(62) Let V be a real unitary space, W_1, W_2 be strict subspaces of V, and v be a vector of V. Then $v + W_1 = v + W_2$ if and only if $W_1 = W_2$.

(63) Let V be a real unitary space, W_1, W_2 be strict subspaces of V, and u, v be vectors of V. If $v + W_1 = u + W_2$, then $W_1 = W_2$.

(64) Let V be a real unitary space, W be a subspace of V, and C be a coset of W. Then C is linearly closed if and only if $C =$ the carrier of W.

(65) Let V be a real unitary space, W_1, W_2 be strict subspaces of V, C_1 be a coset of W_1, and C_2 be a coset of W_2. If $C_1 = C_2$, then $W_1 = W_2$.

(66) Let V be a real unitary space, W be a subspace of V, C be a coset of W, and v be a vector of V. Then $\{v\}$ is a coset of $\mathbf{0}_V$.

(67) Let V be a real unitary space, W be a subspace of V, V_1 be a subset of the carrier of V, and v be a vector of V. If V_1 is a coset of $\mathbf{0}_V$, then there exists a vector v of V such that $V_1 = \{v\}$.

(68) For every real unitary space V and for every subspace W of V holds the carrier of W is a coset of W.

(69) For every real unitary space V holds the carrier of V is a coset of Ω_V.

(70) Let V be a real unitary space, W be a subspace of V, and V_1 be a subset of the carrier of V. If V_1 is a coset of Ω_V, then $V_1 =$ the carrier of V.

(71) Let V be a real unitary space, W be a subspace of V, and C be a coset of W. Then $0_V \in C$ if and only if $C =$ the carrier of W.

(72) Let V be a real unitary space, W be a subspace of V, C be a coset of W, and u be a vector of V. Then $u \in C$ if and only if $C = u + W$.

(73) Let V be a real unitary space, W be a subspace of V, C be a coset of W, and u, v be vectors of V. If $u \in C$ and $v \in C$, then there exists a vector v_1 of V such that $v_1 \in W$ and $u + v_1 = v$.

(74) Let V be a real unitary space, W be a subspace of V, C be a coset of W,

and u, v be vectors of V. If $u \in C$ and $v \in C$, then there exists a vector v_1 of V such that $v_1 \in W$ and $u - v_1 = v$.

(75) Let V be a real unitary space, W be a subspace of V, and v_1, v_2 be vectors of V. Then there exists a coset C of W such that $v_1 \in C$ and $v_2 \in C$ if and only if $v_1 - v_2 \in W$.

(76) Let V be a real unitary space, W be a subspace of V, u be a vector of V, and B, C be cosets of W. If $u \in B$ and $u \in C$, then $B = C$.

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Operations on Subspaces in Real Unitary Space

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Summary. In this article, we extend an operation of real linear space to real unitary space. We show theorems proved in [8] on real unitary space.

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The terminology and notation used here are introduced in the following articles: [7], [3], [10], [11], [2], [1], [13], [12], [6], [9], [5], and [4].

1. Definitions of Sum and Intersection of Subspaces

Let V be a real unitary space and let W_1, W_2 be subspaces of V. The functor $W_1 + W_2$ yields a strict subspace of V and is defined as follows:

(Def. 1) The carrier of $W_1 + W_2 = \{v + u; v$ ranges over vectors of V, u ranges over vectors of V: $v \in W_1 \ \wedge \ u \in W_2\}$.

Let V be a real unitary space and let W_1, W_2 be subspaces of V. The functor $W_1 \cap W_2$ yields a strict subspace of V and is defined by:

(Def. 2) The carrier of $W_1 \cap W_2 =$ (the carrier of W_1) $\cap$ (the carrier of W_2).

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2. Theorems of Sum and Intersecton of Subspaces

One can prove the following propositions:

(1) Let V be a real unitary space, W_1, W_2 be subspaces of V, and x be a set. Then $x \in W_1 + W_2$ if and only if there exist vectors v_1, v_2 of V such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

(2) Let V be a real unitary space, W_1, W_2 be subspaces of V, and v be a vector of V. If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(3) Let V be a real unitary space, W_1, W_2 be subspaces of V, and x be a set. Then $x \in W_1 \cap W_2$ if and only if $x \in W_1$ and $x \in W_2$.

(4) For every real unitary space V and for every strict subspace W of V holds $W + W = W$.

(5) For every real unitary space V and for all subspaces W_1, W_2 of V holds $W_1 + W_2 = W_2 + W_1$.

(6) For every real unitary space V and for all subspaces W_1, W_2, W_3 of V holds $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(7) Let V be a real unitary space and W_1, W_2 be subspaces of V. Then W_1 is a subspace of $W_1 + W_2$ and W_2 is a subspace of $W_1 + W_2$.

(8) Let V be a real unitary space, W_1 be a subspace of V, and W_2 be a strict subspace of V. Then W_1 is a subspace of W_2 if and only if $W_1 + W_2 = W_2$.

(9) For every real unitary space V and for every strict subspace W of V holds $\mathbf{0}_V + W = W$ and $W + \mathbf{0}_V = W$.

(10) Let V be a real unitary space. Then $\mathbf{0}_V + \Omega_V =$ the unitary space structure of V and $\Omega_V + \mathbf{0}_V =$ the unitary space structure of V.

(11) Let V be a real unitary space and W be a subspace of V. Then $\Omega_V + W =$ the unitary space structure of V and $W + \Omega_V =$ the unitary space structure of V.

(12) For every strict real unitary space V holds $\Omega_V + \Omega_V = V$.

(13) For every real unitary space V and for every strict subspace W of V holds $W \cap W = W$.

(14) For every real unitary space V and for all subspaces W_1, W_2 of V holds $W_1 \cap W_2 = W_2 \cap W_1$.

(15) For every real unitary space V and for all subspaces W_1, W_2, W_3 of V holds $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(16) Let V be a real unitary space and W_1, W_2 be subspaces of V. Then $W_1 \cap W_2$ is a subspace of W_1 and $W_1 \cap W_2$ is a subspace of W_2.

(17) Let V be a real unitary space, W_2 be a subspace of V, and W_1 be a strict subspace of V. Then W_1 is a subspace of W_2 if and only if $W_1 \cap W_2 = W_1$.

(18) For every real unitary space V and for every subspace W of V holds $\mathbf{0}_V \cap W = \mathbf{0}_V$ and $W \cap \mathbf{0}_V = \mathbf{0}_V$.

(19) For every real unitary space V holds $\mathbf{0}_V \cap \Omega_V = \mathbf{0}_V$ and $\Omega_V \cap \mathbf{0}_V = \mathbf{0}_V$.

(20) For every real unitary space V and for every strict subspace W of V holds $\Omega_V \cap W = W$ and $W \cap \Omega_V = W$.

(21) For every strict real unitary space V holds $\Omega_V \cap \Omega_V = V$.

(22) For every real unitary space V and for all subspaces W_1, W_2 of V holds $W_1 \cap W_2$ is a subspace of $W_1 + W_2$.

(23) For every real unitary space V and for every subspace W_1 of V and for every strict subspace W_2 of V holds $W_1 \cap W_2 + W_2 = W_2$.

(24) For every real unitary space V and for every subspace W_1 of V and for every strict subspace W_2 of V holds $W_2 \cap (W_2 + W_1) = W_2$.

(25) For every real unitary space V and for all subspaces W_1, W_2, W_3 of V holds $W_1 \cap W_2 + W_2 \cap W_3$ is a subspace of $W_2 \cap (W_1 + W_3)$.

(26) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a subspace of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(27) For every real unitary space V and for all subspaces W_1, W_2, W_3 of V holds $W_2 + W_1 \cap W_3$ is a subspace of $(W_1 + W_2) \cap (W_2 + W_3)$.

(28) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a subspace of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(29) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a strict subspace of W_3, then $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(30) For every real unitary space V and for all strict subspaces W_1, W_2 of V holds $W_1 + W_2 = W_2$ iff $W_1 \cap W_2 = W_1$.

(31) Let V be a real unitary space, W_1 be a subspace of V, and W_2, W_3 be strict subspaces of V. If W_1 is a subspace of W_2, then $W_1 + W_3$ is a subspace of $W_2 + W_3$.

(32) Let V be a real unitary space and W_1, W_2 be subspaces of V. Then there exists a subspace W of V such that the carrier of $W =$ (the carrier of W_1) $\cup$ (the carrier of W_2) if and only if W_1 is a subspace of W_2 or W_2 is a subspace of W_1.

3. Introduction of a Set of Subspaces of Real Unitary Space

Let V be a real unitary space. The functor Subspaces V yielding a set is defined as follows:

(Def. 3) For every set x holds $x \in$ Subspaces V iff x is a strict subspace of V.

Let V be a real unitary space. Observe that Subspaces V is non empty.

The following proposition is true

(33) For every strict real unitary space V holds $V \in \mathrm{Subspaces}\,V$.

4. Definition of the Direct Sum and Linear Complement of Subspaces

Let V be a real unitary space and let W_1, W_2 be subspaces of V. We say that V is the direct sum of W_1 and W_2 if and only if:

(Def. 4) The unitary space structure of $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

Let V be a real unitary space and let W be a subspace of V. A subspace of V is called a linear complement of W if:

(Def. 5) V is the direct sum of it and W.

Let V be a real unitary space and let W be a subspace of V. Observe that there exists a linear complement of W which is strict.

Next we state two propositions:

(34) Let V be a real unitary space and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Then W_2 is a linear complement of W_1.

(35) Let V be a real unitary space, W be a subspace of V, and L be a linear complement of W. Then V is the direct sum of L and W and the direct sum of W and L.

5. Theorems Concerning the Sum, Linear Complement and Coset of Subspace

The following propositions are true:

(36) Let V be a real unitary space, W be a subspace of V, and L be a linear complement of W. Then $W + L =$ the unitary space structure of V and $L + W =$ the unitary space structure of V.

(37) Let V be a real unitary space, W be a subspace of V, and L be a linear complement of W. Then $W \cap L = \mathbf{0}_V$ and $L \cap W = \mathbf{0}_V$.

(38) Let V be a real unitary space and W_1, W_2 be subspaces of V. If V is the direct sum of W_1 and W_2, then V is the direct sum of W_2 and W_1.

(39) Every real unitary space V is the direct sum of $\mathbf{0}_V$ and Ω_V and the direct sum of Ω_V and $\mathbf{0}_V$.

(40) Let V be a real unitary space, W be a subspace of V, and L be a linear complement of W. Then W is a linear complement of L.

(41) For every real unitary space V holds $\mathbf{0}_V$ is a linear complement of Ω_V and Ω_V is a linear complement of $\mathbf{0}_V$.

(42) Let V be a real unitary space, W_1, W_2 be subspaces of V, C_1 be a coset of W_1, and C_2 be a coset of W_2. If C_1 meets C_2, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(43) Let V be a real unitary space and W_1, W_2 be subspaces of V. Then V is the direct sum of W_1 and W_2 if and only if for every coset C_1 of W_1 and for every coset C_2 of W_2 there exists a vector v of V such that $C_1 \cap C_2 = \{v\}$.

6. Decomposition of a Vector of Real Unitary Space

Next we state three propositions:

(44) Let V be a real unitary space and W_1, W_2 be subspaces of V. Then $W_1 + W_2 =$ the unitary space structure of V if and only if for every vector v of V there exist vectors v_1, v_2 of V such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

(45) Let V be a real unitary space, W_1, W_2 be subspaces of V, and v, v_1, v_2, u_1, u_2 be vectors of V. Suppose V is the direct sum of W_1 and W_2 and $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$. Then $v_1 = u_1$ and $v_2 = u_2$.

(46) Let V be a real unitary space and W_1, W_2 be subspaces of V. Suppose that

(i) $V = W_1 + W_2$, and

(ii) there exists a vector v of V such that for all vectors v_1, v_2, u_1, u_2 of V such that $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$.

Then V is the direct sum of W_1 and W_2.

Let V be a real unitary space, let v be a vector of V, and let W_1, W_2 be subspaces of V. Let us assume that V is the direct sum of W_1 and W_2. The functor $v_{\langle W_1,W_2\rangle}$ yielding an element of $[\!:$the carrier of V, the carrier of $V:\!]$ is defined as follows:

(Def. 6) $v = (v_{\langle W_1,W_2\rangle})_{\mathbf{1}} + (v_{\langle W_1,W_2\rangle})_{\mathbf{2}}$ and $(v_{\langle W_1,W_2\rangle})_{\mathbf{1}} \in W_1$ and $(v_{\langle W_1,W_2\rangle})_{\mathbf{2}} \in W_2$.

We now state several propositions:

(47) Let V be a real unitary space, v be a vector of V, and W_1, W_2 be subspaces of V. If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1,W_2\rangle})_{\mathbf{1}} = (v_{\langle W_2,W_1\rangle})_{\mathbf{2}}$.

(48) Let V be a real unitary space, v be a vector of V, and W_1, W_2 be subspaces of V. If V is the direct sum of W_1 and W_2, then $(v_{\langle W_1,W_2\rangle})_{\mathbf{2}} = (v_{\langle W_2,W_1\rangle})_{\mathbf{1}}$.

(49) Let V be a real unitary space, W be a subspace of V, L be a linear complement of W, v be a vector of V, and t be an element of $[\!:$ the carrier of V, the carrier of $V\,:\!]$. If $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W$ and $t_{\mathbf{2}} \in L$, then $t = v_{\langle W,L \rangle}$.

(50) Let V be a real unitary space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} + (v_{\langle W,L \rangle})_{\mathbf{2}} = v$.

(51) Let V be a real unitary space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} \in W$ and $(v_{\langle W,L \rangle})_{\mathbf{2}} \in L$.

(52) Let V be a real unitary space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{1}} = (v_{\langle L,W \rangle})_{\mathbf{2}}$.

(53) Let V be a real unitary space, W be a subspace of V, L be a linear complement of W, and v be a vector of V. Then $(v_{\langle W,L \rangle})_{\mathbf{2}} = (v_{\langle L,W \rangle})_{\mathbf{1}}$.

7. Introduction of Operations on Set of Subspaces

Let V be a real unitary space. The functor SubJoin V yields a binary operation on Subspaces V and is defined by:

(Def. 7) For all elements A_1, A_2 of Subspaces V and for all subspaces W_1, W_2 of V such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\,V)(A_1, A_2) = W_1 + W_2$.

Let V be a real unitary space. The functor SubMeet V yielding a binary operation on Subspaces V is defined as follows:

(Def. 8) For all elements A_1, A_2 of Subspaces V and for all subspaces W_1, W_2 of V such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}\,V)(A_1, A_2) = W_1 \cap W_2$.

8. Theorems of Functions SubJoin, SubMeet

We now state the proposition

(54) For every real unitary space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is a lattice.

Let V be a real unitary space. Note that $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is lattice-like.

The following propositions are true:

(55) For every real unitary space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is lower-bounded.

(56) For every real unitary space V holds $\langle \mathrm{Subspaces}\,V, \mathrm{SubJoin}\,V, \mathrm{SubMeet}\,V \rangle$ is upper-bounded.

(57) For every real unitary space V holds $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a bound lattice.

(58) For every real unitary space V holds $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is modular.

(59) For every real unitary space V holds $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is complemented.

Let V be a real unitary space.

Observe that $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is lower-bounded, upper-bounded, modular, and complemented.

One can prove the following proposition

(60) Let V be a real unitary space and W_1, W_2, W_3 be strict subspaces of V. If W_1 is a subspace of W_2, then $W_1 \cap W_3$ is a subspace of $W_2 \cap W_3$.

9. Auxiliary Theorems in Real Unitary Space

We now state three propositions:

(61) Let V be a real unitary space and W be a strict subspace of V. Suppose that for every vector v of V holds $v \in W$. Then $W =$ the unitary space structure of V.

(62) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. Then there exists a coset C of W such that $v \in C$.

(63) Let V be a real unitary space, W be a subspace of V, v be a vector of V, and x be a set. Then $x \in v + W$ if and only if there exists a vector u of V such that $u \in W$ and $x = v + u$.

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Linear Combinations in Real Unitary Space

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Summary. In this article, we mainly discuss linear combination of vectors in Real Unitary Space and dimension of the space. As the result, we obtain some theorems that are similar to those in Real Linear Space.

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The articles [11], [5], [16], [2], [17], [1], [3], [4], [15], [10], [6], [14], [13], [9], [12], [8], and [7] provide the terminology and notation for this paper.

1. Definition and Fundamental Properties of Linear Combination

Let V be a real unitary space and let A be a subset of the carrier of V. The functor $\mathrm{Lin}(A)$ yielding a strict subspace of V is defined by:

(Def. 1) The carrier of $\mathrm{Lin}(A) = \{\sum l : l$ ranges over linear combinations of $A\}$.

We now state a number of propositions:

(1) Let V be a real unitary space, A be a subset of the carrier of V, and x be a set. Then $x \in \mathrm{Lin}(A)$ if and only if there exists a linear combination l of A such that $x = \sum l$.

(2) Let V be a real unitary space, A be a subset of the carrier of V, and x be a set. If $x \in A$, then $x \in \mathrm{Lin}(A)$.

(3) For every real unitary space V holds $\mathrm{Lin}(\emptyset_{\text{the carrier of } V}) = \mathbf{0}_V$.

(4) For every real unitary space V and for every subset A of the carrier of V such that $\mathrm{Lin}(A) = \mathbf{0}_V$ holds $A = \emptyset$ or $A = \{0_V\}$.

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(5) Let V be a real unitary space, W be a strict subspace of V, and A be a subset of the carrier of V. If $A =$ the carrier of W, then $\mathrm{Lin}(A) = W$.

(6) Let V be a strict real unitary space and A be a subset of the carrier of V. If $A =$ the carrier of V, then $\mathrm{Lin}(A) = V$.

(7) Let V be a real unitary space and A, B be subsets of the carrier of V. If $A \subseteq B$, then $\mathrm{Lin}(A)$ is a subspace of $\mathrm{Lin}(B)$.

(8) Let V be a strict real unitary space and A, B be subsets of the carrier of V. If $\mathrm{Lin}(A) = V$ and $A \subseteq B$, then $\mathrm{Lin}(B) = V$.

(9) For every real unitary space V and for all subsets A, B of the carrier of V holds $\mathrm{Lin}(A \cup B) = \mathrm{Lin}(A) + \mathrm{Lin}(B)$.

(10) For every real unitary space V and for all subsets A, B of the carrier of V holds $\mathrm{Lin}(A \cap B)$ is a subspace of $\mathrm{Lin}(A) \cap \mathrm{Lin}(B)$.

(11) Let V be a real unitary space and A be a subset of the carrier of V. Suppose A is linearly independent. Then there exists a subset B of the carrier of V such that $A \subseteq B$ and B is linearly independent and $\mathrm{Lin}(B) =$ the unitary space structure of V.

(12) Let V be a real unitary space and A be a subset of the carrier of V. Suppose $\mathrm{Lin}(A) = V$. Then there exists a subset B of the carrier of V such that $B \subseteq A$ and B is linearly independent and $\mathrm{Lin}(B) = V$.

2. Definition of the Basis of Real Unitary Space

Let V be a real unitary space. A subset of the carrier of V is said to be a basis of V if:

(Def. 2) It is linearly independent and $\mathrm{Lin}(\mathrm{it}) =$ the unitary space structure of V.

One can prove the following three propositions:

(13) Let V be a strict real unitary space and A be a subset of the carrier of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

(14) Let V be a real unitary space and A be a subset of the carrier of V. If $\mathrm{Lin}(A) = V$, then there exists a basis I of V such that $I \subseteq A$.

(15) Let V be a real unitary space and A be a subset of V. If A is linearly independent, then there exists a basis I of V such that $A \subseteq I$.

3. Some Theorems of Lin, Sum, Carrier

We now state a number of propositions:

(16) Let V be a real unitary space, L be a linear combination of V, A be a subset of V, and F be a finite sequence of elements of the carrier of V. Suppose $\operatorname{rng} F \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum(L\,F) = \sum K$.

(17) Let V be a real unitary space, L be a linear combination of V, and A be a subset of V. Suppose the support of $L \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum L = \sum K$.

(18) Let V be a real unitary space, W be a subspace of V, and L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Let K be a linear combination of W. Suppose $K = L \restriction$the carrier of W. Then the support of $L =$ the support of K and $\sum L = \sum K$.

(19) Let V be a real unitary space, W be a subspace of V, and K be a linear combination of W. Then there exists a linear combination L of V such that the support of $K =$ the support of L and $\sum K = \sum L$.

(20) Let V be a real unitary space, W be a subspace of V, and L be a linear combination of V. Suppose the support of $L \subseteq$ the carrier of W. Then there exists a linear combination K of W such that the support of $K =$ the support of L and $\sum K = \sum L$.

(21) For every real unitary space V and for every basis I of V and for every vector v of V holds $v \in \operatorname{Lin}(I)$.

(22) Let V be a real unitary space, W be a subspace of V, and A be a subset of W. Suppose A is linearly independent. Then there exists a subset B of V such that B is linearly independent and $B = A$.

(23) Let V be a real unitary space, W be a subspace of V, and A be a subset of V. Suppose A is linearly independent and $A \subseteq$ the carrier of W. Then there exists a subset B of W such that B is linearly independent and $B = A$.

(24) Let V be a real unitary space, W be a subspace of V, and A be a basis of W. Then there exists a basis B of V such that $A \subseteq B$.

(25) Let V be a real unitary space and A be a subset of V. Suppose A is linearly independent. Let v be a vector of V. If $v \in A$, then for every subset B of V such that $B = A \setminus \{v\}$ holds $v \notin \operatorname{Lin}(B)$.

(26) Let V be a real unitary space, I be a basis of V, and A be a non empty subset of V. Suppose A misses I. Let B be a subset of V. If $B = I \cup A$, then B is linearly dependent.

(27) Let V be a real unitary space, W be a subspace of V, and A be a subset of V. If $A \subseteq$ the carrier of W, then $\operatorname{Lin}(A)$ is a subspace of W.

(28) Let V be a real unitary space, W be a subspace of V, A be a subset of V, and B be a subset of W. If $A = B$, then $\mathrm{Lin}(A) = \mathrm{Lin}(B)$.

4. Subspaces of Real Unitary Space Generated by One, Two, or Three Vectors

We now state a number of propositions:

(29) Let V be a real unitary space, v be a vector of V, and x be a set. Then $x \in \mathrm{Lin}(\{v\})$ if and only if there exists a real number a such that $x = a \cdot v$.

(30) For every real unitary space V and for every vector v of V holds $v \in \mathrm{Lin}(\{v\})$.

(31) Let V be a real unitary space, v, w be vectors of V, and x be a set. Then $x \in v + \mathrm{Lin}(\{w\})$ if and only if there exists a real number a such that $x = v + a \cdot w$.

(32) Let V be a real unitary space, w_1, w_2 be vectors of V, and x be a set. Then $x \in \mathrm{Lin}(\{w_1, w_2\})$ if and only if there exist real numbers a, b such that $x = a \cdot w_1 + b \cdot w_2$.

(33) For every real unitary space V and for all vectors w_1, w_2 of V holds $w_1 \in \mathrm{Lin}(\{w_1, w_2\})$ and $w_2 \in \mathrm{Lin}(\{w_1, w_2\})$.

(34) Let V be a real unitary space, v, w_1, w_2 be vectors of V, and x be a set. Then $x \in v + \mathrm{Lin}(\{w_1, w_2\})$ if and only if there exist real numbers a, b such that $x = v + a \cdot w_1 + b \cdot w_2$.

(35) Let V be a real unitary space, v_1, v_2, v_3 be vectors of V, and x be a set. Then $x \in \mathrm{Lin}(\{v_1, v_2, v_3\})$ if and only if there exist real numbers a, b, c such that $x = a \cdot v_1 + b \cdot v_2 + c \cdot v_3$.

(36) For every real unitary space V and for all vectors w_1, w_2, w_3 of V holds $w_1 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$ and $w_2 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$ and $w_3 \in \mathrm{Lin}(\{w_1, w_2, w_3\})$.

(37) Let V be a real unitary space, v, w_1, w_2, w_3 be vectors of V, and x be a set. Then $x \in v + \mathrm{Lin}(\{w_1, w_2, w_3\})$ if and only if there exist real numbers a, b, c such that $x = v + a \cdot w_1 + b \cdot w_2 + c \cdot w_3$.

(38) For every real unitary space V and for all vectors v, w of V such that $v \in \mathrm{Lin}(\{w\})$ and $v \neq 0_V$ holds $\mathrm{Lin}(\{v\}) = \mathrm{Lin}(\{w\})$.

(39) Let V be a real unitary space and v_1, v_2, w_1, w_2 be vectors of V. Suppose $v_1 \neq v_2$ and $\{v_1, v_2\}$ is linearly independent and $v_1 \in \mathrm{Lin}(\{w_1, w_2\})$ and $v_2 \in \mathrm{Lin}(\{w_1, w_2\})$. Then $\mathrm{Lin}(\{w_1, w_2\}) = \mathrm{Lin}(\{v_1, v_2\})$ and $\{w_1, w_2\}$ is linearly independent and $w_1 \neq w_2$.

5. Auxiliary Theorems

We now state several propositions:

(40) For every real unitary space V and for every set x holds $x \in \mathbf{0}_V$ iff $x = 0_V$.

(41) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a subspace of W_3, then $W_1 \cap W_2$ is a subspace of W_3.

(42) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. Suppose W_1 is a subspace of W_2 and a subspace of W_3. Then W_1 is a subspace of $W_2 \cap W_3$.

(43) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. Suppose W_1 is a subspace of W_3 and W_2 is a subspace of W_3. Then W_1+W_2 is a subspace of W_3.

(44) Let V be a real unitary space and W_1, W_2, W_3 be subspaces of V. If W_1 is a subspace of W_2, then W_1 is a subspace of $W_2 + W_3$.

(45) Let V be a real unitary space, W_1, W_2 be subspaces of V, and v be a vector of V. If W_1 is a subspace of W_2, then $v + W_1 \subseteq v + W_2$.

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Dimension of Real Unitary Space

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Summary. In this article we describe the dimension of real unitary space. Most of theorems are restricted from real linear space. In the last section, we introduce affine subset of real unitary space.

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The papers [14], [13], [19], [2], [3], [4], [1], [5], [11], [18], [6], [10], [17], [16], [12], [15], [9], [8], and [7] provide the terminology and notation for this paper.

1. Finite-dimensional Real Unitary Space

One can prove the following two propositions:

(1) Let V be a real unitary space, A, B be finite subsets of V, and v be a vector of V. Suppose $v \in \mathrm{Lin}(A \cup B)$ and $v \notin \mathrm{Lin}(B)$. Then there exists a vector w of V such that $w \in A$ and $w \in \mathrm{Lin}(((A \cup B) \setminus \{w\}) \cup \{v\})$.

(2) Let V be a real unitary space and A, B be finite subsets of V. Suppose the unitary space structure of $V = \mathrm{Lin}(A)$ and B is linearly independent. Then $\overline{\overline{B}} \leqslant \overline{\overline{A}}$ and there exists a finite subset C of V such that $C \subseteq A$ and $\overline{\overline{C}} = \overline{\overline{A}} - \overline{\overline{B}}$ and the unitary space structure of $V = \mathrm{Lin}(B \cup C)$.

Let V be a real unitary space. We say that V is finite dimensional if and only if:

(Def. 1) There exists a finite subset of the carrier of V which is a basis of V.

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Let us mention that there exists a real unitary space which is strict and finite dimensional.

Let V be a real unitary space. Let us observe that V is finite dimensional if and only if:

(Def. 2) There exists a finite subset of V which is a basis of V.

We now state several propositions:

(3) For every real unitary space V such that V is finite dimensional holds every basis of V is finite.

(4) Let V be a real unitary space and A be a subset of V. Suppose V is finite dimensional and A is linearly independent. Then A is finite.

(5) For every real unitary space V and for all bases A, B of V such that V is finite dimensional holds $\overline{\overline{A}} = \overline{\overline{B}}$.

(6) For every real unitary space V holds $\mathbf{0}_V$ is finite dimensional.

(7) Let V be a real unitary space and W be a subspace of V. If V is finite dimensional, then W is finite dimensional.

Let V be a real unitary space. Note that there exists a subspace of V which is finite dimensional and strict.

Let V be a finite dimensional real unitary space. Observe that every subspace of V is finite dimensional.

Let V be a finite dimensional real unitary space. Observe that there exists a subspace of V which is strict.

2. Dimension of Real Unitary Space

Let V be a real unitary space. Let us assume that V is finite dimensional. The functor $\dim(V)$ yielding a natural number is defined by:

(Def. 3) For every basis I of V holds $\dim(V) = \overline{\overline{I}}$.

One can prove the following propositions:

(8) For every finite dimensional real unitary space V and for every subspace W of V holds $\dim(W) \leqslant \dim(V)$.

(9) Let V be a finite dimensional real unitary space and A be a subset of V. If A is linearly independent, then $\overline{\overline{A}} = \dim(\mathrm{Lin}(A))$.

(10) For every finite dimensional real unitary space V holds $\dim(V) = \dim(\Omega_V)$.

(11) Let V be a finite dimensional real unitary space and W be a subspace of V. Then $\dim(V) = \dim(W)$ if and only if $\Omega_V = \Omega_W$.

(12) For every finite dimensional real unitary space V holds $\dim(V) = 0$ iff $\Omega_V = \mathbf{0}_V$.

(13) Let V be a finite dimensional real unitary space. Then $\dim(V) = 1$ if and only if there exists a vector v of V such that $v \neq 0_V$ and $\Omega_V = \mathrm{Lin}(\{v\})$.

(14) Let V be a finite dimensional real unitary space. Then $\dim(V) = 2$ if and only if there exist vectors u, v of V such that $u \neq v$ and $\{u, v\}$ is linearly independent and $\Omega_V = \mathrm{Lin}(\{u, v\})$.

(15) For every finite dimensional real unitary space V and for all subspaces W_1, W_2 of V holds $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$.

(16) For every finite dimensional real unitary space V and for all subspaces W_1, W_2 of V holds $\dim(W_1 \cap W_2) \geqslant (\dim(W_1) + \dim(W_2)) - \dim(V)$.

(17) Let V be a finite dimensional real unitary space and W_1, W_2 be subspaces of V. If V is the direct sum of W_1 and W_2, then $\dim(V) = \dim(W_1) + \dim(W_2)$.

3. Fixed-dimensional Subspace Family

We now state the proposition

(18) Let V be a finite dimensional real unitary space, W be a subspace of V, and n be a natural number. Then $n \leqslant \dim(V)$ if and only if there exists a strict subspace W of V such that $\dim(W) = n$.

Let V be a finite dimensional real unitary space and let n be a natural number. The functor $\mathrm{Sub}_n(V)$ yields a set and is defined as follows:

(Def. 4) For every set x holds $x \in \mathrm{Sub}_n(V)$ iff there exists a strict subspace W of V such that $W = x$ and $\dim(W) = n$.

Next we state three propositions:

(19) Let V be a finite dimensional real unitary space and n be a natural number. If $n \leqslant \dim(V)$, then $\mathrm{Sub}_n(V)$ is non empty.

(20) For every finite dimensional real unitary space V and for every natural number n such that $\dim(V) < n$ holds $\mathrm{Sub}_n(V) = \emptyset$.

(21) Let V be a finite dimensional real unitary space, W be a subspace of V, and n be a natural number. Then $\mathrm{Sub}_n(W) \subseteq \mathrm{Sub}_n(V)$.

4. Affine Set

Let V be a non empty RLS structure and let S be a subset of V. We say that S is Affine if and only if:

(Def. 5) For all vectors x, y of V and for every real number a such that $x \in S$ and $y \in S$ holds $(1 - a) \cdot x + a \cdot y \in S$.

One can prove the following propositions:

(22) For every non empty RLS structure V holds Ω_V is Affine and $\emptyset_V$ is Affine.

(23) For every real linear space-like non empty RLS structure V and for every vector v of V holds $\{v\}$ is Affine.

Let V be a non empty RLS structure. Observe that there exists a subset of V which is non empty and Affine and there exists a subset of V which is empty and Affine.

Let V be a real linear space and let W be a subspace of V. The functor $\mathrm{Up}(W)$ yielding a non empty subset of V is defined by:

(Def. 6) $\mathrm{Up}(W) =$ the carrier of W.

Let V be a real unitary space and let W be a subspace of V. The functor $\mathrm{Up}(W)$ yielding a non empty subset of V is defined by:

(Def. 7) $\mathrm{Up}(W) =$ the carrier of W.

We now state two propositions:

(24) For every real linear space V and for every subspace W of V holds $\mathrm{Up}(W)$ is Affine and $0_V \in$ the carrier of W.

(25) Let V be a real linear space and A be a Affine subset of V. Suppose $0_V \in A$. Let x be a vector of V and a be a real number. If $x \in A$, then $a \cdot x \in A$.

Let V be a non empty RLS structure and let S be a non empty subset of V. We say that S is Subspace-like if and only if the conditions (Def. 8) are satisfied.

(Def. 8)(i) The zero of $V \in S$, and

(ii) for all elements x, y of the carrier of V and for every real number a such that $x \in S$ and $y \in S$ holds $x + y \in S$ and $a \cdot x \in S$.

One can prove the following propositions:

(26) Let V be a real linear space and A be a non empty Affine subset of V. If $0_V \in A$, then A is Subspace-like and $A =$ the carrier of $\mathrm{Lin}(A)$.

(27) For every real linear space V and for every subspace W of V holds $\mathrm{Up}(W)$ is Subspace-like.

(28) For every real linear space V and for every strict subspace W of V holds $W = \mathrm{Lin}(\mathrm{Up}(W))$.

(29) Let V be a real unitary space and A be a non empty Affine subset of V. If $0_V \in A$, then $A =$ the carrier of $\mathrm{Lin}(A)$.

(30) For every real unitary space V and for every subspace W of V holds $\mathrm{Up}(W)$ is Subspace-like.

(31) For every real unitary space V and for every strict subspace W of V holds $W = \mathrm{Lin}(\mathrm{Up}(W))$.

Let V be a non empty loop structure, let M be a subset of the carrier of V, and let v be an element of the carrier of V. The functor $v + M$ yields a subset of V and is defined as follows:

(Def. 9) $v + M = \{v + u; u$ ranges over elements of the carrier of V: $u \in M\}$.

We now state three propositions:

(32) Let V be a real linear space, W be a strict subspace of V, M be a subset of the carrier of V, and v be a vector of V. If $\mathrm{Up}(W) = M$, then $v + W = v + M$.

(33) Let V be an Abelian add-associative real linear space-like non empty RLS structure, M be a Affine subset of V, and v be a vector of V. Then $v + M$ is Affine.

(34) Let V be a real unitary space, W be a strict subspace of V, M be a subset of the carrier of V, and v be a vector of V. If $\mathrm{Up}(W) = M$, then $v + W = v + M$.

Let V be a non empty loop structure and let M, N be subsets of the carrier of V. The functor $M + N$ yields a subset of V and is defined as follows:

(Def. 10) $M + N = \{u + v; u$ ranges over elements of the carrier of V, v ranges over elements of the carrier of V: $u \in M \ \wedge \ v \in N\}$.

We now state the proposition

(35) For every Abelian non empty loop structure V and for all subsets M, N of V holds $N + M = M + N$.

Let V be an Abelian non empty loop structure and let M, N be subsets of V. Let us observe that the functor $M + N$ is commutative.

Next we state four propositions:

(36) Let V be a non empty loop structure, M be a subset of V, and v be an element of the carrier of V. Then $\{v\} + M = v + M$.

(37) Let V be an Abelian add-associative real linear space-like non empty RLS structure, M be a Affine subset of V, and v be a vector of V. Then $\{v\} + M$ is Affine.

(38) For every non empty RLS structure V and for all Affine subsets M, N of V holds $M \cap N$ is Affine.

(39) Let V be an Abelian add-associative real linear space-like non empty RLS structure and M, N be Affine subsets of V. Then $M + N$ is Affine.

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Trigonometric Functions on Complex Space

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Summary. This article describes definitions of sine, cosine, hyperbolic sine and hyperbolic cosine. Some of their basic properties are discussed.

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The notation and terminology used here are introduced in the following papers: [9], [4], [10], [1], [8], [2], [3], [5], [7], [11], and [6].

1. Definitions of Trigonometric Functions

We adopt the following convention: x, y denote elements of $\mathbb{R}$, z, z_1, z_2 denote elements of $\mathbb{C}$, and n denotes a natural number.

The function $\sin_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined by:

(Def. 1) $\sin_{\mathbb{C}}(z) = \frac{\exp(i \cdot z) - \exp(-i \cdot z)}{(2+0i) \cdot i}$.

The function $\cos_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined by:

(Def. 2) $\cos_{\mathbb{C}}(z) = \frac{\exp(i \cdot z) + \exp(-i \cdot z)}{2+0i}$.

The function $\sinh_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined by:

(Def. 3) $\sinh_{\mathbb{C}}(z) = \frac{\exp z - \exp(-z)}{2+0i}$.

The function $\cosh_{\mathbb{C}}$ from $\mathbb{C}$ into $\mathbb{C}$ is defined by:

(Def. 4) $\cosh_{\mathbb{C}}(z) = \frac{\exp z + \exp(-z)}{2+0i}$.

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2. Properties of Trigonometric Functions on Complex Space

We now state a number of propositions:

(1) For every element z of $\mathbb{C}$ holds $\sin_{\mathbb{C}z} \cdot \sin_{\mathbb{C}z} + \cos_{\mathbb{C}z} \cdot \cos_{\mathbb{C}z} = 1_{\mathbb{C}}$.

(2) $-\sin_{\mathbb{C}z} = \sin_{\mathbb{C}-z}$.

(3) $\cos_{\mathbb{C}z} = \cos_{\mathbb{C}-z}$.

(4) $\sin_{\mathbb{C}z_1+z_2} = \sin_{\mathbb{C}z_1} \cdot \cos_{\mathbb{C}z_2} + \cos_{\mathbb{C}z_1} \cdot \sin_{\mathbb{C}z_2}$.

(5) $\sin_{\mathbb{C}z_1-z_2} = \sin_{\mathbb{C}z_1} \cdot \cos_{\mathbb{C}z_2} - \cos_{\mathbb{C}z_1} \cdot \sin_{\mathbb{C}z_2}$.

(6) $\cos_{\mathbb{C}z_1+z_2} = \cos_{\mathbb{C}z_1} \cdot \cos_{\mathbb{C}z_2} - \sin_{\mathbb{C}z_1} \cdot \sin_{\mathbb{C}z_2}$.

(7) $\cos_{\mathbb{C}z_1-z_2} = \cos_{\mathbb{C}z_1} \cdot \cos_{\mathbb{C}z_2} + \sin_{\mathbb{C}z_1} \cdot \sin_{\mathbb{C}z_2}$.

(8) $\cosh_{\mathbb{C}z} \cdot \cosh_{\mathbb{C}z} - \sinh_{\mathbb{C}z} \cdot \sinh_{\mathbb{C}z} = 1_{\mathbb{C}}$.

(9) $-\sinh_{\mathbb{C}z} = \sinh_{\mathbb{C}-z}$.

(10) $\cosh_{\mathbb{C}z} = \cosh_{\mathbb{C}-z}$.

(11) $\sinh_{\mathbb{C}z_1+z_2} = \sinh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_2} + \cosh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_2}$.

(12) $\sinh_{\mathbb{C}z_1-z_2} = \sinh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_2} - \cosh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_2}$.

(13) $\cosh_{\mathbb{C}z_1-z_2} = \cosh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_2} - \sinh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_2}$.

(14) $\cosh_{\mathbb{C}z_1+z_2} = \cosh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_2} + \sinh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_2}$.

(15) $\sin_{\mathbb{C}i \cdot z} = i \cdot \sinh_{\mathbb{C}z}$.

(16) $\cos_{\mathbb{C}i \cdot z} = \cosh_{\mathbb{C}z}$.

(17) $\sinh_{\mathbb{C}i \cdot z} = i \cdot \sin_{\mathbb{C}z}$.

(18) $\cosh_{\mathbb{C}i \cdot z} = \cos_{\mathbb{C}z}$.

(19) For all elements x, y of $\mathbb{R}$ holds $\exp(x + yi) = \exp(x) \cdot \cos(y) + (\exp(x) \cdot \sin(y))i$.

(20) $\exp(0_{\mathbb{C}}) = 1 + 0i$.

(21) $\sin_{\mathbb{C}0_{\mathbb{C}}} = 0_{\mathbb{C}}$.

(22) $\sinh_{\mathbb{C}0_{\mathbb{C}}} = 0_{\mathbb{C}}$.

(23) $\cos_{\mathbb{C}0_{\mathbb{C}}} = 1 + 0i$.

(24) $\cosh_{\mathbb{C}0_{\mathbb{C}}} = 1 + 0i$.

(25) $\exp z = \cosh_{\mathbb{C}z} + \sinh_{\mathbb{C}z}$.

(26) $\exp(-z) = \cosh_{\mathbb{C}z} - \sinh_{\mathbb{C}z}$.

(27) $\exp(z + (2 \cdot \pi + 0i) \cdot i) = \exp z$ and $\exp(z + (0 + (2 \cdot \pi)i)) = \exp z$.

(28) $\exp(0 + (2 \cdot \pi \cdot n)i) = 1 + 0i$ and $\exp((2 \cdot \pi \cdot n + 0i) \cdot i) = 1 + 0i$.

(29) $\exp(0 + (-2 \cdot \pi \cdot n)i) = 1 + 0i$ and $\exp((-2 \cdot \pi \cdot n + 0i) \cdot i) = 1 + 0i$.

(30) $\exp(0 + ((2 \cdot n + 1) \cdot \pi)i) = -1 + 0i$ and $\exp(((2 \cdot n + 1) \cdot \pi + 0i) \cdot i) = -1 + 0i$.

(31) $\exp(0 + (-(2 \cdot n + 1) \cdot \pi)i) = -1 + 0i$ and $\exp((-(2 \cdot n + 1) \cdot \pi + 0i) \cdot i) = -1 + 0i$.

(32) $\exp(0 + ((2 \cdot n + \frac{1}{2}) \cdot \pi)i) = 0 + 1i$ and $\exp(((2 \cdot n + \frac{1}{2}) \cdot \pi + 0i) \cdot i) = 0 + 1i$.

(33) $\exp(0+(-(2\cdot n+\frac{1}{2})\cdot\pi)i)=0+(-1)i$ and $\exp((-(2\cdot n+\frac{1}{2})\cdot\pi+0i)\cdot i)=0+(-1)i$.

(34) $\sin_{\mathbb{C}z+(2\cdot n\cdot\pi+0i)}=\sin_{\mathbb{C}z}$.

(35) $\cos_{\mathbb{C}z+(2\cdot n\cdot\pi+0i)}=\cos_{\mathbb{C}z}$.

(36) $\exp(i\cdot z)=\cos_{\mathbb{C}z}+i\cdot\sin_{\mathbb{C}z}$.

(37) $\exp(-i\cdot z)=\cos_{\mathbb{C}z}-i\cdot\sin_{\mathbb{C}z}$.

(38) For every element x of $\mathbb{R}$ holds $\sin_{\mathbb{C}x+0i}=\sin(x)+0i$.

(39) For every element x of $\mathbb{R}$ holds $\cos_{\mathbb{C}x+0i}=\cos(x)+0i$.

(40) For every element x of $\mathbb{R}$ holds $\sinh_{\mathbb{C}x+0i}=\sinh(x)+0i$.

(41) For every element x of $\mathbb{R}$ holds $\cosh_{\mathbb{C}x+0i}=\cosh(x)+0i$.

(42) For all elements x, y of $\mathbb{R}$ holds $x+yi=(x+0i)+i\cdot(y+0i)$.

(43) $\sin_{\mathbb{C}x+yi}=\sin(x)\cdot\cosh(y)+(\cos(x)\cdot\sinh(y))i$.

(44) $\sin_{\mathbb{C}x+(-y)i}=\sin(x)\cdot\cosh(y)+(-\cos(x)\cdot\sinh(y))i$.

(45) $\cos_{\mathbb{C}x+yi}=\cos(x)\cdot\cosh(y)+(-\sin(x)\cdot\sinh(y))i$.

(46) $\cos_{\mathbb{C}x+(-y)i}=\cos(x)\cdot\cosh(y)+(\sin(x)\cdot\sinh(y))i$.

(47) $\sinh_{\mathbb{C}x+yi}=\sinh(x)\cdot\cos(y)+(\cosh(x)\cdot\sin(y))i$.

(48) $\sinh_{\mathbb{C}x+(-y)i}=\sinh(x)\cdot\cos(y)+(-\cosh(x)\cdot\sin(y))i$.

(49) $\cosh_{\mathbb{C}x+yi}=\cosh(x)\cdot\cos(y)+(\sinh(x)\cdot\sin(y))i$.

(50) $\cosh_{\mathbb{C}x+(-y)i}=\cosh(x)\cdot\cos(y)+(-\sinh(x)\cdot\sin(y))i$.

(51) For every natural number n and for every element z of $\mathbb{C}$ holds $(\cos_{\mathbb{C}z}+i\cdot\sin_{\mathbb{C}z})^n_{\mathbb{N}}=\cos_{\mathbb{C}(n+0i)\cdot z}+i\cdot\sin_{\mathbb{C}(n+0i)\cdot z}$.

(52) For every natural number n and for every element z of $\mathbb{C}$ holds $(\cos_{\mathbb{C}z}-i\cdot\sin_{\mathbb{C}z})^n_{\mathbb{N}}=\cos_{\mathbb{C}(n+0i)\cdot z}-i\cdot\sin_{\mathbb{C}(n+0i)\cdot z}$.

(53) For every natural number n and for every element z of $\mathbb{C}$ holds $\exp(i\cdot(n+0i)\cdot z)=(\cos_{\mathbb{C}z}+i\cdot\sin_{\mathbb{C}z})^n_{\mathbb{N}}$.

(54) For every natural number n and for every element z of $\mathbb{C}$ holds $\exp(-i\cdot(n+0i)\cdot z)=(\cos_{\mathbb{C}z}-i\cdot\sin_{\mathbb{C}z})^n_{\mathbb{N}}$.

(55) For all elements x, y of $\mathbb{R}$ holds $\frac{1+(-1)i}{2+0i}\cdot\sinh_{\mathbb{C}x+yi}+\frac{1+1i}{2+0i}\cdot\sinh_{\mathbb{C}x+(-y)i}=(\sinh(x)\cdot\cos(y)+\cosh(x)\cdot\sin(y))+0i$.

(56) For all elements x, y of $\mathbb{R}$ holds $\frac{1+(-1)i}{2+0i}\cdot\cosh_{\mathbb{C}x+yi}+\frac{1+1i}{2+0i}\cdot\cosh_{\mathbb{C}x+(-y)i}=(\sinh(x)\cdot\sin(y)+\cosh(x)\cdot\cos(y))+0i$.

(57) $\sinh_{\mathbb{C}z}\cdot\sinh_{\mathbb{C}z}=\frac{\cosh_{\mathbb{C}(2+0i)\cdot z}-(1+0i)}{2+0i}$.

(58) $\cosh_{\mathbb{C}z}\cdot\cosh_{\mathbb{C}z}=\frac{\cosh_{\mathbb{C}(2+0i)\cdot z}+(1+0i)}{2+0i}$.

(59) $\sinh_{\mathbb{C}(2+0i)\cdot z}=(2+0i)\cdot\sinh_{\mathbb{C}z}\cdot\cosh_{\mathbb{C}z}$ and $\cosh_{\mathbb{C}(2+0i)\cdot z}=(2+0i)\cdot\cosh_{\mathbb{C}z}\cdot\cosh_{\mathbb{C}z}-(1+0i)$.

(60) $\sinh_{\mathbb{C}z_1}\cdot\sinh_{\mathbb{C}z_1}-\sinh_{\mathbb{C}z_2}\cdot\sinh_{\mathbb{C}z_2}=\sinh_{\mathbb{C}z_1+z_2}\cdot\sinh_{\mathbb{C}z_1-z_2}$ and $\cosh_{\mathbb{C}z_1}\cdot\cosh_{\mathbb{C}z_1}-\cosh_{\mathbb{C}z_2}\cdot\cosh_{\mathbb{C}z_2}=\sinh_{\mathbb{C}z_1+z_2}\cdot\sinh_{\mathbb{C}z_1-z_2}$ and

$\sinh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_1} - \sinh_{\mathbb{C}z_2} \cdot \sinh_{\mathbb{C}z_2} = \cosh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_1} - \cosh_{\mathbb{C}z_2} \cdot \cosh_{\mathbb{C}z_2}$.

(61) $\cosh_{\mathbb{C}z_1+z_2} \cdot \cosh_{\mathbb{C}z_1-z_2} = \sinh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_1} + \cosh_{\mathbb{C}z_2} \cdot \cosh_{\mathbb{C}z_2}$ and $\cosh_{\mathbb{C}z_1+z_2} \cdot \cosh_{\mathbb{C}z_1-z_2} = \cosh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_1} + \sinh_{\mathbb{C}z_2} \cdot \sinh_{\mathbb{C}z_2}$ and $\sinh_{\mathbb{C}z_1} \cdot \sinh_{\mathbb{C}z_1} + \cosh_{\mathbb{C}z_2} \cdot \cosh_{\mathbb{C}z_2} = \cosh_{\mathbb{C}z_1} \cdot \cosh_{\mathbb{C}z_1} + \sinh_{\mathbb{C}z_2} \cdot \sinh_{\mathbb{C}z_2}$.

(62) $\sinh_{\mathbb{C}(2+0i)\cdot z_1} + \sinh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i) \cdot \sinh_{\mathbb{C}z_1+z_2} \cdot \cosh_{\mathbb{C}z_1-z_2}$ and $\sinh_{\mathbb{C}(2+0i)\cdot z_1} - \sinh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i) \cdot \sinh_{\mathbb{C}z_1-z_2} \cdot \cosh_{\mathbb{C}z_1+z_2}$.

(63) $\cosh_{\mathbb{C}(2+0i)\cdot z_1} + \cosh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i) \cdot \cosh_{\mathbb{C}z_1+z_2} \cdot \cosh_{\mathbb{C}z_1-z_2}$ and $\cosh_{\mathbb{C}(2+0i)\cdot z_1} - \cosh_{\mathbb{C}(2+0i)\cdot z_2} = (2+0i) \cdot \sinh_{\mathbb{C}z_1+z_2} \cdot \sinh_{\mathbb{C}z_1-z_2}$.

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Topology of Real Unitary Space

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Summary. In this article we introduce three subjects in real unitary space: parallelism of subsets, orthogonality of subsets and topology of the space. In particular, to introduce the topology of real unitary space, we discuss the metric topology which is induced by the inner product in the space. As the result, we are able to discuss some topological subjects on real unitary space.

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The articles [8], [12], [3], [5], [4], [11], [10], [9], [6], [7], [2], and [1] provide the terminology and notation for this paper.

1. Parallelism of Subspaces

Let V be a non empty RLS structure and let M, N be Affine subsets of V. We say that M is parallel to N if and only if:

(Def. 1) There exists a vector v of V such that $M = N + \{v\}$.

One can prove the following propositions:

(1) For every right zeroed non empty RLS structure V holds every Affine subset M of V is parallel to M.

(2) Let V be an add-associative right zeroed right complementable non empty RLS structure and M, N be Affine subsets of V. If M is parallel to N, then N is parallel to M.

(3) Let V be an Abelian add-associative right zeroed right complementable non empty RLS structure and M, L, N be Affine subsets of V. If M is parallel to L and L is parallel to N, then M is parallel to N.

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Let V be a non empty loop structure and let M, N be subsets of the carrier of V. The functor $M - N$ yields a subset of V and is defined as follows:

(Def. 2) $M - N = \{u - v; u$ ranges over elements of the carrier of V, v ranges over elements of the carrier of V: $u \in M \ \wedge \ v \in N\}$.

Next we state a number of propositions:

(4) For every real linear space V and for all Affine subsets M, N of V holds $M - N$ is Affine.

(5) For every non empty loop structure V and for all subsets M, N of V such that M is empty or N is empty holds $M + N$ is empty.

(6) For every non empty loop structure V and for all non empty subsets M, N of V holds $M + N$ is non empty.

(7) For every non empty loop structure V and for all subsets M, N of V such that M is empty or N is empty holds $M - N$ is empty.

(8) For every non empty loop structure V and for all non empty subsets M, N of V holds $M - N$ is non empty.

(9) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure, M, N be subsets of V, and v be an element of the carrier of V. Then $M = N + \{v\}$ if and only if $M - \{v\} = N$.

(10) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure, M, N be subsets of V, and v be an element of the carrier of V. If $v \in N$, then $M + \{v\} \subseteq M + N$.

(11) Let V be an Abelian add-associative right zeroed right complementable non empty loop structure, M, N be subsets of V, and v be an element of the carrier of V. If $v \in N$, then $M - \{v\} \subseteq M - N$.

(12) For every real linear space V and for every non empty subset M of V holds $0_V \in M - M$.

(13) Let V be a real linear space, M be a non empty Affine subset of V, and v be a vector of V. If M is Subspace-like and $v \in M$, then $M + \{v\} \subseteq M$.

(14) Let V be a real linear space, M be a non empty Affine subset of V, and N_1, N_2 be non empty Affine subsets of V. Suppose N_1 is Subspace-like and N_2 is Subspace-like and M is parallel to N_1 and parallel to N_2. Then $N_1 = N_2$.

(15) Let V be a real linear space, M be a non empty Affine subset of V, and v be a vector of V. If $v \in M$, then $0_V \in M - \{v\}$.

(16) Let V be a real linear space, M be a non empty Affine subset of V, and v be a vector of V. Suppose $v \in M$. Then there exists a non empty Affine subset N of V such that $N = M - \{v\}$ and M is parallel to N and N is Subspace-like.

(17) Let V be a real linear space, M be a non empty Affine subset of V, and

u, v be vectors of V. If $u \in M$ and $v \in M$, then $M - \{v\} = M - \{u\}$.

(18) For every real linear space V and for every non empty Affine subset M of V holds $M - M = \bigcup\{M - \{v\}; v$ ranges over vectors of V: $v \in M\}$.

(19) Let V be a real linear space, M be a non empty Affine subset of V, and v be a vector of V. If $v \in M$, then $M - \{v\} = \bigcup\{M - \{u\}; u$ ranges over vectors of V: $u \in M\}$.

(20) Let V be a real linear space and M be a non empty Affine subset of V. Then there exists a non empty Affine subset L of V such that $L = M - M$ and L is Subspace-like and M is parallel to L.

2. Orthogonality

Let V be a real unitary space and let W be a subspace of V. The functor Ort_Comp W yielding a strict subspace of V is defined by:

(Def. 3) The carrier of Ort_Comp $W = \{v; v$ ranges over vectors of V: $\bigwedge_{w\,:\,\text{vector of } V} (w \in W \Rightarrow w,\ v$ are orthogonal$)\}$.

Let V be a real unitary space and let M be a non empty subset of V. The functor Ort_Comp M yielding a strict subspace of V is defined by:

(Def. 4) The carrier of Ort_Comp $M = \{v; v$ ranges over vectors of V: $\bigwedge_{w\,:\,\text{vector of } V} (w \in M \Rightarrow w,\ v$ are orthogonal$)\}$.

One can prove the following propositions:

(21) For every real unitary space V and for every subspace W of V holds $0_V \in$ Ort_Comp W.

(22) For every real unitary space V holds Ort_Comp $\mathbf{0}_V = \Omega_V$.

(23) For every real unitary space V holds Ort_Comp $\Omega_V = \mathbf{0}_V$.

(24) Let V be a real unitary space, W be a subspace of V, and v be a vector of V. If $v \neq 0_V$, then if $v \in W$, then $v \notin$ Ort_Comp W.

(25) For every real unitary space V and for every non empty subset M of V holds $M \subseteq$ the carrier of Ort_Comp Ort_Comp M.

(26) Let V be a real unitary space and M, N be non empty subsets of V. If $M \subseteq N$, then the carrier of Ort_Comp $N \subseteq$ the carrier of Ort_Comp M.

(27) Let V be a real unitary space, W be a subspace of V, and M be a non empty subset of V. If $M =$ the carrier of W, then Ort_Comp $M =$ Ort_Comp W.

(28) For every real unitary space V and for every non empty subset M of V holds Ort_Comp $M =$ Ort_Comp Ort_Comp Ort_Comp M.

(29) Let V be a real unitary space and x, y be vectors of V. Then $\|x+y\|^{\mathbf{2}} = \|x\|^{\mathbf{2}} + 2 \cdot (x|y) + \|y\|^{\mathbf{2}}$ and $\|x-y\|^{\mathbf{2}} = (\|x\|^{\mathbf{2}} - 2 \cdot (x|y)) + \|y\|^{\mathbf{2}}$.

(30) Let V be a real unitary space and x, y be vectors of V. If x, y are orthogonal, then $\|x+y\|^{\mathbf{2}} = \|x\|^{\mathbf{2}} + \|y\|^{\mathbf{2}}$.

(31) For every real unitary space V and for all vectors x, y of V holds $\|x+y\|^{\mathbf{2}} + \|x-y\|^{\mathbf{2}} = 2 \cdot \|x\|^{\mathbf{2}} + 2 \cdot \|y\|^{\mathbf{2}}$.

(32) Let V be a real unitary space and v be a vector of V. Then there exists a subspace W of V such that the carrier of $W = \{u; u$ ranges over vectors of V: $(u|v) = 0\}$.

3. Topology of Real Unitary Space

The scheme *SubFamExU* deals with a unitary space structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a family F of subsets of $\mathcal{A}$ such that for every subset B of the carrier of $\mathcal{A}$ holds $B \in F$ iff $\mathcal{P}[B]$

for all values of the parameters.

Let V be a real unitary space. The open set family of V yields a family of subsets of V and is defined by the condition (Def. 5).

(Def. 5) Let M be a subset of the carrier of V. Then $M \in$ the open set family of V if and only if for every point x of V such that $x \in M$ there exists a real number r such that $r > 0$ and $\mathrm{Ball}(x,r) \subseteq M$.

Next we state several propositions:

(33) Let V be a real unitary space, v be a point of V, and r, p be real numbers. If $r \leqslant p$, then $\mathrm{Ball}(v,r) \subseteq \mathrm{Ball}(v,p)$.

(34) Let V be a real unitary space and v be a point of V. Then there exists a real number r such that $r > 0$ and $\mathrm{Ball}(v,r) \subseteq$ the carrier of V.

(35) Let V be a real unitary space, v, u be points of V, and r be a real number. If $u \in \mathrm{Ball}(v,r)$, then there exists a real number p such that $p > 0$ and $\mathrm{Ball}(u,p) \subseteq \mathrm{Ball}(v,r)$.

(36) Let V be a real unitary space, u, v, w be points of V, and r, p be real numbers. If $v \in \mathrm{Ball}(u,r) \cap \mathrm{Ball}(w,p)$, then there exists a real number q such that $\mathrm{Ball}(v,q) \subseteq \mathrm{Ball}(u,r)$ and $\mathrm{Ball}(v,q) \subseteq \mathrm{Ball}(w,p)$.

(37) Let V be a real unitary space, v be a point of V, and r be a real number. Then $\mathrm{Ball}(v,r) \in$ the open set family of V.

(38) For every real unitary space V holds the carrier of $V \in$ the open set family of V.

(39) Let V be a real unitary space and M, N be subsets of the carrier of V. Suppose $M \in$ the open set family of V and $N \in$ the open set family of V. Then $M \cap N \in$ the open set family of V.

(40) Let V be a real unitary space and A be a family of subsets of the carrier of V. Suppose $A \subseteq$ the open set family of V. Then $\bigcup A \in$ the open set family of V.

(41) For every real unitary space V holds $\langle$the carrier of V, the open set family of $V\rangle$ is a topological space.

Let V be a real unitary space. The functor TopUnitSpace V yields a topological structure and is defined by:

(Def. 6) TopUnitSpace $V = \langle$the carrier of V, the open set family of $V\rangle$.

Let V be a real unitary space. Note that TopUnitSpace V is topological space-like.

Let V be a real unitary space. One can verify that TopUnitSpace V is non empty.

We now state a number of propositions:

(42) For every real unitary space V and for every subset M of TopUnitSpace V such that $M = \Omega_V$ holds M is open and closed.

(43) For every real unitary space V and for every subset M of TopUnitSpace V such that $M = \emptyset_V$ holds M is open and closed.

(44) Let V be a real unitary space, v be a vector of V, and r be a real number. If the carrier of $V = \{0_V\}$ and $r \neq 0$, then Sphere(v, r) is empty.

(45) Let V be a real unitary space, v be a vector of V, and r be a real number. If the carrier of $V \neq \{0_V\}$ and $r > 0$, then Sphere(v, r) is non empty.

(46) Let V be a real unitary space, v be a vector of V, and r be a real number. If $r = 0$, then Ball(v, r) is empty.

(47) Let V be a real unitary space, v be a vector of V, and r be a real number. If the carrier of $V = \{0_V\}$ and $r > 0$, then Ball$(v, r) = \{0_V\}$.

(48) Let V be a real unitary space, v be a vector of V, and r be a real number. Suppose the carrier of $V \neq \{0_V\}$ and $r > 0$. Then there exists a vector w of V such that $w \neq v$ and $w \in$ Ball(v, r).

(49) Let V be a real unitary space. Then the carrier of $V =$ the carrier of TopUnitSpace V and the topology of TopUnitSpace $V =$ the open set family of V.

(50) Let V be a real unitary space, M be a subset of TopUnitSpace V, r be a real number, and v be a point of V. If $M =$ Ball(v, r), then M is open.

(51) Let V be a real unitary space and M be a subset of TopUnitSpace V. Then M is open if and only if for every point v of V such that $v \in M$ there exists a real number r such that $r > 0$ and Ball$(v, r) \subseteq M$.

(52) Let V be a real unitary space, v_1, v_2 be points of V, and r_1, r_2 be real numbers. Then there exists a point u of V and there exists a real number r such that Ball$(v_1, r_1) \cup$ Ball$(v_2, r_2) \subseteq$ Ball(u, r).

(53) Let V be a real unitary space, M be a subset of TopUnitSpace V, v be a vector of V, and r be a real number. If $M = \overline{\mathrm{Ball}}(v, r)$, then M is closed.

(54) Let V be a real unitary space, M be a subset of TopUnitSpace V, v be a vector of V, and r be a real number. If $M = \mathrm{Sphere}(v, r)$, then M is closed.

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Armstrong's Axioms[1]

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Summary. We present a formalization of the seminal paper by W. W. Armstrong [1] on functional dependencies in relational data bases. The paper is formalized in its entirety including examples and applications. The formalization was done with a routine effort albeit some new notions were defined which simplified formulation of some theorems and proofs.

The definitive reference to the theory of relational databases is [15], where saturated sets are called closed sets. Armstrong's "axioms" for functional dependencies are still widely taught at all levels of database design, see for instance [13].

MML Identifier: `ARMSTRNG`.

The articles [21], [10], [28], [11], [24], [30], [32], [31], [18], [3], [9], [7], [26], [22], [4], [23], [25], [14], [20], [2], [5], [29], [8], [6], [17], [16], [27], [19], and [12] provide the notation and terminology for this paper.

1. Preliminaries

The following proposition is true

(1) Let B be a set. Suppose B is $\cap$-closed. Let X be a set and S be a finite family of subsets of X. If $X \in B$ and $S \subseteq B$, then $\mathrm{Intersect}(S) \in B$.

Let us observe that there exists a binary relation which is reflexive, antisymmetric, transitive, and non empty.

One can prove the following proposition

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(2) Let R be an antisymmetric transitive non empty binary relation and X be a finite subset of field R. If $X \neq \emptyset$, then there exists an element of X which is maximal w.r.t. X, R.

Let X be a set and let R be a binary relation. The functor $\mathrm{Maximals}_R(X)$ yields a subset of X and is defined by:

(Def. 1) For every set x holds $x \in \mathrm{Maximals}_R(X)$ iff x is maximal w.r.t. X, R.

Let x, X be sets. We say that x is $\cap$-irreducible in X if and only if:

(Def. 2) $x \in X$ and for all sets z, y such that $z \in X$ and $y \in X$ and $x = z \cap y$ holds $x = z$ or $x = y$.

We introduce x is $\cap$-reducible in X as an antonym of x is $\cap$-irreducible in X.

Let X be a non empty set. The functor $\cap$-Irreducibles(X) yields a subset of X and is defined by:

(Def. 3) For every set x holds $x \in \cap$-Irreducibles(X) iff x is $\cap$-irreducible in X.

The scheme *FinIntersect* deals with a non empty finite set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every set x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x]$

provided the parameters meet the following requirements:

- For every set x such that x is $\cap$-irreducible in $\mathcal{A}$ holds $\mathcal{P}[x]$, and
- For all sets x, y such that $x \in \mathcal{A}$ and $y \in \mathcal{A}$ and $\mathcal{P}[x]$ and $\mathcal{P}[y]$ holds $\mathcal{P}[x \cap y]$.

Next we state the proposition

(3) Let X be a non empty finite set and x be an element of X. Then there exists a non empty subset A of X such that $x = \bigcap A$ and for every set s such that $s \in A$ holds s is $\cap$-irreducible in X.

Let X be a set and let B be a family of subsets of X. We say that B is (B1) if and only if:

(Def. 4) $X \in B$.

Let B be a set. We introduce B is (B2) as a synonym of B is $\cap$-closed.

Let X be a set. Observe that there exists a family of subsets of X which is (B1) and (B2).

The following proposition is true

(4) Let X be a set and B be a non empty family of subsets of X. Suppose B is $\cap$-closed. Let x be an element of B. Suppose x is $\cap$-irreducible in B and $x \neq X$. Let S be a finite family of subsets of X. If $S \subseteq B$ and $x = \mathrm{Intersect}(S)$, then $x \in S$.

Let X, D be non empty sets and let n be a natural number. Observe that every function from X into D^n is finite sequence yielding.

Let f be a finite sequence yielding function and let x be a set. Note that $f(x)$ is finite sequence-like.

Let n be a natural number and let p, q be n-tuples of *Boolean*. The functor $p \wedge q$ yielding a n-tuple of *Boolean* is defined as follows:

(Def. 5) For every set i such that $i \in \operatorname{Seg} n$ holds $(p \wedge q)(i) = p_i \wedge q_i$.

Let us notice that the functor $p \wedge q$ is commutative.

One can prove the following propositions:

(5) For every natural number n and for every n-tuple p of *Boolean* holds $(n\text{-BinarySequence}(0)) \wedge p = n\text{-BinarySequence}(0)$.

(6) For every natural number n and for every n-tuple p of *Boolean* holds $\neg(n\text{-BinarySequence}(0)) \wedge p = p$.

(7) For every natural number i holds $(i + 1)\text{-BinarySequence}(2^i) = \langle \underbrace{0, \ldots, 0}_{i} \rangle ^\frown \langle 1 \rangle$.

(8) Let n, i be natural numbers. Suppose $i < n$. Then $(n\text{-BinarySequence}(2^i))(i+1) = 1$ and for every natural number j such that $j \in \operatorname{Seg} n$ and $j \neq i+1$ holds $(n\text{-BinarySequence}(2^i))(j) = 0$.

2. The Relational Model of Data

We consider DB-relationships as systems

$\langle$ attributes, domains, a relationship $\rangle$,

where the attributes constitute a finite non empty set, the domains constitute a non-empty many sorted set indexed by the attributes, and the relationship is a subset of $\prod$ the domains.

3. Dependency Structures

Let X be a set.

(Def. 6) A binary relation on 2^X is said to be a relation on subsets of X.

We introduce dependency set of X as a synonym of a relation on subsets of X.

Let X be a set. Observe that there exists a dependency set of X which is non empty and finite.

Let X be a set. The functor dependencies(X) yields a dependency set of X and is defined by:

(Def. 7) $\text{dependencies}(X) = [\!:2^X, 2^X:\!]$.

Let X be a set. Observe that dependencies(X) is non empty. A dependency of X is an element of dependencies(X).

Let X be a set and let F be a non empty dependency set of X. We see that the element of F is a dependency of X.

The following three propositions are true:

(9) For all sets X, x holds $x \in$ dependencies(X) iff there exist subsets a, b of X such that $x = \langle a, b\rangle$.

(10) For all sets X, x and for every dependency set F of X such that $x \in F$ there exist subsets a, b of X such that $x = \langle a, b\rangle$.

(11) For every set X and for every dependency set F of X holds every subset of F is a dependency set of X.

Let R be a DB-relationship and let A, B be subsets of the attributes of R. The predicate $A \to_R B$ is defined by:

(Def. 8) For all elements f, g of the relationship of R such that $f{\restriction}A = g{\restriction}A$ holds $f{\restriction}B = g{\restriction}B$.

We introduce (A, B) holds in R as a synonym of $A \to_R B$.

In the sequel R denotes a DB-relationship and A, B denote subsets of the attributes of R.

Let us consider R. The functor dependency-structure(R) yields a dependency set of the attributes of R and is defined as follows:

(Def. 9) dependency-structure$(R) = \{\langle A, B\rangle : A \to_R B\}$.

One can prove the following proposition

(12) For every DB-relationship R and for all subsets A, B of the attributes of R holds $\langle A, B\rangle \in$ dependency-structure(R) iff $A \to_R B$.

4. Full Families of Dependencies

Let X be a set and let P, Q be dependencies of X. The predicate $P \geqslant Q$ is defined by:

(Def. 10) $P_{\mathbf{1}} \subseteq Q_{\mathbf{1}}$ and $Q_{\mathbf{2}} \subseteq P_{\mathbf{2}}$.

Let us note that the predicate $P \geqslant Q$ is reflexive. We introduce $Q \leqslant P$ and also P is at least as informative as Q, as synonyms of $P \geqslant Q$.

The following propositions are true:

(13) For every set X and for all dependencies P, Q of X such that $P \leqslant Q$ and $Q \leqslant P$ holds $P = Q$.

(14) For every set X and for all dependencies P, Q, S of X such that $P \leqslant Q$ and $Q \leqslant S$ holds $P \leqslant S$.

Let X be a set and let A, B be subsets of X. Then $\langle A, B\rangle$ is a dependency of X.

We now state the proposition

(15) For every set X and for all subsets A, B, A', B' of X holds $\langle A, B\rangle \geqslant \langle A', B'\rangle$ iff $A \subseteq A'$ and $B' \subseteq B$.

Let X be a set. The functor Dependencies-Order X yielding a binary relation on dependencies(X) is defined as follows:

(Def. 11) Dependencies-Order $X = \{\langle P, Q\rangle; P$ ranges over dependencies of X, Q ranges over dependencies of X: $P \leqslant Q\}$.

We now state four propositions:

(16) For all sets X, x holds $x \in$ Dependencies-Order X iff there exist dependencies P, Q of X such that $x = \langle P, Q\rangle$ and $P \leqslant Q$.

(17) For every set X holds dom Dependencies-Order $X = [: 2^X, 2^X :]$.

(18) For every set X holds rng Dependencies-Order $X = [: 2^X, 2^X :]$.

(19) For every set X holds field Dependencies-Order $X = [: 2^X, 2^X :]$.

Let X be a set. Note that Dependencies-Order X is non empty and Dependencies-Order X is ordering.

Let X be a set and let F be a dependency set of X. We say that F is (F1) if and only if:

(Def. 12) For every subset A of X holds $\langle A, A\rangle \in F$.

We introduce F is (DC2) as a synonym of F is (F1). We introduce F is (F2) and F is (DC1) as synonyms of F is transitive.

The following proposition is true

(20) Let X be a set and F be a dependency set of X. Then F is (F2) if and only if for all subsets A, B, C of X such that $\langle A, B\rangle \in F$ and $\langle B, C\rangle \in F$ holds $\langle A, C\rangle \in F$.

Let X be a set and let F be a dependency set of X. We say that F is (F3) if and only if:

(Def. 13) For all subsets A, B, A', B' of X such that $\langle A, B\rangle \in F$ and $\langle A, B\rangle \geqslant \langle A', B'\rangle$ holds $\langle A', B'\rangle \in F$.

We say that F is (F4) if and only if:

(Def. 14) For all subsets A, B, A', B' of X such that $\langle A, B\rangle \in F$ and $\langle A', B'\rangle \in F$ holds $\langle A \cup A', B \cup B'\rangle \in F$.

The following proposition is true

(21) For every set X holds dependencies(X) is (F1), (F2), (F3), and (F4).

Let X be a set. Observe that there exists a dependency set of X which is (F1), (F2), (F3), (F4), and non empty.

Let X be a set and let F be a dependency set of X. We say that F is full family if and only if:

(Def. 15) F is (F1), (F2), (F3), and (F4).

Let X be a set. One can verify that there exists a dependency set of X which is full family.

Let X be a set. A Full family of X is a full family dependency set of X.

We now state the proposition

(22) For every finite set X holds every dependency set of X is finite.

Let X be a finite set. Observe that there exists a Full family of X which is finite and every dependency set of X is finite.

Let X be a set. Note that every dependency set of X which is full family is also (F1), (F2), (F3), and (F4) and every dependency set of X which is (F1), (F2), (F3), and (F4) is also full family.

Let X be a set and let F be a dependency set of X. We say that F is (DC3) if and only if:

(Def. 16) For all subsets A, B of X such that $B \subseteq A$ holds $\langle A, B\rangle \in F$.

Let X be a set. Observe that every dependency set of X which is (F1) and (F3) is also (DC3) and every dependency set of X which is (DC3) and (F2) is also (F1) and (F3).

Let X be a set. Observe that there exists a dependency set of X which is (DC3), (F2), (F4), and non empty.

We now state two propositions:

(23) For every set X and for every dependency set F of X such that F is (DC3) and (F2) holds F is (F1) and (F3).

(24) For every set X and for every dependency set F of X such that F is (F1) and (F3) holds F is (DC3).

Let X be a set. Observe that every dependency set of X which is (F1) is also non empty.

The following propositions are true:

(25) For every DB-relationship R holds dependency-structure(R) is full family.

(26) Let X be a set and K be a subset of X. Then $\{\langle A, B\rangle; A$ ranges over subsets of X, B ranges over subsets of X: $K \subseteq A \ \vee \ B \subseteq A\}$ is a Full family of X.

5. Maximal Elements of Full Families

Let X be a set and let F be a dependency set of X. The functor Maximals(F) yielding a dependency set of X is defined as follows:

(Def. 17) $\mathrm{Maximals}(F) = \mathrm{Maximals}_{\mathrm{Dependencies\text{-}Order}\,X}(F)$.

We now state the proposition

(27) For every set X and for every dependency set F of X holds $\mathrm{Maximals}(F) \subseteq F$.

Let X be a set, let F be a dependency set of X, and let x, y be sets. The predicate $x \nearrow_F y$ is defined as follows:

(Def. 18) $\langle x, y\rangle \in \mathrm{Maximals}(F)$.

One can prove the following two propositions:

(28) Let X be a finite set, P be a dependency of X, and F be a dependency set of X. If $P \in F$, then there exist subsets A, B of X such that $\langle A, B\rangle \in \mathrm{Maximals}(F)$ and $\langle A, B\rangle \geqslant P$.

(29) Let X be a set, F be a dependency set of X, and A, B be subsets of X. Then $A \nearrow_F B$ if and only if the following conditions are satisfied:

(i) $\langle A, B\rangle \in F$, and

(ii) it is not true that there exist subsets A', B' of X such that $\langle A', B'\rangle \in F$ and $\langle A, B\rangle \leqslant \langle A', B'\rangle$ with $A \neq A'$ or $B \neq B'$.

Let X be a set and let M be a dependency set of X. We say that M is (M1) if and only if:

(Def. 19) For every subset A of X there exist subsets A', B' of X such that $\langle A', B'\rangle \geqslant \langle A, A\rangle$ and $\langle A', B'\rangle \in M$.

We say that M is (M2) if and only if:

(Def. 20) For all subsets A, B, A', B' of X such that $\langle A, B\rangle \in M$ and $\langle A', B'\rangle \in M$ and $\langle A, B\rangle \geqslant \langle A', B'\rangle$ holds $A = A'$ and $B = B'$.

We say that M is (M3) if and only if:

(Def. 21) For all subsets A, B, A', B' of X such that $\langle A, B\rangle \in M$ and $\langle A', B'\rangle \in M$ and $A' \subseteq B$ holds $B' \subseteq B$.

We now state two propositions:

(30) For every finite non empty set X and for every Full family F of X holds $\mathrm{Maximals}(F)$ is (M1), (M2), and (M3).

(31) Let X be a finite set and M, F be dependency sets of X. Suppose that

(i) M is (M1), (M2), and (M3), and

(ii) $F = \{\langle A, B\rangle; A$ ranges over subsets of X, B ranges over subsets of X: $\bigvee_{A',B' : \text{subset of } X} (\langle A', B'\rangle \geqslant \langle A, B\rangle \ \wedge \ \langle A', B'\rangle \in M)\}$.

Then $M = \mathrm{Maximals}(F)$ and F is full family and for every Full family G of X such that $M = \mathrm{Maximals}(G)$ holds $G = F$.

Let X be a non empty finite set and let F be a Full family of X. Note that $\mathrm{Maximals}(F)$ is non empty.

Next we state the proposition

(32) Let X be a finite set, F be a dependency set of X, and K be a subset of X. Suppose $F = \{\langle A, B\rangle; A$ ranges over subsets of X, B ranges over subsets of X: $K \subseteq A \vee B \subseteq A\}$. Then $\{\langle K, X\rangle\} \cup \{\langle A, A\rangle; A$ ranges over subsets of X: $K \not\subseteq A\} = \mathrm{Maximals}(F)$.

6. Saturated Subsets of Attributes

Let X be a set and let F be a dependency set of X.

The functor saturated-subsets(F) yields a family of subsets of X and is defined as follows:

(Def. 22) saturated-subsets(F) =
$\{B; B$ ranges over subsets of X: $\bigvee_{A\,:\,\text{subset of } X}\ A \nearrow_F B\}$.

We introduce closed-attribute-subset(F) as a synonym of saturated-subsets(F).

Let X be a set and let F be a finite dependency set of X. Observe that saturated-subsets(F) is finite.

Next we state two propositions:

(33) Let X, x be sets and F be a dependency set of X. Then $x \in$ saturated-subsets(F) if and only if there exist subsets B, A of X such that $x = B$ and $A \nearrow_F B$.

(34) For every finite non empty set X and for every Full family F of X holds saturated-subsets(F) is (B1) and (B2).

Let X be a set and let B be a set. The functor (B)-enclosed in X yields a dependency set of X and is defined as follows:

(Def. 23) (B)-enclosed in X = $\{\langle a, b\rangle; a$ ranges over subsets of X, b ranges over subsets of X: $\bigwedge_{c\,:\,\text{set}}\ (c \in B\ \wedge\ a \subseteq c\ \Rightarrow\ b \subseteq c)\}$.

The following three propositions are true:

(35) For every set X and for every family B of subsets of X and for every dependency set F of X holds (B)-enclosed in X is full family.

(36) For every finite non empty set X and for every family B of subsets of X holds $B \subseteq$ saturated-subsets((B)-enclosed in X).

(37) Let X be a finite non empty set and B be a family of subsets of X. Suppose B is (B1) and (B2). Then $B =$ saturated-subsets((B)-enclosed in X) and for every Full family G of X such that $B =$ saturated-subsets(G) holds $G =$ (B)-enclosed in X.

Let X be a set and let F be a dependency set of X. The functor (F)-enclosure yielding a family of subsets of X is defined as follows:

(Def. 24) (F)-enclosure $= \{b; b$ ranges over subsets of X: $\bigwedge_{A,B\,:\,\text{subset of } X}\ (\langle A, B\rangle \in F\ \wedge\ A \subseteq b\ \Rightarrow\ B \subseteq b)\}$.

We now state two propositions:

(38) For every finite non empty set X and for every dependency set F of X holds (F)-enclosure is (B1) and (B2).

(39) Let X be a finite non empty set and F be a dependency set of X. Then $F \subseteq$ ((F)-enclosure)-enclosed in X and for every dependency set G of X such that $F \subseteq G$ and G is full family holds ((F)-enclosure)-enclosed in $X \subseteq G$.

Let X be a finite non empty set and let F be a dependency set of X. The functor dependency-closure(F) yields a Full family of X and is defined by:

(Def. 25) $F \subseteq$ dependency-closure(F) and for every dependency set G of X such that $F \subseteq G$ and G is full family holds dependency-closure(F) $\subseteq G$.

Next we state four propositions:

(40) For every finite non empty set X and for every dependency set F of X holds dependency-closure(F) = ((F)-enclosure)-enclosed in X.

(41) Let X be a set, K be a subset of X, and B be a family of subsets of X. If $B = \{X\} \cup \{A; A$ ranges over subsets of X: $K \not\subseteq A\}$, then B is (B1) and (B2).

(42) Let X be a finite non empty set, F be a dependency set of X, and K be a subset of X. Suppose $F = \{\langle A, B\rangle; A$ ranges over subsets of X, B ranges over subsets of X: $K \subseteq A \ \vee \ B \subseteq A\}$. Then $\{X\} \cup \{B; B$ ranges over subsets of X: $K \not\subseteq B\}$ = saturated-subsets(F).

(43) Let X be a finite set, F be a dependency set of X, and K be a subset of X. Suppose $F = \{\langle A, B\rangle; A$ ranges over subsets of X, B ranges over subsets of X: $K \subseteq A \ \vee \ B \subseteq A\}$. Then $\{X\} \cup \{B; B$ ranges over subsets of X: $K \not\subseteq B\}$ = saturated-subsets(F).

Let X, G be sets and let B be a family of subsets of X. We say that G is generator set of B if and only if:

(Def. 26) $G \subseteq B$ and $B = \{$Intersect(S)$; S$ ranges over families of subsets of X: $S \subseteq G\}$.

We now state four propositions:

(44) For every finite non empty set X holds every family G of subsets of X is generator set of saturated-subsets((G)-enclosed in X).

(45) Let X be a finite non empty set and F be a Full family of X. Then there exists a family G of subsets of X such that G is generator set of saturated-subsets(F) and F = (G)-enclosed in X.

(46) Let X be a set and B be a non empty finite family of subsets of X. If B is (B1) and (B2), then $\cap$-Irreducibles(B) is generator set of B.

(47) Let X, G be sets and B be a non empty finite family of subsets of X. If B is (B1) and (B2) and G is generator set of B, then $\cap$-Irreducibles(B) $\subseteq G \cup \{X\}$.

7. Justification of the Axioms

One can prove the following proposition

(48) Let X be a non empty finite set and F be a Full family of X. Then there exists a DB-relationship R such that the attributes of $R = X$ and for every element a of X holds (the domains of R)(a) $= \mathbb{Z}$ and F = dependency-structure(R).

8. Structure of the Family of Candidate Keys

Let X be a set and let F be a dependency set of X.

The functor candidate-keys(F) yields a family of subsets of X and is defined by:

(Def. 27) candidate-keys(F) $=$ $\{A; A$ ranges over subsets of X: $\langle A, X\rangle \in$ Maximals$(F)\}$.

One can prove the following proposition

(49) Let X be a finite set, F be a dependency set of X, and K be a subset of X. Suppose $F = \{\langle A, B\rangle; A$ ranges over subsets of X, B ranges over subsets of X: $K \subseteq A \lor B \subseteq A\}$. Then candidate-keys($F$) $= \{K\}$.

Let X be a set. We introduce X is (C1) as an antonym of X is empty.

Let X be a set. We say that X is without proper subsets if and only if:

(Def. 28) For all sets x, y such that $x \in X$ and $y \in X$ and $x \subseteq y$ holds $x = y$.

We introduce X is (C2) as a synonym of X is without proper subsets.

We now state four propositions:

(50) For every DB-relationship R holds candidate-keys(dependency-structure(R)) is (C1) and (C2).

(51) Let X be a finite set and C be a family of subsets of X. If C is (C1) and (C2), then there exists a Full family F of X such that $C =$ candidate-keys(F).

(52) Let X be a finite set, C be a family of subsets of X, and B be a set. Suppose C is (C1) and (C2) and $B = \{b; b$ ranges over subsets of X: $\bigwedge_{K\,:\,\text{subset of }X}\ (K \in C \Rightarrow K \not\subseteq b)\}$. Then $C =$ candidate-keys((B)-enclosed in X).

(53) Let X be a non empty finite set and C be a family of subsets of X. Suppose C is (C1) and (C2). Then there exists a DB-relationship R such that the attributes of $R = X$ and $C =$ candidate-keys(dependency-structure(R)).

9. Applications

Let X be a set and let F be a dependency set of X. We say that F is (DC4) if and only if:

(Def. 29) For all subsets A, B, C of X such that $\langle A, B\rangle \in F$ and $\langle A, C\rangle \in F$ holds $\langle A, B \cup C\rangle \in F$.

We say that F is (DC5) if and only if:

(Def. 30) For all subsets A, B, C, D of X such that $\langle A, B\rangle \in F$ and $\langle B \cup C, D\rangle \in F$ holds $\langle A \cup C, D\rangle \in F$.

We say that F is (DC6) if and only if:

(Def. 31) For all subsets A, B, C of X such that $\langle A, B\rangle \in F$ holds $\langle A \cup C, B\rangle \in F$.

One can prove the following propositions:

(54) Let X be a set and F be a dependency set of X. Then F is (F1), (F2), (F3), and (F4) if and only if F is (F2), (DC3), and (F4).

(55) Let X be a set and F be a dependency set of X. Then F is (F1), (F2), (F3), and (F4) if and only if F is (DC1), (DC3), and (DC4).

(56) Let X be a set and F be a dependency set of X. Then F is (F1), (F2), (F3), and (F4) if and only if F is (DC2), (DC5), and (DC6).

Let X be a set and let F be a dependency set of X.

The functor characteristic(F) is defined as follows:

(Def. 32) characteristic(F) $= \{A; A$ ranges over subsets of X: $\bigvee_{a,b\,:\,\text{subset of } X} (\langle a, b\rangle \in F \ \wedge \ a \subseteq A \ \wedge \ b \not\subseteq A)\}$.

Next we state several propositions:

(57) Let X, A be sets and F be a dependency set of X. Suppose $A \in$ characteristic(F). Then A is a subset of X and there exist subsets a, b of X such that $\langle a, b\rangle \in F$ and $a \subseteq A$ and $b \not\subseteq A$.

(58) Let X be a set, A be a subset of X, and F be a dependency set of X. If there exist subsets a, b of X such that $\langle a, b\rangle \in F$ and $a \subseteq A$ and $b \not\subseteq A$, then $A \in$ characteristic(F).

(59) Let X be a finite non empty set and F be a dependency set of X. Then
(i) for all subsets A, B of X holds $\langle A, B\rangle \in$ dependency-closure(F) iff for every subset a of X such that $A \subseteq a$ and $B \not\subseteq a$ holds $a \in$ characteristic(F), and
(ii) saturated-subsets(dependency-closure(F)) $= 2^X \setminus$ characteristic(F).

(60) For every finite non empty set X and for all dependency sets F, G of X such that characteristic(F) $=$ characteristic(G) holds dependency-closure(F) $=$ dependency-closure(G).

(61) For every non empty finite set X and for every dependency set F of X holds characteristic(F) $=$ characteristic(dependency-closure(F)).

Let A be a set, let K be a set, and let F be a dependency set of A. We say that K is prime implicant of F with no complemented variables if and only if the conditions (Def. 33) are satisfied.

(Def. 33)(i) For every subset a of A such that $K \subseteq a$ and $a \neq A$ holds $a \in$ characteristic(F), and
(ii) for every set k such that $k \subset K$ there exists a subset a of A such that $k \subseteq a$ and $a \neq A$ and $a \notin$ characteristic(F).

The following proposition is true

(62) Let X be a finite non empty set, F be a dependency set of X, and K be a subset of X. Then $K \in$ candidate-keys(dependency-closure(F)) if and only if K is prime implicant of F with no complemented variables.

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Convex Sets and Convex Combinations

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Summary. Convexity is one of the most important concepts in a study of analysis. Especially, it has been applied around the optimization problem widely. Our purpose is to define the concept of convexity of a set on Mizar, and to develop the generalities of convex analysis. The construction of this article is as follows: Convexity of the set is defined in the section 1. The section 2 gives the definition of convex combination which is a kind of the linear combination and related theorems are proved there. In section 3, we define the convex hull which is an intersection of all convex sets including a given set. The last section is some theorems which are necessary to compose this article.

MML Identifier: `CONVEX1`.

The notation and terminology used in this paper are introduced in the following articles: [13], [12], [17], [9], [10], [3], [1], [8], [4], [2], [16], [15], [14], [5], [11], [6], and [7].

1. Convex Sets

Let V be a non empty RLS structure, let M be a subset of V, and let r be a real number. The functor $r \cdot M$ yielding a subset of V is defined by:

(Def. 1) $r \cdot M = \{r \cdot v; v$ ranges over elements of the carrier of $V\colon v \in M\}$.

Let V be a non empty RLS structure and let M be a subset of V. We say that M is convex if and only if:

(Def. 2) For all vectors u, v of V and for every real number r such that $0 < r$ and $r < 1$ and $u \in M$ and $v \in M$ holds $r \cdot u + (1 - r) \cdot v \in M$.

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We now state a number of propositions:

(1) Let V be a real linear space-like non empty RLS structure, M be a subset of V, and r be a real number. If M is convex, then $r \cdot M$ is convex.

(2) Let V be an Abelian add-associative real linear space-like non empty RLS structure and M, N be subsets of V. If M is convex and N is convex, then $M + N$ is convex.

(3) For every real linear space V and for all subsets M, N of V such that M is convex and N is convex holds $M - N$ is convex.

(4) Let V be a non empty RLS structure and M be a subset of V. Then M is convex if and only if for every real number r such that $0 < r$ and $r < 1$ holds $r \cdot M + (1 - r) \cdot M \subseteq M$.

(5) Let V be an Abelian non empty RLS structure and M be a subset of V. Suppose M is convex. Let r be a real number. If $0 < r$ and $r < 1$, then $(1 - r) \cdot M + r \cdot M \subseteq M$.

(6) Let V be an Abelian add-associative real linear space-like non empty RLS structure and M, N be subsets of V. Suppose M is convex and N is convex. Let r be a real number. Then $r \cdot M + (1 - r) \cdot N$ is convex.

(7) Let V be a real linear space, M be a subset of V, and v be a vector of V. Then M is convex if and only if $v + M$ is convex.

(8) For every real linear space V holds $\mathrm{Up}(\mathbf{0}_V)$ is convex.

(9) For every real linear space V holds $\mathrm{Up}(\Omega_V)$ is convex.

(10) For every non empty RLS structure V and for every subset M of V such that $M = \emptyset$ holds M is convex.

(11) Let V be an Abelian add-associative real linear space-like non empty RLS structure, M_1, M_2 be subsets of V, and r_1, r_2 be real numbers. If M_1 is convex and M_2 is convex, then $r_1 \cdot M_1 + r_2 \cdot M_2$ is convex.

(12) Let V be a real linear space-like non empty RLS structure, M be a subset of V, and r_1, r_2 be real numbers. Then $(r_1 + r_2) \cdot M \subseteq r_1 \cdot M + r_2 \cdot M$.

(13) Let V be a real linear space, M be a subset of V, and r_1, r_2 be real numbers. If $r_1 \geqslant 0$ and $r_2 \geqslant 0$ and M is convex, then $r_1 \cdot M + r_2 \cdot M \subseteq (r_1 + r_2) \cdot M$.

(14) Let V be an Abelian add-associative real linear space-like non empty RLS structure, M_1, M_2, M_3 be subsets of V, and r_1, r_2, r_3 be real numbers. If M_1 is convex and M_2 is convex and M_3 is convex, then $r_1 \cdot M_1 + r_2 \cdot M_2 + r_3 \cdot M_3$ is convex.

(15) Let V be a non empty RLS structure and F be a family of subsets of V. Suppose that for every subset M of V such that $M \in F$ holds M is convex. Then $\bigcap F$ is convex.

(16) For every non empty RLS structure V and for every subset M of V such

that M is Affine holds M is convex.

Let V be a non empty RLS structure. Observe that there exists a subset of V which is convex.

Let V be a non empty RLS structure. Note that there exists a subset of V which is empty and convex.

Let V be a non empty RLS structure. One can check that there exists a subset of V which is non empty and convex.

The following four propositions are true:

(17) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) \geqslant r\}$, then M is convex.

(18) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) > r\}$, then M is convex.

(19) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) \leqslant r\}$, then M is convex.

(20) Let V be a real unitary space-like non empty unitary space structure, M be a subset of V, v be a vector of V, and r be a real number. If $M = \{u; u$ ranges over vectors of V: $(u|v) < r\}$, then M is convex.

2. Convex Combinations

Let V be a real linear space and let L be a linear combination of V. We say that L is convex if and only if the condition (Def. 3) is satisfied.

(Def. 3) There exists a finite sequence F of elements of the carrier of V such that

(i) F is one-to-one,

(ii) $\operatorname{rng} F =$ the support of L, and

(iii) there exists a finite sequence f of elements of $\mathbb{R}$ such that $\operatorname{len} f = \operatorname{len} F$ and $\sum f = 1$ and for every natural number n such that $n \in \operatorname{dom} f$ holds $f(n) = L(F(n))$ and $f(n) \geqslant 0$.

One can prove the following propositions:

(21) Let V be a real linear space and L be a linear combination of V. If L is convex, then the support of $L \neq \emptyset$.

(22) Let V be a real linear space, L be a linear combination of V, and v be a vector of V. If L is convex and $L(v) \leqslant 0$, then $v \notin$ the support of L.

(23) For every real linear space V and for every linear combination L of V such that L is convex holds $L \neq \mathbf{0}_{\mathrm{LC}_V}$.

(24) Let V be a real linear space, v be a vector of V, and L be a linear combination of $\{v\}$. If L is convex, then $L(v) = 1$ and $\sum L = L(v) \cdot v$.

(25) Let V be a real linear space, v_1, v_2 be vectors of V, and L be a linear combination of $\{v_1, v_2\}$. Suppose $v_1 \neq v_2$ and L is convex. Then $L(v_1) + L(v_2) = 1$ and $L(v_1) \geqslant 0$ and $L(v_2) \geqslant 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2$.

(26) Let V be a real linear space, v_1, v_2, v_3 be vectors of V, and L be a linear combination of $\{v_1, v_2, v_3\}$. Suppose $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_3 \neq v_1$ and L is convex. Then $L(v_1) + L(v_2) + L(v_3) = 1$ and $L(v_1) \geqslant 0$ and $L(v_2) \geqslant 0$ and $L(v_3) \geqslant 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2 + L(v_3) \cdot v_3$.

(27) Let V be a real linear space, v be a vector of V, and L be a linear combination of V. If L is convex and the support of $L = \{v\}$, then $L(v) = 1$.

(28) Let V be a real linear space, v_1, v_2 be vectors of V, and L be a linear combination of V. Suppose L is convex and the support of $L = \{v_1, v_2\}$ and $v_1 \neq v_2$. Then $L(v_1) + L(v_2) = 1$ and $L(v_1) \geqslant 0$ and $L(v_2) \geqslant 0$.

(29) Let V be a real linear space, v_1, v_2, v_3 be vectors of V, and L be a linear combination of V. Suppose L is convex and the support of $L = \{v_1, v_2, v_3\}$ and $v_1 \neq v_2$ and $v_2 \neq v_3$ and $v_3 \neq v_1$. Then $L(v_1) + L(v_2) + L(v_3) = 1$ and $L(v_1) \geqslant 0$ and $L(v_2) \geqslant 0$ and $L(v_3) \geqslant 0$ and $\sum L = L(v_1) \cdot v_1 + L(v_2) \cdot v_2 + L(v_3) \cdot v_3$.

3. Convex Hull

In this article we present several logical schemes. The scheme *SubFamExRLS* deals with an RLS structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a family F of subsets of $\mathcal{A}$ such that for every subset B of the carrier of $\mathcal{A}$ holds $B \in F$ iff $\mathcal{P}[B]$

for all values of the parameters.

The scheme *SubFamExRLS2* deals with an RLS structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

> There exists a family F of subsets of $\mathcal{A}$ such that for every subset B of $\mathcal{A}$ holds $B \in F$ iff $\mathcal{P}[B]$

for all values of the parameters.

Let V be a non empty RLS structure and let M be a subset of V. The functor Convex-Family M yields a family of subsets of V and is defined as follows:

(Def. 4) For every subset N of V holds $N \in$ Convex-Family M iff N is convex and $M \subseteq N$.

Let V be a non empty RLS structure and let M be a subset of V. The functor conv M yields a convex subset of V and is defined by:

(Def. 5) $\operatorname{conv} M = \bigcap \text{Convex-Family}\, M$.

The following proposition is true

(30) Let V be a non empty RLS structure, M be a subset of V, and N be a convex subset of V. If $M \subseteq N$, then $\operatorname{conv} M \subseteq N$.

4. Miscellaneous

One can prove the following propositions:

(31) Let p be a finite sequence and x, y, z be sets. Suppose p is one-to-one and $\operatorname{rng} p = \{x, y, z\}$ and $x \neq y$ and $y \neq z$ and $z \neq x$. Then $p = \langle x, y, z \rangle$ or $p = \langle x, z, y \rangle$ or $p = \langle y, x, z \rangle$ or $p = \langle y, z, x \rangle$ or $p = \langle z, x, y \rangle$ or $p = \langle z, y, x \rangle$.

(32) For every real linear space-like non empty RLS structure V and for every subset M of V holds $1 \cdot M = M$.

(33) For every non empty RLS structure V and for every empty subset M of V and for every real number r holds $r \cdot M = \emptyset$.

(34) For every real linear space V and for every non empty subset M of V holds $0 \cdot M = \{0_V\}$.

(35) For every right zeroed non empty loop structure V and for every subset M of V holds $M + \{0_V\} = M$.

(36) For every add-associative non empty loop structure V and for all subsets M_1, M_2, M_3 of V holds $(M_1 + M_2) + M_3 = M_1 + (M_2 + M_3)$.

(37) Let V be a real linear space-like non empty RLS structure, M be a subset of V, and r_1, r_2 be real numbers. Then $r_1 \cdot (r_2 \cdot M) = (r_1 \cdot r_2) \cdot M$.

(38) Let V be a real linear space-like non empty RLS structure, M_1, M_2 be subsets of V, and r be a real number. Then $r \cdot (M_1 + M_2) = r \cdot M_1 + r \cdot M_2$.

(39) Let V be a non empty RLS structure, M, N be subsets of V, and r be a real number. If $M \subseteq N$, then $r \cdot M \subseteq r \cdot N$.

(40) For every non empty loop structure V and for every empty subset M of V and for every subset N of V holds $M + N = \emptyset$.

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Quotient Vector Spaces and Functionals[1]

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Summary. The article presents well known facts about quotient vector spaces and functionals (see [8]). There are repeated theorems and constructions with either weaker assumptions or in more general situations (see [11], [7], [10]). The construction of coefficient functionals and non-degenerate functional in quotient vector space generated by functional in the given vector space are the only new things which are done.

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The articles [15], [5], [21], [13], [3], [1], [20], [2], [17], [7], [22], [4], [6], [14], [19], [12], [18], [16], and [9] provide the notation and terminology for this paper.

1. Auxiliary Facts about Double Loops and Vector Spaces

The following proposition is true

(1) Let K be an add-associative right zeroed right complementable left distributive left unital non empty double loop structure and a be an element of the carrier of K. Then $(-\mathbf{1}_K) \cdot a = -a$.

Let K be a double loop structure. The functor StructVectSp(K) yields a strict vector space structure over K and is defined as follows:

(Def. 1) StructVectSp(K) = $\langle$the carrier of K, the addition of K, the zero of K, the multiplication of $K\rangle$.

Let K be a non empty double loop structure. Note that StructVectSp(K) is non empty.

Let K be an Abelian non empty double loop structure. One can verify that StructVectSp(K) is Abelian.

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Let K be an add-associative non empty double loop structure. Note that StructVectSp(K) is add-associative.

Let K be a right zeroed non empty double loop structure.

Note that StructVectSp(K) is right zeroed.

Let K be a right complementable non empty double loop structure. Observe that StructVectSp(K) is right complementable.

Let K be an associative left unital distributive non empty double loop structure. One can check that StructVectSp(K) is vector space-like.

Let K be a non degenerated non empty double loop structure. Note that StructVectSp(K) is non trivial.

Let K be a non degenerated non empty double loop structure. Note that there exists a non empty vector space structure over K which is non trivial.

Let K be an add-associative right zeroed right complementable non empty double loop structure. Observe that there exists a non empty vector space structure over K which is add-associative, right zeroed, right complementable, and strict.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure. One can check that there exists a non empty vector space structure over K which is add-associative, right zeroed, right complementable, vector space-like, and strict.

Let K be an Abelian add-associative right zeroed right complementable associative left unital distributive non degenerated non empty double loop structure. One can verify that there exists a non empty vector space structure over K which is Abelian, add-associative, right zeroed, right complementable, vector space-like, strict, and non trivial.

Next we state a number of propositions:

(2) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, a be an element of the carrier of K, V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and v be a vector of V. Then $0_K \cdot v = 0_V$ and $a \cdot 0_V = 0_V$.

(3) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, S, T be subspaces of V, and v be a vector of V. If $S \cap T = \mathbf{0}_V$ and $v \in S$ and $v \in T$, then $v = 0_V$.

(4) Let K be a field, V be a vector space over K, x be a set, and v be a vector of V. Then $x \in \mathrm{Lin}(\{v\})$ if and only if there exists an element a of the carrier of K such that $x = a \cdot v$.

(5) Let K be a field, V be a vector space over K, v be a vector of V, and a, b be scalars of V. If $v \neq 0_V$ and $a \cdot v = b \cdot v$, then $a = b$.

(6) Let K be an add-associative right zeroed right complementable associa-

tive Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$, then $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$.

(7) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v = v_1 + v_2$.

(8) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_1 \in W_1$ and $v_2 \in W_2$.

(9) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_{\langle W_2, W_1 \rangle} = \langle v_2, v_1 \rangle$.

(10) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_1$, then $v_{\langle W_1, W_2 \rangle} = \langle v, 0_V \rangle$.

(11) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_2$, then $v_{\langle W_1, W_2 \rangle} = \langle 0_V, v \rangle$.

(12) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, V_1 be a subspace of V, W_1 be a subspace of V_1, and v be a vector of V. If $v \in W_1$, then v is a vector of V_1.

(13) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, V_1, V_2, W be subspaces of V, and W_1, W_2 be subspaces of W. If $W_1 = V_1$ and $W_2 = V_2$, then $W_1 + W_2 = V_1 + V_2$.

(14) Let K be a field, V be a vector space over K, W be a subspace of V, v be a vector of V, and w be a vector of W. If $v = w$, then $\mathrm{Lin}(\{w\}) = \mathrm{Lin}(\{v\})$.

(15) Let K be a field, V be a vector space over K, v be a vector of V, and X be a subspace of V. Suppose $v \notin X$. Let y be a vector of $X + \mathrm{Lin}(\{v\})$ and W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $W = X$, then $X + \mathrm{Lin}(\{v\})$ is the direct sum of W and $\mathrm{Lin}(\{y\})$.

(16) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $X = W$ and $v \notin X$, then $y_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle 0_W, y \rangle$.

(17) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. If $w \in X$, then $w_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle w, 0_V \rangle$.

(18) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, v be a vector of V, and W_1, W_2 be subspaces of V. Then there exist vectors v_1, v_2 of V such that $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$.

(19) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. Then there exists a vector x of X and there exists an element r of the carrier of K such that $w_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x, r \cdot v \rangle$.

(20) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w_1, w_2 be vectors of $X + \mathrm{Lin}(\{v\})$, x_1, x_2 be vectors of X, and r_1, r_2 be elements of the carrier of K. If $(w_1)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x_1, r_1 \cdot v \rangle$ and $(w_2)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x_2, r_2 \cdot v \rangle$, then $(w_1 + w_2)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x_1 + x_2, (r_1 + r_2) \cdot v \rangle$.

(21) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$, x be a vector of X, and t, r be elements of the carrier of K. If $w_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x, r \cdot v \rangle$, then $(t \cdot w)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle t \cdot x, t \cdot r \cdot v \rangle$.

2. Quotient Vector Space for Non-Commutative Double Loop

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor $\mathrm{CosetSet}(V, W)$ yielding a non empty family of subsets of the carrier of V is defined as follows:

(Def. 2) $\mathrm{CosetSet}(V, W) = \{A : A$ ranges over cosets of $W\}$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor $\mathrm{addCoset}(V, W)$ yields a binary operation on $\mathrm{CosetSet}(V, W)$ and is defined by:

(Def. 3) For all elements A, B of $\mathrm{CosetSet}(V, W)$ and for all vectors a, b of V such that $A = a+W$ and $B = b+W$ holds $(\mathrm{addCoset}(V, W))(A, B) = a+b+W$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor $\mathrm{zeroCoset}(V, W)$ yielding an element of $\mathrm{CosetSet}(V, W)$ is defined as follows:

(Def. 4) $\mathrm{zeroCoset}(V, W) =$ the carrier of W.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor $\mathrm{lmultCoset}(V, W)$ yields a function from $[\!:$ the carrier of K, $\mathrm{CosetSet}(V, W) :\!]$ into $\mathrm{CosetSet}(V, W)$ and is defined by the condition (Def. 5).

(Def. 5) Let z be an element of the carrier of K, A be an element of $\mathrm{CosetSet}(V, W)$, and a be a vector of V. If $A = a + W$, then $(\mathrm{lmultCoset}(V, W))(z, A) = z \cdot a + W$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor ${}^{V}/_{W}$ yielding a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over K is defined by the conditions (Def. 6).

(Def. 6)(i) The carrier of ${}^{V}/_{W} = \mathrm{CosetSet}(V, W)$,

(ii) the addition of ${}^{V}/_{W} = \mathrm{addCoset}(V, W)$,

(iii) the zero of ${}^{V}/_{W} = \mathrm{zeroCoset}(V, W)$, and

(iv) the left multiplication of ${}^{V}/_{W} = \mathrm{lmultCoset}(V, W)$.

The following propositions are true:

(22) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, and W be a subspace of V. Then $\mathrm{zeroCoset}(V, W) = 0_V + W$ and $0_{{}^{V}/_{W}} = \mathrm{zeroCoset}(V, W)$.

(23) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, and w be a vector of ${}^{V}/_{W}$. Then w is a coset of W and there exists a vector v of V such that $w = v + W$.

(24) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, and v be a vector of V. Then $v + W$ is a coset of W and $v + W$ is a vector of ${}^{V}/_{W}$.

(25) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, and W be a subspace of V. Then every coset of W is a vector of ${}^{V}/_{W}$.

(26) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, A be a vector of ${}^{V}/_{W}$, v be a vector of V, and a be a scalar of V. If $A = v + W$, then $a \cdot A = a \cdot v + W$.

(27) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, A_1, A_2 be vectors of ${}^{V}/_{W}$, and v_1, v_2 be vectors of V. If $A_1 = v_1 + W$ and $A_2 = v_2 + W$, then $A_1 + A_2 = v_1 + v_2 + W$.

3. Auxiliary Facts about Functionals

Next we state the proposition

(28) Let K be a field, V be a vector space over K, X be a subspace of V, f_1 be a linear functional in X, v be a vector of V, and y be a vector of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $v \notin X$. Let r be an element of the carrier of K. Then there exists a linear functional p_1 in $X + \mathrm{Lin}(\{v\})$ such that $p_1 \restriction$the carrier of $X = f_1$ and $p_1(y) = r$.

Let K be a right zeroed non empty loop structure and let V be a non empty vector space structure over K. One can verify that there exists a functional in V which is additive and 0-preserving.

Let K be an add-associative right zeroed right complementable non empty double loop structure and let V be a right zeroed non empty vector space structure over K. Observe that every functional in V which is additive is also 0-preserving.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure and let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K. One can verify that every functional in V which is homogeneous is also 0-preserving.

Let K be a non empty zero structure and let V be a non empty vector space structure over K. One can check that 0Functional V is constant.

Let K be a non empty zero structure and let V be a non empty vector space structure over K. Note that there exists a functional in V which is constant.

Let K be an add-associative right zeroed right complementable non empty double loop structure, let V be a right zeroed non empty vector space structure over K, and let f be a 0-preserving functional in V. Let us observe that f is constant if and only if:

(Def. 7) $f = 0\mathrm{Functional}\, V$.

Let K be an add-associative right zeroed right complementable non empty double loop structure and let V be a right zeroed non empty vector space structure over K. Note that there exists a functional in V which is constant, additive, and 0-preserving.

Let K be a non empty 1-sorted structure and let V be a non empty vector space structure over K. One can check that every functional in V which is non constant is also non trivial.

Let K be a field and let V be a non trivial vector space over K. Observe that there exists a functional in V which is additive, homogeneous, non constant, and non trivial.

Let K be a field and let V be a non trivial vector space over K. One can check that every functional in V which is trivial is also constant.

Let K be a field, let V be a non trivial vector space over K, let v be a vector of V, and let W be a linear complement of $\mathrm{Lin}(\{v\})$. Let us assume that $v \neq 0_V$. The functor coeffFunctional(v, W) yielding a non constant non trivial linear functional in V is defined as follows:

(Def. 8) $(\mathrm{coeffFunctional}(v, W))(v) = \mathbf{1}_K$ and $\mathrm{coeffFunctional}(v, W)\restriction$the carrier of $W = 0\mathrm{Functional}\, W$.

We now state several propositions:

(29) Let K be a field, V be a non trivial vector space over K, and f be a non constant 0-preserving functional in V. Then there exists a vector v of V such that $v \neq 0_V$ and $f(v) \neq 0_K$.

(30) Let K be a field, V be a non trivial vector space over K, v be a vector of V, a be a scalar of V, and W be a linear complement of $\mathrm{Lin}(\{v\})$. If $v \neq 0_V$, then $(\mathrm{coeffFunctional}(v, W))(a \cdot v) = a$.

(31) Let K be a field, V be a non trivial vector space over K, v, w be vectors of V, and W be a linear complement of $\mathrm{Lin}(\{v\})$. If $v \neq 0_V$ and $w \in W$, then $(\mathrm{coeffFunctional}(v, W))(w) = 0_K$.

(32) Let K be a field, V be a non trivial vector space over K, v, w be vectors of V, a be a scalar of V, and W be a linear complement of $\mathrm{Lin}(\{v\})$. If $v \neq 0_V$ and $w \in W$, then $(\mathrm{coeffFunctional}(v, W))(a \cdot v + w) = a$.

(33) Let K be a non empty loop structure, V be a non empty vector space structure over K, f, g be functionals in V, and v be a vector of V. Then

$(f - g)(v) = f(v) - g(v)$.

Let K be a field and let V be a non trivial vector space over K. Note that $\overline{V}$ is non trivial.

4. Kernel of Additive Functional. Linear Functionals in Quotient Vector Spaces

Let K be a non empty zero structure, let V be a non empty vector space structure over K, and let f be a functional in V. The functor $\ker f$ yields a subset of the carrier of V and is defined by:

(Def. 9) $\ker f = \{v; v$ ranges over vectors of V: $f(v) = 0_K\}$.

Let K be a right zeroed non empty loop structure, let V be a non empty vector space structure over K, and let f be a 0-preserving functional in V. One can check that $\ker f$ is non empty.

One can prove the following proposition

(34) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and f be a linear functional in V. Then $\ker f$ is linearly closed.

Let K be a non empty zero structure, let V be a non empty vector space structure over K, and let f be a functional in V. We say that f is degenerated if and only if:

(Def. 10) $\ker f \neq \{0_V\}$.

Let K be a non degenerated non empty double loop structure and let V be a non trivial non empty vector space structure over K. One can check that every functional in V which is non degenerated and 0-preserving is also non constant.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let f be a linear functional in V. The functor $\operatorname{Ker} f$ yields a strict non empty subspace of V and is defined as follows:

(Def. 11) The carrier of $\operatorname{Ker} f = \ker f$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, let W be a subspace of V, and let f be an additive functional in V. Let us assume that the carrier of $W \subseteq \ker f$. The functor ${}^{f}/_{W}$ yielding an additive functional in ${}^{V}/_{W}$ is defined by:

(Def. 12) For every vector A of ${}^{V}/_{W}$ and for every vector v of V such that $A = v + W$ holds $({}^{f}/_{W})(A) = f(v)$.

One can prove the following proposition

(35) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, and f be a linear functional in V. If the carrier of $W \subseteq \ker f$, then ${}^{f}/_{W}$ is homogeneous.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let f be a linear functional in V. The functor CQFunctional f yielding a linear functional in ${}^{V}/_{\mathrm{Ker}\, f}$ is defined as follows:

(Def. 13) CQFunctional $f = {}^{f}/_{\mathrm{Ker}\, f}$.

One can prove the following proposition

(36) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, f be a linear functional in V, A be a vector of ${}^{V}/_{\mathrm{Ker}\, f}$, and v be a vector of V. If $A = v + \mathrm{Ker}\, f$, then CQFunctional $f(A) = f(v)$.

Let K be a field, let V be a non trivial vector space over K, and let f be a non constant linear functional in V. Observe that CQFunctional f is non constant.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let f be a linear functional in V. One can verify that CQFunctional f is non degenerated.

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Bilinear Functionals in Vector Spaces[1]

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Summary. The main goal of the article is the presentation of the theory of bilinear functionals in vector spaces. It introduces standard operations on bilinear functionals and proves their classical properties. It is shown that quotient functionals are non-degenerate on the left and the right. In the case of symmetric and alternating bilinear functionals it is shown that the left and right kernels are equal.

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The papers [13], [6], [17], [12], [4], [18], [11], [2], [16], [3], [9], [19], [5], [7], [1], [15], [14], [10], and [8] provide the notation and terminology for this paper.

1. Two Form on Vector Spaces and Operations on Them

Let K be a 1-sorted structure and let V, W be vector space structures over K.

(Def. 1) A function from $[\!:$ the carrier of V, the carrier of $W\,:\!]$ into the carrier of K is said to be a form of V, W.

Let K be a non empty zero structure and let V, W be vector space structures over K. The functor $\mathrm{NulForm}(V, W)$ yielding a form of V, W is defined by:

(Def. 2) $\mathrm{NulForm}(V, W) = [\!:$ the carrier of V, the carrier of $W\,:\!] \longmapsto 0_K$.

Let K be a non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be forms of V, W. The functor $f + g$ yields a form of V, W and is defined as follows:

(Def. 3) For every vector v of V and for every vector w of W holds $(f + g)(\langle v, w\rangle) = f(\langle v, w\rangle) + g(\langle v, w\rangle)$.

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Let K be a non empty groupoid, let V, W be non empty vector space structures over K, let f be a form of V, W, and let a be an element of the carrier of K. The functor $a \cdot f$ yields a form of V, W and is defined by:

(Def. 4) For every vector v of V and for every vector w of W holds $(a \cdot f)(\langle v, w\rangle) = a \cdot f(\langle v, w\rangle)$.

Let K be a non empty loop structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. The functor $-f$ yielding a form of V, W is defined as follows:

(Def. 5) For every vector v of V and for every vector w of W holds $(-f)(\langle v, w\rangle) = -f(\langle v, w\rangle)$.

Let K be an add-associative right zeroed right complementable left distributive left unital non empty double loop structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. Then $-f$ is a form of V, W and it can be characterized by the condition:

(Def. 6) $-f = (-\mathbf{1}_K) \cdot f$.

Let K be a non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be forms of V, W. The functor $f - g$ yields a form of V, W and is defined by:

(Def. 7) $f - g = f + -g$.

Let K be a non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be forms of V, W. Then $f - g$ is a form of V, W and it can be characterized by the condition:

(Def. 8) For every vector v of V and for every vector w of W holds $(f - g)(\langle v, w\rangle) = f(\langle v, w\rangle) - g(\langle v, w\rangle)$.

Let K be an Abelian non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be forms of V, W. Let us notice that the functor $f + g$ is commutative.

Next we state several propositions:

(1) Let K be a non empty zero structure, V, W be non empty vector space structures over K, v be a vector of V, and w be a vector of W. Then $(\mathrm{NulForm}(V, W))(\langle v, w\rangle) = 0_K$.

(2) Let K be a right zeroed non empty loop structure, V, W be non empty vector space structures over K, and f be a form of V, W. Then $f + \mathrm{NulForm}(V, W) = f$.

(3) Let K be an add-associative non empty loop structure, V, W be non empty vector space structures over K, and f, g, h be forms of V, W. Then $(f + g) + h = f + (g + h)$.

(4) Let K be an add-associative right zeroed right complementable non empty loop structure, V, W be non empty vector space structures over K, and f be a form of V, W. Then $f - f = \mathrm{NulForm}(V, W)$.

(5) Let K be a right distributive non empty double loop structure, V, W be non empty vector space structures over K, a be an element of the carrier of K, and f, g be forms of V, W. Then $a \cdot (f + g) = a \cdot f + a \cdot g$.

(6) Let K be a left distributive non empty double loop structure, V, W be non empty vector space structures over K, a, b be elements of the carrier of K, and f be a form of V, W. Then $(a + b) \cdot f = a \cdot f + b \cdot f$.

(7) Let K be an associative non empty double loop structure, V, W be non empty vector space structures over K, a, b be elements of the carrier of K, and f be a form of V, W. Then $(a \cdot b) \cdot f = a \cdot (b \cdot f)$.

(8) Let K be a left unital non empty double loop structure, V, W be non empty vector space structures over K, and f be a form of V, W. Then $\mathbf{1}_K \cdot f = f$.

2. Functional Generated by Two Form when the One of Arguments is Fixed

Let K be a non empty 1-sorted structure, let V, W be non empty vector space structures over K, let f be a form of V, W, and let v be a vector of V. The functor $f(v, \cdot)$ yielding a functional in W is defined as follows:

(Def. 9) $f(v, \cdot) = (\mathrm{curry}\, f)(v)$.

Let K be a non empty 1-sorted structure, let V, W be non empty vector space structures over K, let f be a form of V, W, and let w be a vector of W. The functor $f(\cdot, w)$ yields a functional in V and is defined by:

(Def. 10) $f(\cdot, w) = (\mathrm{curry}'\, f)(w)$.

The following propositions are true:

(9) Let K be a non empty 1-sorted structure, V, W be non empty vector space structures over K, f be a form of V, W, and v be a vector of V. Then $\mathrm{dom}\, f(v, \cdot) =$ the carrier of W and $\mathrm{rng}\, f(v, \cdot) \subseteq$ the carrier of K and for every vector w of W holds $(f(v, \cdot))(w) = f(\langle v, w \rangle)$.

(10) Let K be a non empty 1-sorted structure, V, W be non empty vector space structures over K, f be a form of V, W, and w be a vector of W. Then $\mathrm{dom}\, f(\cdot, w) =$ the carrier of V and $\mathrm{rng}\, f(\cdot, w) \subseteq$ the carrier of K and for every vector v of V holds $(f(\cdot, w))(v) = f(\langle v, w \rangle)$.

(11) Let K be a non empty zero structure, V, W be non empty vector space structures over K, f be a form of V, W, and v be a vector of V. Then $\mathrm{NulForm}(V, W)(v, \cdot) = \mathrm{0Functional}\, W$.

(12) Let K be a non empty zero structure, V, W be non empty vector space structures over K, f be a form of V, W, and w be a vector of W. Then $\mathrm{NulForm}(V, W)(\cdot, w) = \mathrm{0Functional}\, V$.

(13) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f, g be forms of V, W, and w be a vector of W. Then $(f+g)(\cdot, w) = f(\cdot, w) + g(\cdot, w)$.

(14) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f, g be forms of V, W, and v be a vector of V. Then $(f+g)(v, \cdot) = f(v, \cdot) + g(v, \cdot)$.

(15) Let K be a non empty double loop structure, V, W be non empty vector space structures over K, f be a form of V, W, a be an element of the carrier of K, and w be a vector of W. Then $(a \cdot f)(\cdot, w) = a \cdot f(\cdot, w)$.

(16) Let K be a non empty double loop structure, V, W be non empty vector space structures over K, f be a form of V, W, a be an element of the carrier of K, and v be a vector of V. Then $(a \cdot f)(v, \cdot) = a \cdot f(v, \cdot)$.

(17) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f be a form of V, W, and w be a vector of W. Then $(-f)(\cdot, w) = -f(\cdot, w)$.

(18) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f be a form of V, W, and v be a vector of V. Then $(-f)(v, \cdot) = -f(v, \cdot)$.

(19) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f, g be forms of V, W, and w be a vector of W. Then $(f-g)(\cdot, w) = f(\cdot, w) - g(\cdot, w)$.

(20) Let K be a non empty loop structure, V, W be non empty vector space structures over K, f, g be forms of V, W, and v be a vector of V. Then $(f-g)(v, \cdot) = f(v, \cdot) - g(v, \cdot)$.

3. Two Form Generated by Functionals

Let K be a non empty groupoid, let V, W be non empty vector space structures over K, let f be a functional in V, and let g be a functional in W. The functor $f \otimes g$ yields a form of V, W and is defined as follows:

(Def. 11) For every vector v of V and for every vector w of W holds $f \otimes g(\langle v, w\rangle) = f(v) \cdot g(w)$.

One can prove the following propositions:

(21) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V, W be non empty vector space structures over K, f be a functional in V, v be a vector of V, and w be a vector of W. Then $f \otimes (0\mathrm{Functional}\, W)(\langle v, w\rangle) = 0_K$.

(22) Let K be an add-associative right zeroed right complementable left distributive non empty double loop structure, V, W be non empty vector

space structures over K, g be a functional in W, v be a vector of V, and w be a vector of W. Then $(0\mathrm{Functional}\,V) \otimes g(\langle v, w\rangle) = 0_K$.

(23) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V, W be non empty vector space structures over K, and f be a functional in V. Then $f \otimes (0\mathrm{Functional}\,W) = \mathrm{NulForm}(V, W)$.

(24) Let K be an add-associative right zeroed right complementable left distributive non empty double loop structure, V, W be non empty vector space structures over K, and g be a functional in W. Then $(0\mathrm{Functional}\,V) \otimes g = \mathrm{NulForm}(V, W)$.

(25) Let K be a non empty groupoid, V, W be non empty vector space structures over K, f be a functional in V, g be a functional in W, and v be a vector of V. Then $(f \otimes g)(v, \cdot) = f(v) \cdot g$.

(26) Let K be a commutative non empty groupoid, V, W be non empty vector space structures over K, f be a functional in V, g be a functional in W, and w be a vector of W. Then $(f \otimes g)(\cdot, w) = g(w) \cdot f$.

4. Bilinear Forms and their Properties

Let K be a non empty loop structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. We say that f is additive wrt. second argument if and only if:

(Def. 12) For every vector v of V holds $f(v, \cdot)$ is additive.

We say that f is additive wrt. first argument if and only if:

(Def. 13) For every vector w of W holds $f(\cdot, w)$ is additive.

Let K be a non empty groupoid, let V, W be non empty vector space structures over K, and let f be a form of V, W. We say that f is homogeneous wrt. second argument if and only if:

(Def. 14) For every vector v of V holds $f(v, \cdot)$ is homogeneous.

We say that f is homogeneous wrt. first argument if and only if:

(Def. 15) For every vector w of W holds $f(\cdot, w)$ is homogeneous.

Let K be a right zeroed non empty loop structure and let V, W be non empty vector space structures over K. Note that $\mathrm{NulForm}(V, W)$ is additive wrt. second argument and $\mathrm{NulForm}(V, W)$ is additive wrt. first argument.

Let K be a right zeroed non empty loop structure and let V, W be non empty vector space structures over K. Note that there exists a form of V, W which is additive wrt. second argument and additive wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure and let V, W be non empty vector space structures over K. Observe that $\mathrm{NulForm}(V, W)$ is homogeneous wrt. second argument and $\mathrm{NulForm}(V, W)$ is homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure and let V, W be non empty vector space structures over K. One can verify that there exists a form of V, W which is additive wrt. second argument, homogeneous wrt. second argument, additive wrt. first argument, and homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure and let V, W be non empty vector space structures over K. A bilinear form of V, W is an additive wrt. first argument homogeneous wrt. first argument additive wrt. second argument homogeneous wrt. second argument form of V, W.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be an additive wrt. second argument form of V, W, and let v be a vector of V. Note that $f(v, \cdot)$ is additive.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be an additive wrt. first argument form of V, W, and let w be a vector of W. One can check that $f(\cdot, w)$ is additive.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a homogeneous wrt. second argument form of V, W, and let v be a vector of V. Note that $f(v, \cdot)$ is homogeneous.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a homogeneous wrt. first argument form of V, W, and let w be a vector of W. One can verify that $f(\cdot, w)$ is homogeneous.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a functional in V, and let g be an additive functional in W. One can check that $f \otimes g$ is additive wrt. second argument.

Let K be an add-associative right zeroed right complementable commutative right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be an additive functional in V, and let g be a functional in W. Note that $f \otimes g$ is additive wrt. first argument.

Let K be an add-associative right zeroed right complementable commutative associative right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a functional in V, and let g be a homogeneous functional in W. Note that $f \otimes g$ is homogeneous wrt. second argument.

Let K be an add-associative right zeroed right complementable commutative associative right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a homogeneous functional in V, and let g be a functional in W. Note that $f \otimes g$ is homogeneous wrt. first argument.

Let K be a non degenerated non empty double loop structure, let V be a non trivial non empty vector space structure over K, let W be a non empty vector space structure over K, let f be a functional in V, and let g be a functional in W. One can verify that $f \otimes g$ is non trivial.

Let K be a non degenerated non empty double loop structure, let V be a non empty vector space structure over K, let W be a non trivial non empty vector space structure over K, let f be a functional in V, and let g be a functional in W. One can verify that $f \otimes g$ is non trivial.

Let K be a field, let V, W be non trivial vector spaces over K, let f be a non constant 0-preserving functional in V, and let g be a non constant 0-preserving functional in W. Observe that $f \otimes g$ is non constant.

Let K be a field and let V, W be non trivial vector spaces over K. Observe that there exists a form of V, W which is non trivial, non constant, additive wrt. second argument, homogeneous wrt. second argument, additive wrt. first argument, and homogeneous wrt. first argument.

Let K be an Abelian add-associative right zeroed non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be additive wrt. first argument forms of V, W. Observe that $f + g$ is additive wrt. first argument.

Let K be an Abelian add-associative right zeroed non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be additive wrt. second argument forms of V, W. Observe that $f + g$ is additive wrt. second argument.

Let K be a right distributive right zeroed non empty double loop structure, let V, W be non empty vector space structures over K, let f be an additive wrt. first argument form of V, W, and let a be an element of the carrier of K. Observe that $a \cdot f$ is additive wrt. first argument.

Let K be a right distributive right zeroed non empty double loop structure, let V, W be non empty vector space structures over K, let f be an additive wrt. second argument form of V, W, and let a be an element of the carrier of K. Observe that $a \cdot f$ is additive wrt. second argument.

Let K be an Abelian add-associative right zeroed right complementable non empty loop structure, let V, W be non empty vector space structures over K, and let f be an additive wrt. first argument form of V, W. One can verify that $-f$ is additive wrt. first argument.

Let K be an Abelian add-associative right zeroed right complementable non empty loop structure, let V, W be non empty vector space structures over K, and let f be an additive wrt. second argument form of V, W. One can check that $-f$ is additive wrt. second argument.

Let K be an Abelian add-associative right zeroed right complementable non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be additive wrt. first argument forms of V, W. Observe that $f-g$ is additive wrt. first argument.

Let K be an Abelian add-associative right zeroed right complementable non empty loop structure, let V, W be non empty vector space structures over K, and let f, g be additive wrt. second argument forms of V, W. Note that $f-g$ is additive wrt. second argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f, g be homogeneous wrt. first argument forms of V, W. One can verify that $f+g$ is homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f, g be homogeneous wrt. second argument forms of V, W. Note that $f+g$ is homogeneous wrt. second argument.

Let K be an add-associative right zeroed right complementable associative commutative right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a homogeneous wrt. first argument form of V, W, and let a be an element of the carrier of K. One can check that $a \cdot f$ is homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable associative commutative right distributive non empty double loop structure, let V, W be non empty vector space structures over K, let f be a homogeneous wrt. second argument form of V, W, and let a be an element of the carrier of K. One can check that $a \cdot f$ is homogeneous wrt. second argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f be a homogeneous wrt. first argument form of V, W. One can verify that $-f$ is homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f be a homogeneous wrt. second argument form of V, W. Note that $-f$ is homogeneous wrt. second argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f, g be homogeneous wrt. first argument forms of V, W. One can check that $f-g$ is homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V, W be non empty vector space structures over K, and let f, g be homogeneous wrt. second argument forms of

V, W. One can check that $f - g$ is homogeneous wrt. second argument.

We now state a number of propositions:

(27) Let K be a non empty loop structure, V, W be non empty vector space structures over K, v, u be vectors of V, w be a vector of W, and f be a form of V, W. If f is additive wrt. first argument, then $f(\langle v + u, w\rangle) = f(\langle v, w\rangle) + f(\langle u, w\rangle)$.

(28) Let K be a non empty loop structure, V, W be non empty vector space structures over K, v be a vector of V, u, w be vectors of W, and f be a form of V, W. If f is additive wrt. second argument, then $f(\langle v, u+w\rangle) = f(\langle v, u\rangle) + f(\langle v, w\rangle)$.

(29) Let K be a right zeroed non empty loop structure, V, W be non empty vector space structures over K, v, u be vectors of V, w, t be vectors of W, and f be an additive wrt. first argument additive wrt. second argument form of V, W. Then $f(\langle v + u, w + t\rangle) = f(\langle v, w\rangle) + f(\langle v, t\rangle) + (f(\langle u, w\rangle) + f(\langle u, t\rangle))$.

(30) Let K be an add-associative right zeroed right complementable non empty double loop structure, V, W be right zeroed non empty vector space structures over K, f be an additive wrt. second argument form of V, W, and v be a vector of V. Then $f(\langle v, 0_W\rangle) = 0_K$.

(31) Let K be an add-associative right zeroed right complementable non empty double loop structure, V, W be right zeroed non empty vector space structures over K, f be an additive wrt. first argument form of V, W, and w be a vector of W. Then $f(\langle 0_V, w\rangle) = 0_K$.

(32) Let K be a non empty groupoid, V, W be non empty vector space structures over K, v be a vector of V, w be a vector of W, a be an element of the carrier of K, and f be a form of V, W. If f is homogeneous wrt. first argument, then $f(\langle a \cdot v, w\rangle) = a \cdot f(\langle v, w\rangle)$.

(33) Let K be a non empty groupoid, V, W be non empty vector space structures over K, v be a vector of V, w be a vector of W, a be an element of the carrier of K, and f be a form of V, W. If f is homogeneous wrt. second argument, then $f(\langle v, a \cdot w\rangle) = a \cdot f(\langle v, w\rangle)$.

(34) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, V, W be add-associative right zeroed right complementable vector space-like non empty vector space structures over K, f be a homogeneous wrt. first argument form of V, W, and w be a vector of W. Then $f(\langle 0_V, w\rangle) = 0_K$.

(35) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, V, W be add-associative right zeroed right complementable vector space-like non empty vector space structures over K, f be a homogeneous wrt. second

argument form of V, W, and v be a vector of V. Then $f(\langle v, 0_W\rangle) = 0_K$.

(36) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, v, u be vectors of V, w be a vector of W, and f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. Then $f(\langle v-u, w\rangle) = f(\langle v, w\rangle) - f(\langle u, w\rangle)$.

(37) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, v be a vector of V, w, t be vectors of W, and f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. Then $f(\langle v, w-t\rangle) = f(\langle v, w\rangle) - f(\langle v, t\rangle)$.

(38) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, v, u be vectors of V, w, t be vectors of W, and f be a bilinear form of V, W. Then $f(\langle v-u, w-t\rangle) = f(\langle v, w\rangle) - f(\langle v, t\rangle) - (f(\langle u, w\rangle) - f(\langle u, t\rangle))$.

(39) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, V, W be add-associative right zeroed right complementable vector space-like non empty vector space structures over K, v, u be vectors of V, w, t be vectors of W, a, b be elements of the carrier of K, and f be a bilinear form of V, W. Then $f(\langle v + a\cdot u, w + b\cdot t\rangle) = f(\langle v, w\rangle) + b\cdot f(\langle v, t\rangle) + (a\cdot f(\langle u, w\rangle) + a\cdot(b\cdot f(\langle u, t\rangle)))$.

(40) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, v, u be vectors of V, w, t be vectors of W, a, b be elements of the carrier of K, and f be a bilinear form of V, W. Then $f(\langle v - a\cdot u, w - b\cdot t\rangle) = f(\langle v, w\rangle) - b\cdot f(\langle v, t\rangle) - (a\cdot f(\langle u, w\rangle) - a\cdot(b\cdot f(\langle u, t\rangle)))$.

(41) Let K be an add-associative right zeroed right complementable non empty double loop structure, V, W be right zeroed non empty vector space structures over K, and f be a form of V, W. Suppose f is additive wrt. second argument and additive wrt. first argument. Then f is constant if and only if for every vector v of V and for every vector w of W holds $f(\langle v, w\rangle) = 0_K$.

5. Left and Right Kernel of Form. Kernel of "Diagonal"

Let K be a zero structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. The functor leftker f yields a subset of the carrier of V and is defined as follows:

(Def. 16) leftker $f = \{v; v$ ranges over vectors of V: $\bigwedge_{w\,:\,\text{vector of } W} f(\langle v, w\rangle) = 0_K\}$.

Let K be a zero structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. The functor rightker f yielding a subset of the carrier of W is defined by:

(Def. 17) rightker $f = \{w; w$ ranges over vectors of W: $\bigwedge_{v\,:\,\text{vector of } V} f(\langle v, w\rangle) = 0_K\}$.

Let K be a zero structure, let V be a non empty vector space structure over K, and let f be a form of V, V. The functor diagker f yielding a subset of the carrier of V is defined by:

(Def. 18) diagker $f = \{v; v$ ranges over vectors of V: $f(\langle v, v\rangle) = 0_K\}$.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V be a right zeroed non empty vector space structure over K, let W be a non empty vector space structure over K, and let f be an additive wrt. first argument form of V, W. Note that leftker f is non empty.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, let W be a non empty vector space structure over K, and let f be a homogeneous wrt. first argument form of V, W. Observe that leftker f is non empty.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V be a non empty vector space structure over K, let W be a right zeroed non empty vector space structure over K, and let f be an additive wrt. second argument form of V, W. One can verify that rightker f is non empty.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, let V be a non empty vector space structure over K, let W be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and let f be a homogeneous wrt. second argument form of V, W. One can check that rightker f is non empty.

Let K be an add-associative right zeroed right complementable non empty double loop structure, let V be a right zeroed non empty vector space structure over K, and let f be an additive wrt. second argument form of V, V. Note that diagker f is non empty.

Let K be an add-associative right zeroed right complementable non empty double loop structure, let V be a right zeroed non empty vector space structure over K, and let f be an additive wrt. first argument form of V, V. Note that diagker f is non empty.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and let f be a homogeneous wrt. second argument form of V, V. One can check that diagker f is non empty.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and let f be a homogeneous wrt. first argument form of V, V. One can check that diagker f is non empty.

We now state three propositions:

(42) Let K be a zero structure, V be a non empty vector space structure over K, and f be a form of V, V. Then leftker $f \subseteq$ diagker f and rightker $f \subseteq$ diagker f.

(43) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V, W be non empty vector space structures over K, and f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. Then leftker f is linearly closed.

(44) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V, W be non empty vector space structures over K, and f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. Then rightker f is linearly closed.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, let W be a non empty vector space structure over K, and let f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. The functor LKer f yielding a strict non empty subspace of V is defined by:

(Def. 19) The carrier of LKer f = leftker f.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a non empty vector space structure over K, let W be a vector space over K, and let f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. The functor RKer f yielding a strict non empty subspace of W is defined by:

(Def. 20) The carrier of RKer f = rightker f.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, let W be a non empty vector space structure over K, and let f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. The functor LQForm(f) yields an additive wrt. first argument homogeneous wrt. first argument form of ${}^{V}/_{\mathrm{LKer}\, f}$, W and is defined by the condition (Def. 21).

(Def. 21) Let A be a vector of ${}^{V}/_{\mathrm{LKer}\, f}$, w be a vector of W, and v be a vector of V. If $A = v + \mathrm{LKer}\, f$, then $(\mathrm{LQForm}(f))(\langle A, w\rangle) = f(\langle v, w\rangle)$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a non empty vector space structure over K, let W be a vector space over K, and let f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. The functor RQForm(f) yielding an additive wrt. second argument homogeneous wrt. second argument form of V, ${}^{W}/_{\mathrm{RKer}\, f}$ is defined by the condition (Def. 22).

(Def. 22) Let A be a vector of ${}^{W}/_{\mathrm{RKer}\, f}$, v be a vector of V, and w be a vector of W. If $A = w + \mathrm{RKer}\, f$, then $(\mathrm{RQForm}(f))(\langle v, A\rangle) = f(\langle v, w\rangle)$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V, W be vector spaces over K, and let f be a bilinear form of V, W. Note that LQForm(f) is additive wrt. second argument and homogeneous wrt. second argument and RQForm(f) is additive wrt. first argument and homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V, W be vector spaces over K, and let f be a bilinear form of V, W. The functor QForm(f) yields a bilinear form of ${}^{V}/_{\mathrm{LKer}\, f}$, ${}^{W}/_{\mathrm{RKer}\, f}$ and is defined by the condition (Def. 23).

(Def. 23) Let A be a vector of ${}^{V}/_{\mathrm{LKer}\, f}$, B be a vector of ${}^{W}/_{\mathrm{RKer}\, f}$, v be a vector of V, and w be a vector of W. If $A = v + \mathrm{LKer}\, f$ and $B = w + \mathrm{RKer}\, f$, then $(\mathrm{QForm}(f))(\langle A, B\rangle) = f(\langle v, w\rangle)$.

One can prove the following propositions:

(45) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a non empty vector space structure over K, and f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. Then $\mathrm{rightker}\, f = \mathrm{rightker}(\mathrm{LQForm}(f))$.

(46) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a non empty vector space structure over K, W be a vector space over K, and f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. Then $\mathrm{leftker}\, f = \mathrm{leftker}(\mathrm{RQForm}(f))$.

(47) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, and f be a bilinear form of V, W. Then $\mathrm{RKer}\, f = \mathrm{RKer}(\mathrm{LQForm}(f))$.

(48) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, and f be a bilinear form of V, W. Then $\mathrm{LKer}\, f = \mathrm{LKer}(\mathrm{RQForm}(f))$.

(49) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, and f be a bilinear form of V, W. Then $\mathrm{QForm}(f) = \mathrm{RQForm}(\mathrm{LQForm}(f))$ and $\mathrm{QForm}(f) = \mathrm{LQForm}(\mathrm{RQForm}(f))$.

(50) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, W be vector spaces over K, and f be a bilinear form of V, W. Then $\mathrm{leftker}(\mathrm{QForm}(f)) = \mathrm{leftker}(\mathrm{RQForm}(\mathrm{LQForm}(f)))$ and $\mathrm{rightker}(\mathrm{QForm}(f)) = \mathrm{rightker}(\mathrm{RQForm}(\mathrm{LQForm}(f)))$ and $\mathrm{leftker}(\mathrm{QForm}(f)) = \mathrm{leftker}(\mathrm{LQForm}(\mathrm{RQForm}(f)))$ and $\mathrm{rightker}(\mathrm{QForm}(f)) = \mathrm{rightker}(\mathrm{LQForm}(\mathrm{RQForm}(f)))$.

(51) Let K be an add-associative right zeroed right complementable distributive non empty double loop structure, V, W be non empty vector space structures over K, f be a functional in V, and g be a functional in W. Then $\ker f \subseteq \mathrm{leftker}(f \otimes g)$.

(52) Let K be an add-associative right zeroed right complementable associative commutative left unital field-like distributive non empty double loop structure, V, W be non empty vector space structures over K, f be a functional in V, and g be a functional in W. If $g \neq 0\mathrm{Functional}\, W$, then $\mathrm{leftker}(f \otimes g) = \ker f$.

(53) Let K be an add-associative right zeroed right complementable distributive non empty double loop structure, V, W be non empty vector space structures over K, f be a functional in V, and g be a functional in W. Then $\ker g \subseteq \mathrm{rightker}(f \otimes g)$.

(54) Let K be an add-associative right zeroed right complementable associative commutative left unital field-like distributive non empty double loop structure, V, W be non empty vector space structures over K, f be a functional in V, and g be a functional in W. If $f \neq 0\mathrm{Functional}\, V$, then $\mathrm{rightker}(f \otimes g) = \ker g$.

(55) Let K be an add-associative right zeroed right complementable commutative Abelian associative left unital distributive field-like non empty double loop structure, V be a vector space over K, W be a non empty

vector space structure over K, f be a linear functional in V, and g be a functional in W. If $g \neq 0\mathrm{Functional}\,W$, then $\mathrm{LKer}(f \otimes g) = \mathrm{Ker}\,f$ and $\mathrm{LQForm}(f \otimes g) = (\mathrm{CQFunctional}\,f) \otimes g$.

(56) Let K be an add-associative right zeroed right complementable commutative Abelian associative left unital distributive field-like non empty double loop structure, V be a non empty vector space structure over K, W be a vector space over K, f be a functional in V, and g be a linear functional in W. If $f \neq 0\mathrm{Functional}\,V$, then $\mathrm{RKer}(f \otimes g) = \mathrm{Ker}\,g$ and $\mathrm{RQForm}(f \otimes g) = f \otimes (\mathrm{CQFunctional}\,g)$.

(57) Let K be a field, V, W be non trivial vector spaces over K, f be a non constant linear functional in V, and g be a non constant linear functional in W. Then $\mathrm{QForm}(f \otimes g) = (\mathrm{CQFunctional}\,f) \otimes (\mathrm{CQFunctional}\,g)$.

Let K be a zero structure, let V, W be non empty vector space structures over K, and let f be a form of V, W. We say that f is degenerated on left if and only if:

(Def. 24) $\mathrm{leftker}\,f \neq \{0_V\}$.

We say that f is degenerated on right if and only if:

(Def. 25) $\mathrm{rightker}\,f \neq \{0_W\}$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, let W be a right zeroed non empty vector space structure over K, and let f be an additive wrt. first argument homogeneous wrt. first argument form of V, W. Note that $\mathrm{LQForm}(f)$ is non degenerated on left.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a right zeroed non empty vector space structure over K, let W be a vector space over K, and let f be an additive wrt. second argument homogeneous wrt. second argument form of V, W. Note that $\mathrm{RQForm}(f)$ is non degenerated on right.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V, W be vector spaces over K, and let f be a bilinear form of V, W. Observe that $\mathrm{QForm}(f)$ is non degenerated on left and non degenerated on right.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V, W be vector spaces over K, and let f be a bilinear form of V, W. One can verify that $\mathrm{RQForm}(\mathrm{LQForm}(f))$ is non degenerated on left and non degenerated on right and $\mathrm{LQForm}(\mathrm{RQForm}(f))$ is non degenerated on left and non degenerated on right.

Let K be a field, let V, W be non trivial vector spaces over K, and let f be a non constant bilinear form of V, W. Note that $\mathrm{QForm}(f)$ is non constant.

6. Bilinear Symmetric and Alternating Forms

Let K be a 1-sorted structure, let V be a vector space structure over K, and let f be a form of V, V. We say that f is symmetric if and only if:

(Def. 26) For all vectors v, w of V holds $f(\langle v, w\rangle) = f(\langle w, v\rangle)$.

Let K be a zero structure, let V be a vector space structure over K, and let f be a form of V, V. We say that f is alternating if and only if:

(Def. 27) For every vector v of V holds $f(\langle v, v\rangle) = 0_K$.

Let K be a non empty zero structure and let V be a non empty vector space structure over K. Observe that $\mathrm{NulForm}(V, V)$ is symmetric and $\mathrm{NulForm}(V, V)$ is alternating.

Let K be a non empty zero structure and let V be a non empty vector space structure over K. Observe that there exists a form of V, V which is symmetric and there exists a form of V, V which is alternating.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure and let V be a non empty vector space structure over K. Note that there exists a form of V, V which is symmetric, additive wrt. second argument, homogeneous wrt. second argument, additive wrt. first argument, and homogeneous wrt. first argument and there exists a form of V, V which is alternating, additive wrt. second argument, homogeneous wrt. second argument, additive wrt. first argument, and homogeneous wrt. first argument.

Let K be a field and let V be a non trivial vector space over K. Observe that there exists a form of V, V which is symmetric, non trivial, non constant, additive wrt. second argument, homogeneous wrt. second argument, additive wrt. first argument, and homogeneous wrt. first argument.

Let K be an add-associative right zeroed right complementable non empty loop structure and let V be a non empty vector space structure over K. Note that there exists a form of V, V which is alternating, additive wrt. second argument, and additive wrt. first argument.

Let K be a non empty loop structure, let V be a non empty vector space structure over K, and let f, g be symmetric forms of V, V. One can check that $f + g$ is symmetric.

Let K be a non empty double loop structure, let V be a non empty vector space structure over K, let f be a symmetric form of V, V, and let a be an element of the carrier of K. One can check that $a \cdot f$ is symmetric.

Let K be a non empty loop structure, let V be a non empty vector space structure over K, and let f be a symmetric form of V, V. Note that $-f$ is symmetric.

Let K be a non empty loop structure, let V be a non empty vector space structure over K, and let f, g be symmetric forms of V, V. Observe that $f - g$

is symmetric.

Let K be a right zeroed non empty loop structure, let V be a non empty vector space structure over K, and let f, g be alternating forms of V, V. One can check that $f+g$ is alternating.

Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, let V be a non empty vector space structure over K, let f be an alternating form of V, V, and let a be a scalar of K. One can verify that $a \cdot f$ is alternating.

Let K be an add-associative right zeroed right complementable non empty loop structure, let V be a non empty vector space structure over K, and let f be an alternating form of V, V. Note that $-f$ is alternating.

Let K be an add-associative right zeroed right complementable non empty loop structure, let V be a non empty vector space structure over K, and let f, g be alternating forms of V, V. Observe that $f-g$ is alternating.

One can prove the following two propositions:

(58) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V be a non empty vector space structure over K, and f be a symmetric bilinear form of V, V. Then $\operatorname{leftker} f = \operatorname{rightker} f$.

(59) Let K be an add-associative right zeroed right complementable non empty loop structure, V be a non empty vector space structure over K, f be an alternating additive wrt. first argument additive wrt. second argument form of V, V, and v, w be vectors of V. Then $f(\langle v, w\rangle) = -f(\langle w, v\rangle)$.

Let K be a Fanoian field, let V be a non empty vector space structure over K, and let f be an additive wrt. first argument additive wrt. second argument form of V, V. Let us observe that f is alternating if and only if:

(Def. 28) For all vectors v, w of V holds $f(\langle v, w\rangle) = -f(\langle w, v\rangle)$.

Next we state the proposition

(60) Let K be an add-associative right zeroed right complementable right distributive non empty double loop structure, V be a non empty vector space structure over K, and f be an alternating bilinear form of V, V. Then $\operatorname{leftker} f = \operatorname{rightker} f$.

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Hermitan Functionals. Canonical Construction of Scalar Product in Quotient Vector Space[1]

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Summary. In the article we present antilinear functionals, sesquilinear and hermitan forms. We prove Schwarz and Minkowski inequalities, and Parallelogram Law for non-negative hermitan form. The proof of Schwarz inequality is based on [14]. The incorrect proof of this fact can be found in [11]. The construction of scalar product in quotient vector space from non-negative hermitan functions is the main result of the article.

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The notation and terminology used in this paper have been introduced in the following articles: [16], [5], [20], [6], [15], [3], [1], [19], [10], [21], [4], [17], [2], [7], [18], [12], [13], [9], and [8].

1. Auxiliary Facts about Complex Numbers

The following propositions are true:

(1) For every element a of $\mathbb{C}$ such that $a = \overline{a}$ holds $\Im(a) = 0$.

(2) For every element a of $\mathbb{C}$ such that $a \neq 0_{\mathbb{C}}$ holds $|\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i| = 1$ and $\Re((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i) \cdot a) = |a|$ and $\Im((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|}i) \cdot a) = 0$.

(3) For every element a of $\mathbb{C}$ there exists an element b of $\mathbb{C}$ such that $|b| = 1$ and $\Re(b \cdot a) = |a|$ and $\Im(b \cdot a) = 0$.

(4) For every element a of $\mathbb{C}$ holds $a \cdot \overline{a} = |a|^{\mathbf{2}} + 0i$.

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(5) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $a = \overline{a}$ holds $\Im(a) = 0$.

(6) $\overline{i_{\mathbb{C}_{\mathrm{F}}}} = (i)^{-1}$.

(7) $i_{\mathbb{C}_{\mathrm{F}}} \cdot \overline{i_{\mathbb{C}_{\mathrm{F}}}} = \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}$.

(8) Let a be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. Suppose $a \neq 0_{\mathbb{C}_{\mathrm{F}}}$. Then $|\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|} i_{\mathbb{C}_{\mathrm{F}}}| = 1$ and $\Re((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|} i_{\mathbb{C}_{\mathrm{F}}}) \cdot a) = |a|$ and $\Im((\frac{\Re(a)}{|a|} + \frac{-\Im(a)}{|a|} i_{\mathbb{C}_{\mathrm{F}}}) \cdot a) = 0$.

(9) Let a be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. Then there exists an element b of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $|b| = 1$ and $\Re(b \cdot a) = |a|$ and $\Im(b \cdot a) = 0$.

(10) For all elements a, b of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(a - b) = \Re(a) - \Re(b)$ and $\Im(a - b) = \Im(a) - \Im(b)$.

(11) For all elements a, b of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(a) = 0$ holds $\Re(a \cdot b) = \Re(a) \cdot \Re(b)$ and $\Im(a \cdot b) = \Re(a) \cdot \Im(b)$.

(12) For all elements a, b of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(a) = 0$ and $\Im(b) = 0$ holds $\Im(a \cdot b) = 0$.

(13) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(a) = \Re(\overline{a})$.

(14) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $\Im(a) = 0$ holds $a = \overline{a}$.

(15) For all real numbers r, s holds $(r + 0 i_{\mathbb{C}_{\mathrm{F}}}) \cdot (s + 0 i_{\mathbb{C}_{\mathrm{F}}}) = r \cdot s + 0 i_{\mathbb{C}_{\mathrm{F}}}$.

(16) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $a \cdot \overline{a} = |a|^{\mathbf{2}} + 0 i_{\mathbb{C}_{\mathrm{F}}}$.

(17) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ such that $0 \leqslant \Re(a)$ and $\Im(a) = 0$ holds $|a| = \Re(a)$.

(18) For every element a of the carrier of $\mathbb{C}_{\mathrm{F}}$ holds $\Re(a) + \Re(\overline{a}) = 2 \cdot \Re(a)$.

2. Antilinear Functionals in Complex Vector Spaces

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f be a functional in V. We say that f is complex-homogeneous if and only if:

(Def. 1) For every vector v of V and for every scalar a of V holds $f(a \cdot v) = \overline{a} \cdot f(v)$.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that $0\mathrm{Functional}\, V$ is complex-homogeneous.

Let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can verify that every functional in V which is complex-homogeneous is also 0-preserving.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can check that there exists a functional in V which is additive, complex-homogeneous, and 0-preserving.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. An antilinear functional of V is an additive complex-homogeneous functional in V.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f, g be complex-homogeneous functionals in V. Observe that $f + g$ is complex-homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f be a complex-homogeneous functional in V. One can verify that $-f$ is complex-homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, let a be a scalar of V, and let f be a complex-homogeneous functional in V. One can verify that $a \cdot f$ is complex-homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f, g be complex-homogeneous functionals in V. One can check that $f - g$ is complex-homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f be a functional in V. The functor $\overline{f}$ yields a functional in V and is defined by:

(Def. 2) For every vector v of V holds $\overline{f}(v) = \overline{f(v)}$.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f be an additive functional in V. Note that $\overline{f}$ is additive.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f be a homogeneous functional in V. Note that $\overline{f}$ is complex-homogeneous.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f be a complex-homogeneous functional in V. Note that $\overline{f}$ is homogeneous.

Let V be a non trivial vector space over $\mathbb{C}_{\mathrm{F}}$ and let f be a non constant functional in V. One can check that $\overline{f}$ is non constant.

Let V be a non trivial vector space over $\mathbb{C}_{\mathrm{F}}$. One can check that there exists a functional in V which is additive, complex-homogeneous, non constant, and non trivial.

The following propositions are true:

(19) For every non empty vector space structure V over $\mathbb{C}_{\mathrm{F}}$ and for every functional f in V holds $\overline{\overline{f}} = f$.

(20) For every non empty vector space structure V over $\mathbb{C}_{\mathrm{F}}$ holds $\overline{0\mathrm{Functional}\, V} = 0\mathrm{Functional}\, V$.

(21) For every non empty vector space structure V over $\mathbb{C}_{\mathrm{F}}$ and for all functionals f, g in V holds $\overline{f+g} = \overline{f} + \overline{g}$.

(22) For every non empty vector space structure V over $\mathbb{C}_{\mathrm{F}}$ and for every functional f in V holds $\overline{-f} = -\overline{f}$.

(23) Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, f be a functional in V, and a be a scalar of V. Then $\overline{a \cdot f} = \overline{a} \cdot \overline{f}$.

(24) For every non empty vector space structure V over $\mathbb{C}_{\mathrm{F}}$ and for all functionals f, g in V holds $\overline{f-g} = \overline{f} - \overline{g}$.

(25) Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, f be a functional in V, and v be a vector of V. Then $f(v) = 0_{\mathbb{C}_{\mathrm{F}}}$ if and only if $\overline{f}(v) = 0_{\mathbb{C}_{\mathrm{F}}}$.

(26) For every non empty vector space structure V over $\mathbb{C}_{\mathrm{F}}$ and for every functional f in V holds $\ker f = \ker \overline{f}$.

(27) Let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\rm F}$ and f be an antilinear functional of V. Then $\ker f$ is linearly closed.

(28) Let V be a vector space over $\mathbb{C}_{\rm F}$, W be a subspace of V, and f be an antilinear functional of V. If the carrier of $W \subseteq \ker \overline{f}$, then ${}^{f}/_{W}$ is complex-homogeneous.

Let V be a vector space over $\mathbb{C}_{\rm F}$ and let f be an antilinear functional of V. The functor QcFunctional f yields an antilinear functional of ${}^{V}/_{\mathrm{Ker}\,\overline{f}}$ and is defined as follows:

(Def. 3) QcFunctional $f = {}^{f}/_{\mathrm{Ker}\,\overline{f}}$.

We now state the proposition

(29) Let V be a vector space over $\mathbb{C}_{\rm F}$, f be an antilinear functional of V, A be a vector of ${}^{V}/_{\mathrm{Ker}\,\overline{f}}$, and v be a vector of V. If $A = v + \mathrm{Ker}\,\overline{f}$, then $(\mathrm{QcFunctional}\,f)(A) = f(v)$.

Let V be a non trivial vector space over $\mathbb{C}_{\rm F}$ and let f be a non constant antilinear functional of V. One can check that QcFunctional f is non constant.

Let V be a vector space over $\mathbb{C}_{\rm F}$ and let f be an antilinear functional of V. Observe that QcFunctional f is non degenerated.

3. Sesquilinear Forms in Complex Vector Spaces

Let V, W be non empty vector space structures over $\mathbb{C}_{\rm F}$ and let f be a form of V, W. We say that f is complex-homogeneous wrt. second argument if and only if:

(Def. 4) For every vector v of V holds $f(v, \cdot)$ is complex-homogeneous.

We now state the proposition

(30) Let V, W be non empty vector space structures over $\mathbb{C}_{\rm F}$, v be a vector of V, w be a vector of W, a be an element of the carrier of $\mathbb{C}_{\rm F}$, and f be a form of V, W. Suppose f is complex-homogeneous wrt. second argument. Then $f(\langle v, a \cdot w\rangle) = \overline{a} \cdot f(\langle v, w\rangle)$.

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f be a form of V, V. We say that f is hermitan if and only if:

(Def. 5) For all vectors v, u of V holds $f(\langle v, u\rangle) = \overline{f(\langle u, v\rangle)}$.

We say that f is diagonal real valued if and only if:

(Def. 6) For every vector v of V holds $\Im(f(\langle v, v\rangle)) = 0$.

We say that f is diagonal plus-real valued if and only if:

(Def. 7) For every vector v of V holds $0 \leqslant \Re(f(\langle v, v\rangle))$.

Let V, W be non empty vector space structures over $\mathbb{C}_{\rm F}$. Observe that $\mathrm{NulForm}(V, W)$ is complex-homogeneous wrt. second argument.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that $\mathrm{NulForm}(V, V)$ is hermitan and $\mathrm{NulForm}(V, V)$ is diagonal plus-real valued.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that every form of V, V which is hermitan is also diagonal real valued.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can check that there exists a form of V, V which is diagonal plus-real valued, hermitan, diagonal real valued, additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, and complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$. One can check that there exists a form of V, W which is additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, and complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$. A sesquilinear form of V, W is an additive wrt. first argument homogeneous wrt. first argument additive wrt. second argument complex-homogeneous wrt. second argument form of V, W.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. One can check that every form of V, V which is hermitan and additive wrt. second argument is also additive wrt. first argument.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that every form of V, V which is hermitan and additive wrt. first argument is also additive wrt. second argument.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Observe that every form of V, V which is hermitan and homogeneous wrt. first argument is also complex-homogeneous wrt. second argument.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. Note that every form of V, V which is hermitan and complex-homogeneous wrt. second argument is also homogeneous wrt. first argument.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$. A hermitan form of V is a hermitan additive wrt. first argument homogeneous wrt. first argument form of V, V.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$, let f be a functional in V, and let g be a complex-homogeneous functional in W. Note that $f \otimes g$ is complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$, let f be a complex-homogeneous wrt. second argument form of V, W, and let v be a vector of V. One can verify that $f(v, \cdot)$ is complex-homogeneous.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$ and let f, g be complex-homogeneous wrt. second argument forms of V, W. One can verify that $f + g$ is complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$, let f be a complex-homogeneous wrt. second argument form of V, W, and let a be a scalar of V. Observe that $a \cdot f$ is complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$ and let f be a complex-homogeneous wrt. second argument form of V, W. One can check that $-f$ is complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$ and let f, g be complex-homogeneous wrt. second argument forms of V, W. Observe that $f - g$ is complex-homogeneous wrt. second argument.

Let V, W be non trivial vector spaces over $\mathbb{C}_{\mathrm{F}}$. Observe that there exists a form of V, W which is additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, complex-homogeneous wrt. second argument, non constant, and non trivial.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$ and let f be a form of V, W. The functor $\overline{f}$ yielding a form of V, W is defined by:

(Def. 8) For every vector v of V and for every vector w of W holds $\overline{f}(\langle v, w\rangle) = \overline{f(\langle v, w\rangle)}$.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$ and let f be an additive wrt. second argument form of V, W. Note that $\overline{f}$ is additive wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$ and let f be an additive wrt. first argument form of V, W. Note that $\overline{f}$ is additive wrt. first argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$ and let f be a homogeneous wrt. second argument form of V, W. One can check that $\overline{f}$ is complex-homogeneous wrt. second argument.

Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$ and let f be a complex-homogeneous wrt. second argument form of V, W. Note that $\overline{f}$ is homogeneous wrt. second argument.

Let V, W be non trivial vector spaces over $\mathbb{C}_{\mathrm{F}}$ and let f be a non constant form of V, W. One can verify that $\overline{f}$ is non constant.

The following proposition is true

(31) Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, f be a functional in V, and v be a vector of V. Then $f \otimes \overline{f}(\langle v, v\rangle) = |f(v)|^{\mathbf{2}} + 0i_{\mathbb{C}_{\mathrm{F}}}$.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f be a functional in V. One can verify that $f \otimes \overline{f}$ is diagonal plus-real valued, hermitan, and diagonal real valued.

Let V be a non trivial vector space over $\mathbb{C}_{\mathrm{F}}$. Note that there exists a form of V, V which is diagonal plus-real valued, hermitan, diagonal real valued, additive wrt. first argument, homogeneous wrt. first argument, additive wrt. second argument, complex-homogeneous wrt. second argument, non constant, and non trivial.

We now state a number of propositions:

(32) For all non empty vector space structures V, W over $\mathbb{C}_{\mathrm{F}}$ and for every form f of V, W holds $\overline{\overline{f}} = f$.

(33) For all non empty vector space structures V, W over $\mathbb{C}_{\mathrm{F}}$ holds $\overline{\mathrm{NulForm}(V, W)} = \mathrm{NulForm}(V, W)$.

(34) For all non empty vector space structures V, W over $\mathbb{C}_{\mathrm{F}}$ and for all forms f, g of V, W holds $\overline{f + g} = \overline{f} + \overline{g}$.

(35) For all non empty vector space structures V, W over $\mathbb{C}_{\mathrm{F}}$ and for every form f of V, W holds $\overline{-f} = -\overline{f}$.

(36) Let V, W be non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$, f be a form of V, W, and a be an element of $\mathbb{C}_{\mathrm{F}}$. Then $\overline{a \cdot f} = \overline{a} \cdot \overline{f}$.

(37) For all non empty vector space structures V, W over $\mathbb{C}_{\mathrm{F}}$ and for all forms f, g of V, W holds $\overline{f - g} = \overline{f} - \overline{g}$.

(38) Let V, W be vector spaces over $\mathbb{C}_{\mathrm{F}}$, v be a vector of V, w, t be vectors of W, and f be an additive wrt. second argument complex-homogeneous wrt. second argument form of V, W. Then $f(\langle v, w - t \rangle) = f(\langle v, w \rangle) - f(\langle v, t \rangle)$.

(39) Let V, W be vector spaces over $\mathbb{C}_{\mathrm{F}}$, v, u be vectors of V, w, t be vectors of W, and f be a sesquilinear form of V, W. Then $f(\langle v - u, w - t \rangle) = f(\langle v, w \rangle) - f(\langle v, t \rangle) - (f(\langle u, w \rangle) - f(\langle u, t \rangle))$.

(40) Let V, W be add-associative right zeroed right complementable vector space-like non empty vector space structures over $\mathbb{C}_{\mathrm{F}}$, v, u be vectors of V, w, t be vectors of W, a, b be elements of the carrier of $\mathbb{C}_{\mathrm{F}}$, and f be a sesquilinear form of V, W. Then $f(\langle v + a \cdot u, w + b \cdot t \rangle) = f(\langle v, w \rangle) + \overline{b} \cdot f(\langle v, t \rangle) + (a \cdot f(\langle u, w \rangle) + a \cdot (\overline{b} \cdot f(\langle u, t \rangle)))$.

(41) Let V, W be vector spaces over $\mathbb{C}_{\mathrm{F}}$, v, u be vectors of V, w, t be vectors of W, a, b be elements of the carrier of $\mathbb{C}_{\mathrm{F}}$, and f be a sesquilinear form of V, W. Then $f(\langle v - a \cdot u, w - b \cdot t \rangle) = f(\langle v, w \rangle) - \overline{b} \cdot f(\langle v, t \rangle) - (a \cdot f(\langle u, w \rangle) - a \cdot (\overline{b} \cdot f(\langle u, t \rangle)))$.

(42) Let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, f be a complex-homogeneous wrt. second argument form of V, V, and v be a vector of V. Then $f(\langle v, 0_V \rangle) = 0_{\mathbb{C}_{\mathrm{F}}}$.

(43) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, v, w be vectors of V, and f be a hermitan form of V. Then $f(\langle v, w \rangle) + f(\langle v, w \rangle) + f(\langle v, w \rangle) + f(\langle v, w \rangle) = ((f(\langle v + w, v + w \rangle) - f(\langle v - w, v - w \rangle)) + i_{\mathbb{C}_{\mathrm{F}}} \cdot f(\langle v + i_{\mathbb{C}_{\mathrm{F}}} \cdot w, v + i_{\mathbb{C}_{\mathrm{F}}} \cdot w \rangle)) - i_{\mathbb{C}_{\mathrm{F}}} \cdot f(\langle v - i_{\mathbb{C}_{\mathrm{F}}} \cdot w, v - i_{\mathbb{C}_{\mathrm{F}}} \cdot w \rangle)$.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, let f be a form of V, V, and let v be a vector of V. The functor $||v||^2_f$ yields a real number and is defined as follows:

(Def. 9) $||v||_f^2 = \Re(f(\langle v, v\rangle))$.

The following propositions are true:

(44) Let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, f be a diagonal plus-real valued diagonal real valued form of V, V, and v be a vector of V. Then $|f(\langle v, v\rangle)| = \Re(f(\langle v, v\rangle))$ and $||v||_f^2 = |f(\langle v, v\rangle)|$.

(45) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, v, w be vectors of V, f be a sesquilinear form of V, V, r be a real number, and a be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. Suppose $|a| = 1$ and $\Re(a \cdot f(\langle w, v\rangle)) = |f(\langle w, v\rangle)|$ and $\Im(a \cdot f(\langle w, v\rangle)) = 0$. Then $f(\langle v - (r + 0i_{\mathbb{C}_{\mathrm{F}}}) \cdot a \cdot w, v - (r + 0i_{\mathbb{C}_{\mathrm{F}}}) \cdot a \cdot w\rangle) = (f(\langle v, v\rangle) - (r + 0i_{\mathbb{C}_{\mathrm{F}}}) \cdot (a \cdot f(\langle w, v\rangle)) - (r + 0i_{\mathbb{C}_{\mathrm{F}}}) \cdot (\overline{a} \cdot f(\langle v, w\rangle))) + (r^{\mathbf{2}} + 0i_{\mathbb{C}_{\mathrm{F}}}) \cdot f(\langle w, w\rangle)$.

(46) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, v, w be vectors of V, f be a diagonal plus-real valued hermitan form of V, r be a real number, and a be an element of the carrier of $\mathbb{C}_{\mathrm{F}}$. Suppose $|a| = 1$ and $\Re(a \cdot f(\langle w, v\rangle)) = |f(\langle w, v\rangle)|$ and $\Im(a \cdot f(\langle w, v\rangle)) = 0$. Then $\Re(f(\langle v - (r + 0i_{\mathbb{C}_{\mathrm{F}}}) \cdot a \cdot w, v - (r + 0i_{\mathbb{C}_{\mathrm{F}}}) \cdot a \cdot w\rangle)) = (||v||_f^2 - 2 \cdot |f(\langle w, v\rangle)| \cdot r) + ||w||_f^2 \cdot r^{\mathbf{2}}$ and $0 \leqslant (||v||_f^2 - 2 \cdot |f(\langle w, v\rangle)| \cdot r) + ||w||_f^2 \cdot r^{\mathbf{2}}$.

(47) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, v, w be vectors of V, and f be a diagonal plus-real valued hermitan form of V. If $||w||_f^2 = 0$, then $|f(\langle w, v\rangle)| = 0$.

(48) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, v, w be vectors of V, and f be a diagonal plus-real valued hermitan form of V. Then $|f(\langle v, w\rangle)|^{\mathbf{2}} \leqslant ||v||_f^2 \cdot ||w||_f^2$.

(49) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, f be a diagonal plus-real valued hermitan form of V, and v, w be vectors of V. Then $|f(\langle v, w\rangle)|^{\mathbf{2}} \leqslant |f(\langle v, v\rangle)| \cdot |f(\langle w, w\rangle)|$.

(50) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, f be a diagonal plus-real valued hermitan form of V, and v, w be vectors of V. Then $||v + w||_f^2 \leqslant (\sqrt{||v||_f^2} + \sqrt{||w||_f^2})^{\mathbf{2}}$.

(51) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, f be a diagonal plus-real valued hermitan form of V, and v, w be vectors of V. Then $|f(\langle v + w, v + w\rangle)| \leqslant (\sqrt{|f(\langle v, v\rangle)|} + \sqrt{|f(\langle w, w\rangle)|})^{\mathbf{2}}$.

(52) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, f be a hermitan form of V, and v, w be elements of the carrier of V. Then $||v+w||_f^2 + ||v-w||_f^2 = 2 \cdot ||v||_f^2 + 2 \cdot ||w||_f^2$.

(53) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, f be a diagonal plus-real valued hermitan form of V, and v, w be elements of the carrier of V. Then $|f(\langle v + w, v + w\rangle)| + |f(\langle v - w, v - w\rangle)| = 2 \cdot |f(\langle v, v\rangle)| + 2 \cdot |f(\langle w, w\rangle)|$.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f be a form of V, V. The functor $||\cdot||_f$ yields a RFunctional of V and is defined as follows:

(Def. 10) For every element v of the carrier of V holds $(||\cdot||_f)(v) = \sqrt{||v||_f^2}$.

Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$ and let f be a diagonal plus-real valued hermitan form of V. Then $||\cdot||_f$ is a Semi-Norm of V.

4. Kernel of Hermitan Forms and Hermitan Forms in Quotient Vector Spaces

Let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$ and let f be a complex-homogeneous wrt. second argument form of V, V. Note that diagker f is non empty.

We now state several propositions:

(54) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$ and f be a diagonal plus-real valued hermitan form of V. Then diagker f is linearly closed.

(55) For every vector space V over $\mathbb{C}_{\mathrm{F}}$ and for every diagonal plus-real valued hermitan form f of V holds diagker f = leftker f.

(56) For every vector space V over $\mathbb{C}_{\mathrm{F}}$ and for every diagonal plus-real valued hermitan form f of V holds diagker f = rightker f.

(57) For every non empty vector space structure V over $\mathbb{C}_{\mathrm{F}}$ and for every form f of V, V holds diagker f = diagker $\overline{f}$.

(58) For all non empty vector space structures V, W over $\mathbb{C}_{\mathrm{F}}$ and for every form f of V, W holds leftker f = leftker $\overline{f}$ and rightker f = rightker $\overline{f}$.

(59) For every vector space V over $\mathbb{C}_{\mathrm{F}}$ and for every diagonal plus-real valued hermitan form f of V holds LKer f = RKer $\overline{f}$.

(60) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, f be a diagonal plus-real valued diagonal real valued form of V, V, and v be a vector of V. If $\Re(f(\langle v, v\rangle)) = 0$, then $f(\langle v, v\rangle) = 0_{\mathbb{C}_{\mathrm{F}}}$.

(61) Let V be a vector space over $\mathbb{C}_{\mathrm{F}}$, f be a diagonal plus-real valued hermitan form of V, and v be a vector of V. Suppose $\Re(f(\langle v, v\rangle)) = 0$ and f is non degenerated on left and non degenerated on right. Then $v = 0_V$.

Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, let W be a vector space over $\mathbb{C}_{\mathrm{F}}$, and let f be an additive wrt. second argument complex-homogeneous wrt. second argument form of V, W. The functor RQForm$^*(f)$ yielding an additive wrt. second argument complex-homogeneous wrt. second argument form of V, ${}^{W}/_{\mathrm{RKer}\,\overline{f}}$ is defined as follows:

(Def. 11) $\mathrm{RQForm}^*(f) = \overline{\mathrm{RQForm}(\overline{f})}$.

We now state the proposition

(62) Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, W be a vector space over $\mathbb{C}_{\mathrm{F}}$, f be an additive wrt. second argument complex-

homogeneous wrt. second argument form of V, W, v be a vector of V, and w be a vector of W. Then $(\mathrm{RQForm}^*(f))(\langle v, w + \mathrm{RKer}\,\overline{f}\rangle) = f(\langle v, w\rangle)$.

Let V, W be vector spaces over $\mathbb{C}_{\mathrm{F}}$ and let f be a sesquilinear form of V, W. Note that $\mathrm{LQForm}(f)$ is additive wrt. second argument and complex-homogeneous wrt. second argument and $\mathrm{RQForm}^*(f)$ is additive wrt. first argument and homogeneous wrt. first argument.

Let V, W be vector spaces over $\mathbb{C}_{\mathrm{F}}$ and let f be a sesquilinear form of V, W. The functor $\mathrm{QForm}^*\,f$ yields a sesquilinear form of ${}^{V}/_{\mathrm{LKer}\,f}$, ${}^{W}/_{\mathrm{RKer}\,\overline{f}}$ and is defined by the condition (Def. 12).

(Def. 12) Let A be a vector of ${}^{V}/_{\mathrm{LKer}\,f}$, B be a vector of ${}^{W}/_{\mathrm{RKer}\,\overline{f}}$, v be a vector of V, and w be a vector of W. If $A = v + \mathrm{LKer}\,f$ and $B = w + \mathrm{RKer}\,\overline{f}$, then $(\mathrm{QForm}^*\,f)(\langle A, B\rangle) = f(\langle v, w\rangle)$.

Let V, W be non trivial vector spaces over $\mathbb{C}_{\mathrm{F}}$ and let f be a non constant sesquilinear form of V, W. Observe that $\mathrm{QForm}^*\,f$ is non constant.

Let V be a right zeroed non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, let W be a vector space over $\mathbb{C}_{\mathrm{F}}$, and let f be an additive wrt. second argument complex-homogeneous wrt. second argument form of V, W. One can verify that $\mathrm{RQForm}^*(f)$ is non degenerated on right.

One can prove the following propositions:

(63) Let V be a non empty vector space structure over $\mathbb{C}_{\mathrm{F}}$, W be a vector space over $\mathbb{C}_{\mathrm{F}}$, and f be an additive wrt. second argument complex-homogeneous wrt. second argument form of V, W. Then $\mathrm{leftker}\,f = \mathrm{leftker}(\mathrm{RQForm}^*(f))$.

(64) For all vector spaces V, W over $\mathbb{C}_{\mathrm{F}}$ and for every sesquilinear form f of V, W holds $\mathrm{RKer}\,\overline{f} = \mathrm{RKer}\,\overline{\mathrm{LQForm}(f)}$.

(65) For all vector spaces V, W over $\mathbb{C}_{\mathrm{F}}$ and for every sesquilinear form f of V, W holds $\mathrm{LKer}\,f = \mathrm{LKer}(\mathrm{RQForm}^*(f))$.

(66) For all vector spaces V, W over $\mathbb{C}_{\mathrm{F}}$ and for every sesquilinear form f of V, W holds $\mathrm{QForm}^*\,f = \mathrm{RQForm}^*(\mathrm{LQForm}(f))$ and $\mathrm{QForm}^*\,f = \mathrm{LQForm}(\mathrm{RQForm}^*(f))$.

(67) Let V, W be vector spaces over $\mathbb{C}_{\mathrm{F}}$ and f be a sesquilinear form of V, W. Then $\mathrm{leftker}(\mathrm{QForm}^*\,f) = \mathrm{leftker}(\mathrm{RQForm}^*(\mathrm{LQForm}(f)))$ and
$\mathrm{rightker}(\mathrm{QForm}^*\,f) = \mathrm{rightker}(\mathrm{RQForm}^*(\mathrm{LQForm}(f)))$ and
$\mathrm{leftker}(\mathrm{QForm}^*\,f) = \mathrm{leftker}(\mathrm{LQForm}(\mathrm{RQForm}^*(f)))$ and
$\mathrm{rightker}(\mathrm{QForm}^*\,f) = \mathrm{rightker}(\mathrm{LQForm}(\mathrm{RQForm}^*(f)))$.

Let V, W be vector spaces over $\mathbb{C}_{\mathrm{F}}$ and let f be a sesquilinear form of V, W. Note that $\mathrm{RQForm}^*(\mathrm{LQForm}(f))$ is non degenerated on left and non degenerated on right and $\mathrm{LQForm}(\mathrm{RQForm}^*(f))$ is non degenerated on left and non degenerated on right.

Let V, W be vector spaces over $\mathbb{C}_{\rm F}$ and let f be a sesquilinear form of V, W. Note that $\mathrm{QForm}^* f$ is non degenerated on left and non degenerated on right.

5. Scalar Product in Quotient Vector Space Generated by Non-Negative Hermitan Form

Let V be a non empty vector space structure over $\mathbb{C}_{\rm F}$ and let f be a form of V, V. We say that f is positive diagonal valued if and only if:

(Def. 13) For every vector v of V such that $v \neq 0_V$ holds $0 < \Re(f(\langle v, v\rangle))$.

Let V be a right zeroed non empty vector space structure over $\mathbb{C}_{\rm F}$. Note that every form of V, V which is positive diagonal valued and additive wrt. first argument is also diagonal plus-real valued.

Let V be a right zeroed non empty vector space structure over $\mathbb{C}_{\rm F}$. One can verify that every form of V, V which is positive diagonal valued and additive wrt. second argument is also diagonal plus-real valued.

Let V be a vector space over $\mathbb{C}_{\rm F}$ and let f be a diagonal plus-real valued hermitan form of V. The functor $\langle\cdot|\cdot\rangle_f$ yields a diagonal plus-real valued hermitan form of ${}^{V}/_{\mathrm{LKer}\, f}$ and is defined as follows:

(Def. 14) $\langle\cdot|\cdot\rangle_f = \mathrm{QForm}^* f$.

Next we state three propositions:

(68) Let V be a vector space over $\mathbb{C}_{\rm F}$, f be a diagonal plus-real valued hermitan form of V, A, B be vectors of ${}^{V}/_{\mathrm{LKer}\, f}$, and v, w be vectors of V. If $A = v + \mathrm{LKer}\, f$ and $B = w + \mathrm{LKer}\, f$, then $(\langle\cdot|\cdot\rangle_f)(\langle A, B\rangle) = f(\langle v, w\rangle)$.

(69) For every vector space V over $\mathbb{C}_{\rm F}$ and for every diagonal plus-real valued hermitan form f of V holds $\mathrm{leftker}(\langle\cdot|\cdot\rangle_f) = \mathrm{leftker}(\mathrm{QForm}^* f)$.

(70) For every vector space V over $\mathbb{C}_{\rm F}$ and for every diagonal plus-real valued hermitan form f of V holds $\mathrm{rightker}(\langle\cdot|\cdot\rangle_f) = \mathrm{rightker}(\mathrm{QForm}^* f)$.

Let V be a vector space over $\mathbb{C}_{\rm F}$ and let f be a diagonal plus-real valued hermitan form of V. Observe that $\langle\cdot|\cdot\rangle_f$ is non degenerated on left, non degenerated on right, and positive diagonal valued.

Let V be a non trivial vector space over $\mathbb{C}_{\rm F}$ and let f be a diagonal plus-real valued non constant hermitan form of V. Note that $\langle\cdot|\cdot\rangle_f$ is non constant.

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The Class of Series-Parallel Graphs. Part I

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Summary. The paper introduces some preliminary notions concerning the graph theory according to [20]. We define Necklace n as a graph with vertex $\{1, 2, 3, \ldots, n\}$ and edge set $\{(1, 2), (2, 3), \ldots, (n-1, n)\}$. The goal of the article is to prove that Necklace n and Complement of Necklace n are isomorphic for $n = 0, 1, 4$.

MML Identifier: `NECKLACE`.

The terminology and notation used in this paper are introduced in the following papers: [23], [22], [25], [12], [1], [15], [5], [11], [2], [24], [26], [28], [18], [6], [7], [21], [13], [19], [27], [8], [9], [10], [17], [3], [4], [14], and [16].

1. Preliminaries

We adopt the following rules: n is a natural number and x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3 are sets.

Let x_1, x_2, x_3, x_4, x_5 be sets. We say that x_1, x_2, x_3, x_4, x_5 are mutually different if and only if:

(Def. 1) $x_1 \neq x_2$ and $x_1 \neq x_3$ and $x_1 \neq x_4$ and $x_1 \neq x_5$ and $x_2 \neq x_3$ and $x_2 \neq x_4$ and $x_2 \neq x_5$ and $x_3 \neq x_4$ and $x_3 \neq x_5$ and $x_4 \neq x_5$.

Next we state several propositions:

(1) If x_1, x_2, x_3, x_4, x_5 are mutually different, then $\operatorname{card}\{x_1, x_2, x_3, x_4, x_5\} = 5$.

(2) $4 = \{0, 1, 2, 3\}$.

(3) $[\!:\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}:\!] = \{\langle x_1, y_1\rangle, \langle x_1, y_2\rangle, \langle x_1, y_3\rangle, \langle x_2, y_1\rangle, \langle x_2, y_2\rangle, \langle x_2, y_3\rangle, \langle x_3, y_1\rangle, \langle x_3, y_2\rangle, \langle x_3, y_3\rangle\}$.

(4) For every set x and for every natural number n such that $x \in n$ holds x is a natural number.

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(5) For every non empty natural number x holds $0 \in x$.

Let us observe that every set which is natural is also cardinal.

Let X be a set. One can check that δ_X is one-to-one.

Next we state the proposition

(6) For every set X holds $\overline{\overline{\triangle_X}} = \overline{\overline{X}}$.

Let R be a trivial binary relation. Observe that $\operatorname{dom} R$ is trivial.

Let us observe that every function which is trivial is also one-to-one.

We now state several propositions:

(7) For all functions f, g such that $\operatorname{dom} f$ misses $\operatorname{dom} g$ holds $\operatorname{rng}(f{+}{\cdot}g) = \operatorname{rng} f \cup \operatorname{rng} g$.

(8) For all one-to-one functions f, g such that $\operatorname{dom} f$ misses $\operatorname{dom} g$ and $\operatorname{rng} f$ misses $\operatorname{rng} g$ holds $(f{+}{\cdot}g)^{-1} = f^{-1}{+}{\cdot}g^{-1}$.

(9) For all sets A, a, b holds $(A \longmapsto a){+}{\cdot}(A \longmapsto b) = A \longmapsto b$.

(10) For all sets a, b holds $(a \dot{\longmapsto} b)^{-1} = b \dot{\longmapsto} a$.

(11) For all sets a, b, c, d such that $a = b$ iff $c = d$ holds $[a \longmapsto c, b \longmapsto d]^{-1} = [c \longmapsto a, d \longmapsto b]$.

The scheme *Convers* deals with a non empty set $\mathcal{A}$, a binary relation $\mathcal{B}$, two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, and a unary predicate $\mathcal{P}$, and states that:

$$\mathcal{B}^{\smile} = \{\langle \mathcal{F}(x), \mathcal{G}(x) \rangle; x \text{ ranges over elements of } \mathcal{A} : \mathcal{P}[x]\}$$

provided the parameters meet the following condition:

- $\mathcal{B} = \{\langle \mathcal{G}(x), \mathcal{F}(x) \rangle; x \text{ ranges over elements of } \mathcal{A} : \mathcal{P}[x]\}$.

Next we state the proposition

(12) For all natural numbers i, j, n such that $i < j$ and $j \in n$ holds $i \in n$.

2. Auxiliary Concepts

Let R, S be non empty relational structures. We say that S embeds R if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exists a map f from R into S such that

(i) f is one-to-one, and

(ii) for all elements x, y of the carrier of R holds $\langle x, y \rangle \in$ the internal relation of R iff $\langle f(x), f(y) \rangle \in$ the internal relation of S.

Let us note that the predicate S embeds R is reflexive.

One can prove the following proposition

(13) For all non empty relational structures R, S, T such that R embeds S and S embeds T holds R embeds T.

Let R, S be non empty relational structures. We say that R is equimorphic to S if and only if:

(Def. 3) R embeds S and S embeds R.

Let us notice that the predicate R is equimorphic to S is reflexive and symmetric.

The following proposition is true

(14) Let R, S, T be non empty relational structures. Suppose R is equimorphic to S and S is equimorphic to T. Then R is equimorphic to T.

Let R be a non empty relational structure. We introduce R is parallel as an antonym of R is connected.

Let R be a relational structure. We say that R is symmetric if and only if:

(Def. 4) The internal relation of R is symmetric in the carrier of R.

Let R be a relational structure. We say that R is asymmetric if and only if:

(Def. 5) The internal relation of R is asymmetric.

We now state the proposition

(15) Let R be a relational structure. Suppose R is asymmetric. Then the internal relation of R misses (the internal relation of $R)^{\smile}$.

Let R be a relational structure. We say that R is irreflexive if and only if:

(Def. 6) For every set x such that $x \in$ the carrier of R holds $\langle x, x\rangle \notin$ the internal relation of R.

Let n be a natural number. The functor n-SuccRelStr yielding a strict relational structure is defined as follows:

(Def. 7) The carrier of n-SuccRelStr $= n$ and the internal relation of n-SuccRelStr $= \{\langle i, i+1\rangle; i$ ranges over natural numbers: $i+1 < n\}$.

The following propositions are true:

(16) For every natural number n holds n-SuccRelStr is asymmetric.

(17) If $n > 0$, then $\overline{\overline{\text{the internal relation of } n\text{-SuccRelStr}}} = n-1$.

Let R be a relational structure. The functor SymRelStr R yielding a strict relational structure is defined by the conditions (Def. 8).

(Def. 8)(i) The carrier of SymRelStr $R =$ the carrier of R, and

(ii) the internal relation of SymRelStr $R =$ (the internal relation of $R) \cup$ (the internal relation of $R)^{\smile}$.

Let R be a relational structure. Note that SymRelStr R is symmetric.

Let us mention that there exists a relational structure which is non empty and symmetric.

Let R be a symmetric relational structure. One can verify that the internal relation of R is symmetric.

Let R be a relational structure. The functor ComplRelStr R yielding a strict relational structure is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of ComplRelStr $R =$ the carrier of R, and

(ii) the internal relation of ComplRelStr $R =$ (the internal relation of $R)^{\rm c} \setminus \triangle_{\text{the carrier of } R}$.

Let R be a non empty relational structure. Observe that ComplRelStr R is non empty.

Next we state the proposition

(18) Let S, R be relational structures. Suppose S and R are isomorphic. Then $\overline{\overline{\text{the internal relation of } S}} = \overline{\overline{\text{the internal relation of } R}}$.

3. Necklace n

Let n be a natural number. The functor Necklace n yielding a strict relational structure is defined as follows:

(Def. 10) Necklace n = SymRelStr n-SuccRelStr .

Let n be a natural number. One can check that Necklace n is symmetric.

We now state two propositions:

(19) The internal relation of Necklace $n = \{\langle i, i+1\rangle; i$ ranges over natural numbers: $i+1 < n\} \cup \{\langle i+1, i\rangle; i$ ranges over natural numbers: $i+1 < n\}$.

(20) Let x be a set. Then $x \in$ the internal relation of Necklace n if and only if there exists a natural number i such that $i+1 < n$ but $x = \langle i, i+1\rangle$ or $x = \langle i+1, i\rangle$.

Let n be a natural number. Observe that Necklace n is irreflexive.

Next we state the proposition

(21) For every natural number n holds the carrier of Necklace $n = n$.

Let n be a non empty natural number. Observe that Necklace n is non empty.

Let n be a natural number. Observe that the carrier of Necklace n is finite.

One can prove the following propositions:

(22) For all natural numbers n, i such that $i+1 < n$ holds $\langle i, i+1\rangle \in$ the internal relation of Necklace n.

(23) For every natural number n and for every natural number i such that $i \in$ the carrier of Necklace n holds $i < n$.

(24) For every non empty natural number n holds Necklace n is connected.

(25) For all natural numbers i, j such that $\langle i, j\rangle \in$ the internal relation of Necklace n holds $i = j+1$ or $j = i+1$.

(26) Let i, j be natural numbers. Suppose $i = j+1$ or $j = i+1$ but $i \in$ the carrier of Necklace n but $j \in$ the carrier of Necklace n. Then $\langle i, j\rangle \in$ the internal relation of Necklace n.

(27) If $n > 0$, then $\overline{\overline{\{\langle i+1, i\rangle; i \text{ ranges over natural numbers: } i+1 < n\}}} = n - 1$.

(28) If $n > 0$, then $\overline{\overline{\text{the internal relation of Necklace } n}} = 2 \cdot (n-1)$.

(29) Necklace 1 and ComplRelStr Necklace 1 are isomorphic.

(30) Necklace 4 and ComplRelStr Necklace 4 are isomorphic.

(31) If Necklace n and ComplRelStr Necklace n are isomorphic, then $n = 0$ or $n = 1$ or $n = 4$.

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Term Orders

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Summary. We continue the formalization of [5] towards Gröbner Bases. Here we deal with term orders, that is with orders on bags. We introduce the notions of head term, head coefficient, and head monomial necessary to define reduction for polynomials.

MML Identifier: `TERMORD`.

The papers [16], [21], [22], [1], [10], [23], [7], [8], [3], [2], [12], [20], [17], [4], [6], [9], [11], [24], [14], [13], [18], [19], and [15] provide the terminology and notation for this paper.

1. Preliminaries

One can check that there exists a loop structure which is non trivial.

Let us mention that there exists a non trivial loop structure which is add-associative, right complementable, and right zeroed.

Let X be a set and let b be a bag of X. We say that b is non-zero if and only if:

(Def. 1) $b \neq \mathrm{EmptyBag}\, X$.

Next we state two propositions:

(1) For every set X and for all bags b_1, b_2 of X holds $b_1 \mid b_2$ iff there exists a bag b of X such that $b_2 = b_1 + b$.

(2) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive non empty double loop structure, and p be a series of n, L. Then $0\_(n, L) * p = 0\_(n, L)$.

Let n be an ordinal number, let L be an add-associative right complementable right zeroed unital distributive non empty double loop structure, and let m_1, m_2 be monomials of n, L. Note that $m_1 * m_2$ is monomial-like.

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Let n be an ordinal number, let L be an add-associative right complementable right zeroed distributive non empty double loop structure, and let c_1, c_2 be constant polynomials of n, L. One can verify that $c_1 * c_2$ is constant.

One can prove the following two propositions:

(3) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, b, b' be bags of n, and a, a' be non-zero elements of L. Then $\mathrm{Monom}(a \cdot a', b + b') = \mathrm{Monom}(a, b) * \mathrm{Monom}(a', b')$.

(4) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, and a, a' be elements of L. Then $a \cdot a' \_(n, L) = (a \_(n, L)) * (a' \_(n, L))$.

2. Term Orders

Let n be an ordinal number. One can verify that there exists a term order of n which is admissible and connected.

Let n be a natural number. Observe that every admissible term order of n is well founded.

Let n be an ordinal number, let T be a term order of n, and let b, b' be bags of n. The predicate $b \leqslant_T b'$ is defined by:

(Def. 2) $\langle b, b' \rangle \in T$.

Let n be an ordinal number, let T be a term order of n, and let b, b' be bags of n. The predicate $b <_T b'$ is defined by:

(Def. 3) $b \leqslant_T b'$ and $b \neq b'$.

Let n be an ordinal number, let T be a term order of n, and let b_1, b_2 be bags of n. The functor $\min_T(b_1, b_2)$ yields a bag of n and is defined as follows:

(Def. 4) $\min_T(b_1, b_2) = \begin{cases} b_1, & \text{if } b_1 \leqslant_T b_2, \\ b_2, & \text{otherwise.} \end{cases}$

The functor $\max_T(b_1, b_2)$ yields a bag of n and is defined as follows:

(Def. 5) $\max_T(b_1, b_2) = \begin{cases} b_1, & \text{if } b_2 \leqslant_T b_1, \\ b_2, & \text{otherwise.} \end{cases}$

We now state a number of propositions:

(5) Let n be an ordinal number, T be a connected term order of n, and b_1, b_2 be bags of n. Then $b_1 \leqslant_T b_2$ if and only if $b_2 \not<_T b_1$.

(6) For every ordinal number n and for every term order T of n and for every bag b of n holds $b \leqslant_T b$.

(7) Let n be an ordinal number, T be a term order of n, and b_1, b_2 be bags of n. If $b_1 \leqslant_T b_2$ and $b_2 \leqslant_T b_1$, then $b_1 = b_2$.

(8) Let n be an ordinal number, T be a term order of n, and b_1, b_2, b_3 be bags of n. If $b_1 \leqslant_T b_2$ and $b_2 \leqslant_T b_3$, then $b_1 \leqslant_T b_3$.

(9) For every ordinal number n and for every admissible term order T of n and for every bag b of n holds $\mathrm{EmptyBag}\, n \leqslant_T b$.

(10) Let n be an ordinal number, T be an admissible term order of n, and b_1, b_2 be bags of n. If $b_1 \mid b_2$, then $b_1 \leqslant_T b_2$.

(11) For every ordinal number n and for every term order T of n and for all bags b_1, b_2 of n holds $\min_T(b_1, b_2) = b_1$ or $\min_T(b_1, b_2) = b_2$.

(12) For every ordinal number n and for every term order T of n and for all bags b_1, b_2 of n holds $\max_T(b_1, b_2) = b_1$ or $\max_T(b_1, b_2) = b_2$.

(13) Let n be an ordinal number, T be a connected term order of n, and b_1, b_2 be bags of n. Then $\min_T(b_1, b_2) \leqslant_T b_1$ and $\min_T(b_1, b_2) \leqslant_T b_2$.

(14) Let n be an ordinal number, T be a connected term order of n, and b_1, b_2 be bags of n. Then $b_1 \leqslant_T \max_T(b_1, b_2)$ and $b_2 \leqslant_T \max_T(b_1, b_2)$.

(15) Let n be an ordinal number, T be a connected term order of n, and b_1, b_2 be bags of n. Then $\min_T(b_1, b_2) = \min_T(b_2, b_1)$ and $\max_T(b_1, b_2) = \max_T(b_2, b_1)$.

(16) Let n be an ordinal number, T be a connected term order of n, and b_1, b_2 be bags of n. Then $\min_T(b_1, b_2) = b_1$ if and only if $\max_T(b_1, b_2) = b_2$.

3. Head Terms, Head Monomials, and Head Coefficients

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p be a polynomial of n, L. The functor $\mathrm{HT}(p, T)$ yields an element of $\mathrm{Bags}\, n$ and is defined as follows:

(Def. 6) $\mathrm{Support}\, p = \emptyset$ and $\mathrm{HT}(p, T) = \mathrm{EmptyBag}\, n$ or $\mathrm{HT}(p, T) \in \mathrm{Support}\, p$ and for every bag b of n such that $b \in \mathrm{Support}\, p$ holds $b \leqslant_T \mathrm{HT}(p, T)$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p be a polynomial of n, L. The functor $\mathrm{HC}(p, T)$ yielding an element of L is defined as follows:

(Def. 7) $\mathrm{HC}(p, T) = p(\mathrm{HT}(p, T))$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p be a polynomial of n, L. The functor $\mathrm{HM}(p, T)$ yielding a monomial of n, L is defined by:

(Def. 8) $\mathrm{HM}(p, T) = \mathrm{Monom}(\mathrm{HC}(p, T), \mathrm{HT}(p, T))$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non trivial zero structure, and let p be a non-zero polynomial of n, L. Observe that $\mathrm{HM}(p, T)$ is non-zero and $\mathrm{HC}(p, T)$ is non-zero.

The following propositions are true:

(17) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and p be a polynomial of n, L. Then $\mathrm{HC}(p,T) = 0_L$ if and only if $p = 0\_(n,L)$.

(18) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $(\mathrm{HM}(p,T))(\mathrm{HT}(p,T)) = p(\mathrm{HT}(p,T))$.

(19) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, p be a polynomial of n, L, and b be a bag of n. If $b \neq \mathrm{HT}(p,T)$, then $(\mathrm{HM}(p,T))(b) = 0_L$.

(20) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $\mathrm{Support}\,\mathrm{HM}(p,T) \subseteq \mathrm{Support}\,p$.

(21) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $\mathrm{Support}\,\mathrm{HM}(p,T) = \emptyset$ or $\mathrm{Support}\,\mathrm{HM}(p,T) = \{\mathrm{HT}(p,T)\}$.

(22) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $\mathrm{term}\,\mathrm{HM}(p,T) = \mathrm{HT}(p,T)$ and $\mathrm{coefficient}\,\mathrm{HM}(p,T) = \mathrm{HC}(p,T)$.

(23) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and m be a monomial of n, L. Then $\mathrm{HT}(m,T) = \mathrm{term}\,m$ and $\mathrm{HC}(m,T) = \mathrm{coefficient}\,m$ and $\mathrm{HM}(m,T) = m$.

(24) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and c be a constant polynomial of n, L. Then $\mathrm{HT}(c,T) = \mathrm{EmptyBag}\,n$ and $\mathrm{HC}(c,T) = c(\mathrm{EmptyBag}\,n)$.

(25) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and a be an element of L. Then $\mathrm{HT}(a\,\_(n,L),T) = \mathrm{EmptyBag}\,n$ and $\mathrm{HC}(a\,\_(n,L),T) = a$.

(26) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $\mathrm{HT}(\mathrm{HM}(p,T),T) = \mathrm{HT}(p,T)$.

(27) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a polynomial of n, L. Then $\mathrm{HC}(\mathrm{HM}(p,T),T) = \mathrm{HC}(p,T)$.

(28) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and p be a polynomial of n, L. Then $\mathrm{HM}(\mathrm{HM}(p,T),T) = \mathrm{HM}(p,T)$.

(29) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable left zeroed right zeroed unital distributive integral domain-like non trivial double loop structure, and p, q be non-zero polynomials of n, L. Then $\mathrm{HT}(p,T) + \mathrm{HT}(q,T) \in$

Support$(p * q)$.

(30) Let n be an ordinal number, L be an add-associative right complementable right zeroed unital distributive non empty double loop structure, and p, q be polynomials of n, L. Then $\text{Support}(p * q) \subseteq \{s + t; s$ ranges over elements of Bags n, t ranges over elements of Bags $n : s \in \text{Support}\, p \wedge t \in \text{Support}\, q\}$.

(31) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, and p, q be non-zero polynomials of n, L. Then $\text{HT}(p*q, T) = \text{HT}(p, T) + \text{HT}(q, T)$.

(32) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, and p, q be non-zero polynomials of n, L. Then $\text{HC}(p*q, T) = \text{HC}(p, T) \cdot \text{HC}(q, T)$.

(33) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, and p, q be non-zero polynomials of n, L. Then $\text{HM}(p*q, T) = \text{HM}(p, T) * \text{HM}(q, T)$.

(34) Let n be an ordinal number, T be an admissible connected term order of n, L be a right zeroed non empty loop structure, and p, q be polynomials of n, L. Then $\text{HT}(p + q, T) \leqslant_T \max_T(\text{HT}(p, T), \text{HT}(q, T))$.

4. Reductum of a Polynomial

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed non empty loop structure, and let p be a polynomial of n, L. The functor $\text{Red}(p, T)$ yielding a polynomial of n, L is defined by:

(Def. 9) $\text{Red}(p, T) = p - \text{HM}(p, T)$.

The following propositions are true:

(35) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a polynomial of n, L. Then $\text{Support}\,\text{Red}(p, T) \subseteq \text{Support}\, p$.

(36) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a polynomial of n, L. Then $\text{Support}\,\text{Red}(p, T) = \text{Support}\, p \setminus \{\text{HT}(p, T)\}$.

(37) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop

structure, and p be a polynomial of n, L. Then $\operatorname{Support}(\mathrm{HM}(p,T) + \mathrm{Red}(p,T)) = \operatorname{Support} p$.

(38) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a polynomial of n, L. Then $\mathrm{HM}(p,T) + \mathrm{Red}(p,T) = p$.

(39) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a polynomial of n, L. Then $(\mathrm{Red}(p,T))(\mathrm{HT}(p,T)) = 0_L$.

(40) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, p be a polynomial of n, L, and b be a bag of n. If $b \in \operatorname{Support} p$ and $b \neq \mathrm{HT}(p,T)$, then $(\mathrm{Red}(p,T))(b) = p(b)$.

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Polynomial Reduction

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Summary. We continue the formalization of [8] towards Gröbner Bases. In this article we introduce reduction of polynomials and prove its termination, its adequateness for ideal congruence as well as the translation lemma used later to show confluence of reduction.

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The notation and terminology used here are introduced in the following papers: [21], [26], [12], [27], [29], [28], [10], [11], [4], [3], [17], [6], [22], [13], [5], [25], [2], [7], [24], [9], [16], [14], [19], [1], [23], [18], [15], and [20].

1. Preliminaries

Let n be an ordinal number and let R be a non trivial zero structure. One can verify that there exists a monomial of n, R which is non-zero.

Let us observe that there exists a field which is non trivial.

Let us note that every left zeroed add-right-cancelable right distributive left unital commutative associative non empty double loop structure which is field-like is also integral domain-like.

Let n be an ordinal number, let L be an add-associative right complementable left zeroed right zeroed unital distributive integral domain-like non trivial double loop structure, and let p, q be non-zero finite-Support series of n, L. Note that $p * q$ is non-zero.

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2. More on Polynomials and Monomials

The following propositions are true:

(1) Let X be a set, L be an Abelian add-associative right zeroed right complementable non empty loop structure, and p, q be series of X, L. Then $-(p+q) = -p + -q$.

(2) For every set X and for every left zeroed non empty loop structure L and for every series p of X, L holds $0\_(X, L) + p = p$.

(3) Let X be a set, L be an add-associative right zeroed right complementable non empty loop structure, and p be a series of X, L. Then $-p + p = 0\_(X, L)$ and $p + -p = 0\_(X, L)$.

(4) Let n be a set, L be an add-associative right zeroed right complementable non empty loop structure, and p be a series of n, L. Then $p - 0\_(n, L) = p$.

(5) Let n be an ordinal number, L be an add-associative right complementable right zeroed add-left-cancelable left distributive non empty double loop structure, and p be a series of n, L. Then $0\_(n, L) * p = 0\_(n, L)$.

(6) Let n be an ordinal number, L be an Abelian right zeroed add-associative right complementable unital distributive associative commutative non trivial double loop structure, and p, q be polynomials of n, L. Then $-p * q = (-p) * q$ and $-p * q = p * -q$.

(7) Let n be an ordinal number, L be an add-associative right complementable right zeroed distributive non empty double loop structure, p be a polynomial of n, L, m be a monomial of n, L, and b be a bag of n. Then $(m * p)(\mathrm{term}\, m + b) = m(\mathrm{term}\, m) \cdot p(b)$.

(8) Let X be a set, L be a right zeroed add-left-cancelable left distributive non empty double loop structure, and p be a series of X, L. Then $0_L \cdot p = 0\_(X, L)$.

(9) Let X be a set, L be an add-associative right zeroed right complementable distributive non empty double loop structure, p be a series of X, L, and a be an element of L. Then $-a \cdot p = (-a) \cdot p$ and $-a \cdot p = a \cdot -p$.

(10) Let X be a set, L be a left distributive non empty double loop structure, p be a series of X, L, and a, a' be elements of L. Then $a \cdot p + a' \cdot p = (a + a') \cdot p$.

(11) Let X be a set, L be an associative non empty multiplicative loop with zero structure, p be a series of X, L, and a, a' be elements of L. Then $(a \cdot a') \cdot p = a \cdot (a' \cdot p)$.

(12) Let n be an ordinal number, L be an add-associative right zeroed right complementable unital associative commutative distributive non empty double loop structure, p, p' be series of n, L, and a be an element of L. Then $a \cdot (p * p') = p * (a \cdot p')$.

3. Multiplication of Polynomials with Bags

Let n be an ordinal number, let b be a bag of n, let L be a non empty zero structure, and let p be a series of n, L. The functor $b * p$ yielding a series of n, L is defined as follows:

(Def. 1) For every bag b' of n such that $b \mid b'$ holds $(b*p)(b') = p(b' -' b)$ and for every bag b' of n such that $b \nmid b'$ holds $(b*p)(b') = 0_L$.

Let n be an ordinal number, let b be a bag of n, let L be a non empty zero structure, and let p be a finite-Support series of n, L. Note that $b * p$ is finite-Support.

We now state a number of propositions:

(13) Let n be an ordinal number, b, b' be bags of n, L be a non empty zero structure, and p be a series of n, L. Then $(b*p)(b'+b) = p(b')$.

(14) Let n be an ordinal number, L be a non empty zero structure, p be a polynomial of n, L, and b be a bag of n. Then $\mathrm{Support}(b*p) \subseteq \{b+b'; b'$ ranges over elements of $\mathrm{Bags}\, n : b' \in \mathrm{Support}\, p\}$.

(15) Let n be an ordinal number, T be a connected admissible term order of n, L be a non trivial zero structure, p be a non-zero polynomial of n, L, and b be a bag of n. Then $\mathrm{HT}(b*p, T) = b + \mathrm{HT}(p, T)$.

(16) Let n be an ordinal number, T be a connected admissible term order of n, L be a non empty zero structure, p be a polynomial of n, L, and b, b' be bags of n. If $b' \in \mathrm{Support}(b*p)$, then $b' \leqslant_T b + \mathrm{HT}(p, T)$.

(17) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and p be a series of n, L. Then $\mathrm{EmptyBag}\, n * p = p$.

(18) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, p be a series of n, L, and b_1, b_2 be bags of n. Then $(b_1+b_2)*p = b_1*(b_2*p)$.

(19) Let n be an ordinal number, L be an add-associative right zeroed right complementable distributive non trivial double loop structure, p be a polynomial of n, L, and a be an element of L. Then $\mathrm{Support}(a \cdot p) \subseteq \mathrm{Support}\, p$.

(20) Let n be an ordinal number, L be an integral domain-like non trivial double loop structure, p be a polynomial of n, L, and a be a non-zero element of L. Then $\mathrm{Support}\, p \subseteq \mathrm{Support}(a \cdot p)$.

(21) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right zeroed right complementable distributive integral domain-like non trivial double loop structure, p be a polynomial of n, L, and a be a non-zero element of L. Then $\mathrm{HT}(a \cdot p, T) = \mathrm{HT}(p, T)$.

(22) Let n be an ordinal number, L be an add-associative right complementable right zeroed distributive non trivial double loop structure, p

be a series of n, L, b be a bag of n, and a be an element of L. Then $a \cdot (b * p) = \mathrm{Monom}(a, b) * p$.

(23) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed unital distributive integral domain-like non trivial double loop structure, p be a non-zero polynomial of n, L, q be a polynomial of n, L, and m be a non-zero monomial of n, L. If $\mathrm{HT}(p, T) \in \mathrm{Support}\, q$, then $\mathrm{HT}(m * p, T) \in \mathrm{Support}(m * q)$.

4. Orders on Polynomials

Let n be an ordinal number and let T be a connected term order of n. Observe that $\langle \mathrm{Bags}\, n, T \rangle$ is connected.

Let n be a natural number and let T be an admissible term order of n. Note that $\langle \mathrm{Bags}\, n, T \rangle$ is well founded.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p, q be polynomials of n, L. The predicate $p \leqslant_T q$ is defined as follows:

(Def. 2) $\langle \mathrm{Support}\, p, \mathrm{Support}\, q \rangle \in \mathrm{FinOrd} \langle \mathrm{Bags}\, n, T \rangle$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p, q be polynomials of n, L. The predicate $p <_T q$ is defined as follows:

(Def. 3) $p \leqslant_T q$ and $\mathrm{Support}\, p \neq \mathrm{Support}\, q$.

Let n be an ordinal number, let T be a connected term order of n, let L be a non empty zero structure, and let p be a polynomial of n, L. The functor $\mathrm{Support}(p, T)$ yielding an element of Fin (the carrier of $\langle \mathrm{Bags}\, n, T \rangle$) is defined by:

(Def. 4) $\mathrm{Support}(p, T) = \mathrm{Support}\, p$.

Next we state a number of propositions:

(24) Let n be an ordinal number, T be a connected term order of n, L be a non trivial zero structure, and p be a non-zero polynomial of n, L. Then $\mathrm{PosetMax}\, \mathrm{Support}(p, T) = \mathrm{HT}(p, T)$.

(25) Let n be an ordinal number, T be a connected term order of n, L be a non empty loop structure, and p be a polynomial of n, L. Then $p \leqslant_T p$.

(26) Let n be an ordinal number, T be a connected term order of n, L be a non empty loop structure, and p, q be polynomials of n, L. Then $p \leqslant_T q$ and $q \leqslant_T p$ if and only if $\mathrm{Support}\, p = \mathrm{Support}\, q$.

(27) Let n be an ordinal number, T be a connected term order of n, L be a non empty loop structure, and p, q, r be polynomials of n, L. If $p \leqslant_T q$ and $q \leqslant_T r$, then $p \leqslant_T r$.

(28) Let n be an ordinal number, T be a connected term order of n, L be a non empty loop structure, and p, q be polynomials of n, L. Then $p \leqslant_T q$ or $q \leqslant_T p$.

(29) Let n be an ordinal number, T be a connected term order of n, L be a non empty loop structure, and p, q be polynomials of n, L. Then $p \leqslant_T q$ if and only if $q \not<_T p$.

(30) Let n be an ordinal number, T be a connected term order of n, L be a non empty zero structure, and p be a polynomial of n, L. Then $0\_(n, L) \leqslant_T p$.

(31) Let n be a natural number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed unital distributive non trivial double loop structure, and P be a non empty subset of Polynom-Ring(n, L). Then there exists a polynomial p of n, L such that $p \in P$ and for every polynomial q of n, L such that $q \in P$ holds $p \leqslant_T q$.

(32) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p, q be polynomials of n, L. Then $p <_T q$ if and only if one of the following conditions is satisfied:

(i) $p = 0\_(n, L)$ and $q \neq 0\_(n, L)$, or

(ii) $\mathrm{HT}(p, T) <_T \mathrm{HT}(q, T)$, or

(iii) $\mathrm{HT}(p, T) = \mathrm{HT}(q, T)$ and $\mathrm{Red}(p, T) <_T \mathrm{Red}(q, T)$.

(33) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a non-zero polynomial of n, L. Then $\mathrm{Red}(p, T) <_T \mathrm{HM}(p, T)$.

(34) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a polynomial of n, L. Then $\mathrm{HM}(p, T) \leqslant_T p$.

(35) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed non trivial loop structure, and p be a non-zero polynomial of n, L. Then $\mathrm{Red}(p, T) <_T p$.

5. Polynomial Reduction

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, let f, p, g be polynomials of n, L, and let b be a bag of n. We say that f reduces to g, p, b, T if and only if:

(Def. 5) $f \neq 0\_(n, L)$ and $p \neq 0\_(n, L)$ and $b \in \mathrm{Support}\, f$ and there exists a bag s of n such that $s + \mathrm{HT}(p, T) = b$ and $g = f - \frac{f(b)}{\mathrm{HC}(p,T)} \cdot (s * p)$.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let f, p, g be polynomials of n, L. We say that f reduces to g, p, T if and only if:

(Def. 6) There exists a bag b of n such that f reduces to g, p, b, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, let f, g be polynomials of n, L, and let P be a subset of Polynom-Ring(n, L). We say that f reduces to g, P, T if and only if:

(Def. 7) There exists a polynomial p of n, L such that $p \in P$ and f reduces to g, p, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let f, p be polynomials of n, L. We say that f is reducible wrt p, T if and only if:

(Def. 8) There exists a polynomial g of n, L such that f reduces to g, p, T.

We introduce f is irreducible wrt p, T and f is in normal form wrt p, T as antonyms of f is reducible wrt p, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, let f be a polynomial of n, L, and let P be a subset of Polynom-Ring(n, L). We say that f is reducible wrt P, T if and only if:

(Def. 9) There exists a polynomial g of n, L such that f reduces to g, P, T.

We introduce f is irreducible wrt P, T and f is in normal form wrt P, T as antonyms of f is reducible wrt P, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let f, p, g be polynomials of n, L. We say that f top reduces to g, p, T if and only if:

(Def. 10) f reduces to g, p, $\mathrm{HT}(f, T)$, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let f, p be polynomials of n, L. We say that f is top reducible wrt p, T if and only if:

(Def. 11) There exists a polynomial g of n, L such that f top reduces to g, p, T.

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, let f be a polynomial of n, L, and let P be a subset of Polynom-Ring(n, L). We say that f is top reducible wrt P, T if and only if:

(Def. 12) There exists a polynomial p of n, L such that $p \in P$ and f is top reducible wrt p, T.

Next we state several propositions:

(36) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, f be a polynomial of n, L, and p be a non-zero polynomial of n, L. Then f is reducible wrt p, T if and only if there exists a bag b of n such that $b \in \mathrm{Support}\, f$ and $\mathrm{HT}(p, T) \mid b$.

(37) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and p be a polynomial of n, L. Then $0\_(n, L)$ is irreducible wrt p, T.

(38) Let n be an ordinal number, T be an admissible connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non degenerated non empty double loop structure, f, p be polynomials of n, L, and m be a non-zero monomial of n, L. If f reduces to $f - m * p$, p, T, then $\mathrm{HT}(m * p, T) \in \mathrm{Support}\, f$.

(39) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, f, p, g be polynomials of n, L, and b be a bag of n. If f reduces to g, p, b, T, then $b \notin \mathrm{Support}\, g$.

(40) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, f, p, g be polynomials of n, L, and b, b' be bags of n. Suppose $b <_T b'$. If f reduces to g, p, b, T, then $b' \in \mathrm{Support}\, g$ iff $b' \in \mathrm{Support}\, f$.

(41) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, f, p, g be polynomials of n, L, and b, b' be bags of n. If $b <_T b'$, then if f reduces to g, p, b, T, then $f(b') = g(b')$.

(42) Let n be an ordinal number, T be a connected admissible term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and f, p, g be polynomials of n, L. Suppose f

reduces to g, p, T. Let b be a bag of n. If $b \in \operatorname{Support} g$, then $b \leqslant_T \mathrm{HT}(f, T)$.

(43) Let n be an ordinal number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and f, p, g be polynomials of n, L. If f reduces to g, p, T, then $g <_T f$.

6. Polynomial Reduction Relation

Let n be an ordinal number, let T be a connected term order of n, let L be an add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, and let P be a subset of Polynom-Ring(n, L). The functor PolyRedRel(P, T) yields a relation between (the carrier of Polynom-Ring(n, L)) $\setminus \{0\_(n, L)\}$ and the carrier of Polynom-Ring(n, L) and is defined by:

(Def. 13) For all polynomials p, q of n, L holds $\langle p, q \rangle \in$ PolyRedRel(P, T) iff p reduces to q, P, T.

Next we state the proposition

(44) Let n be an ordinal number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, f, g be polynomials of n, L, and P be a subset of Polynom-Ring(n, L). If PolyRedRel(P, T) reduces f to g, then $g \leqslant_T f$ but $g = 0\_(n, L)$ or $\mathrm{HT}(g, T) \leqslant_T \mathrm{HT}(f, T)$.

Let n be a natural number, let T be a connected admissible term order of n, let L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, and let P be a subset of Polynom-Ring(n, L). One can verify that PolyRedRel(P, T) is strongly-normalizing.

One can prove the following propositions:

(45) Let n be a natural number, T be an admissible connected term order of n, L be an add-associative right complementable left zeroed right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, h be polynomials of n, L. If $f \in P$, then PolyRedRel(P, T) reduces $h * f$ to $0\_(n, L)$.

(46) Let n be an ordinal number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated

non empty double loop structure, P be a subset of Polynom-Ring(n, L), f, g be polynomials of n, L, and m be a non-zero monomial of n, L. If f reduces to g, P, T, then $m * f$ reduces to $m * g$, P, T.

(47) Let n be an ordinal number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, P be a subset of Polynom-Ring(n, L), f, g be polynomials of n, L, and m be a monomial of n, L. If PolyRedRel(P, T) reduces f to g, then PolyRedRel(P, T) reduces $m * f$ to $m * g$.

(48) Let n be an ordinal number, T be a connected admissible term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, P be a subset of Polynom-Ring(n, L), f be a polynomial of n, L, and m be a monomial of n, L. If PolyRedRel(P, T) reduces f to $0\_(n, L)$, then PolyRedRel(P, T) reduces $m * f$ to $0\_(n, L)$.

(49) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, g, h, h_1 be polynomials of n, L. Suppose $f - g = h$ and PolyRedRel(P, T) reduces h to h_1. Then there exist polynomials f_1, g_1 of n, L such that $f_1 - g_1 = h_1$ and PolyRedRel(P, T) reduces f to f_1 and PolyRedRel(P, T) reduces g to g_1.

(50) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, g be polynomials of n, L. Suppose PolyRedRel(P, T) reduces $f - g$ to $0\_(n, L)$. Then f and g are convergent w.r.t. PolyRedRel(P, T).

(51) Let n be an ordinal number, T be a connected term order of n, L be an add-associative right complementable right zeroed commutative associative well unital distributive Abelian field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, g be polynomials of n, L. Suppose PolyRedRel(P, T) reduces $f - g$ to $0\_(n, L)$. Then f and g are convertible w.r.t. PolyRedRel(P, T).

Let R be a non empty loop structure, let I be a subset of R, and let a, b be elements of R. The predicate $a \equiv b (\mathrm{mod}\, I)$ is defined as follows:

(Def. 14) $a - b \in I$.

One can prove the following propositions:

(52) Let R be an add-associative left zeroed right zeroed right complementable right distributive non empty double loop structure, I be a right ideal

non empty subset of R, and a be an element of R. Then $a \equiv a(\mathrm{mod}\, I)$.

(53) Let R be an add-associative right zeroed right complementable right unital right distributive non empty double loop structure, I be a right ideal non empty subset of R, and a, b be elements of R. If $a \equiv b(\mathrm{mod}\, I)$, then $b \equiv a(\mathrm{mod}\, I)$.

(54) Let R be an add-associative right zeroed right complementable non empty loop structure, I be an add closed non empty subset of R, and a, b, c be elements of R. If $a \equiv b(\mathrm{mod}\, I)$ and $b \equiv c(\mathrm{mod}\, I)$, then $a \equiv c(\mathrm{mod}\, I)$.

(55) Let R be an Abelian add-associative right zeroed right complementable unital distributive associative non trivial double loop structure, I be an add closed non empty subset of R, and a, b, c, d be elements of R. If $a \equiv b(\mathrm{mod}\, I)$ and $c \equiv d(\mathrm{mod}\, I)$, then $a + c \equiv b + d(\mathrm{mod}\, I)$.

(56) Let R be an add-associative right zeroed right complementable commutative distributive non empty double loop structure, I be an add closed right ideal non empty subset of R, and a, b, c, d be elements of R. If $a \equiv b(\mathrm{mod}\, I)$ and $c \equiv d(\mathrm{mod}\, I)$, then $a \cdot c \equiv b \cdot d(\mathrm{mod}\, I)$.

(57) Let n be an ordinal number, T be a connected term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, g be elements of Polynom-Ring(n, L). If f and g are convertible w.r.t. PolyRedRel(P, T), then $f \equiv g(\mathrm{mod}\, P$–ideal).

(58) Let n be a natural number, T be an admissible connected term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non degenerated non empty double loop structure, P be a non empty subset of Polynom-Ring(n, L), and f, g be elements of Polynom-Ring(n, L). If $f \equiv g(\mathrm{mod}\, P$–ideal), then f and g are convertible w.r.t. PolyRedRel(P, T).

(59) Let n be an ordinal number, T be a connected term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f, g be polynomials of n, L. If PolyRedRel(P, T) reduces f to g, then $f - g \in P$–ideal.

(60) Let n be an ordinal number, T be a connected term order of n, L be an Abelian add-associative right complementable right zeroed commutative associative well unital distributive field-like non trivial double loop structure, P be a subset of Polynom-Ring(n, L), and f be a polynomial of n, L. If PolyRedRel(P, T) reduces f to $0\_(n, L)$, then $f \in P$–ideal.

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Processes in Petri Nets

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Summary. Sequential and concurrent compositions of processes in Petri nets are introduced. A process is formalized as a set of (possible), so called, firing sequences. In the definition of the sequential composition the standard concatenation is used

$$R_1 \operatorname{before} R_2 = \{p_1 \frown p_2 : p_1 \in R_1 \ \wedge \ p_2 \in R_2\}$$

The definition of the concurrent composition is more complicated

$$R_1 \operatorname{concur} R_2 = \{q_1 \cup q_2 : q_1 \text{ misses } q_2 \ \wedge \ \operatorname{Seq} q_1 \in R_1 \ \wedge \ \operatorname{Seq} q_2 \in R_2\}$$

For example,

$$\{\langle t_0 \rangle\} \operatorname{concur} \{\langle t_1, t_2 \rangle\} = \{\langle t_0, t_1, t_2 \rangle, \langle t_1, t_0, t_2 \rangle, \langle t_1, t_2, t_0 \rangle\}$$

The basic properties of the compositions are shown.

MML Identifier: `PNPROC_1`.

The notation and terminology used in this paper are introduced in the following papers: [14], [13], [18], [6], [17], [9], [1], [3], [7], [12], [2], [10], [15], [5], [16], [8], [11], and [4].

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1. Preliminaries

We adopt the following rules: i is a natural number and x, x_1, x_2, y_1, y_2 are sets.

Next we state three propositions:

(1) If $i > 0$, then $\{\langle i, x\rangle\}$ is a finite subsequence.

(2) For every finite subsequence q holds $q = \emptyset$ iff $\mathrm{Seq}\, q = \emptyset$.

(3) For every finite subsequence q such that $q = \{\langle i, x\rangle\}$ holds $\mathrm{Seq}\, q = \langle x\rangle$.

Let us observe that every finite subsequence is finite.

We now state several propositions:

(4) For every finite subsequence q such that $\mathrm{Seq}\, q = \langle x\rangle$ there exists i such that $q = \{\langle i, x\rangle\}$.

(5) If $\{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle\}$ is a finite sequence, then $x_1 = 1$ and $x_2 = 1$ and $y_1 = y_2$ or $x_1 = 1$ and $x_2 = 2$ or $x_1 = 2$ and $x_2 = 1$.

(6) $\langle x_1, x_2\rangle = \{\langle 1, x_1\rangle, \langle 2, x_2\rangle\}$.

(7) For every finite subsequence p holds $\overline{\overline{p}} = \mathrm{len}\, \mathrm{Seq}\, p$.

(8) For all binary relations P, R such that $\mathrm{dom}\, P$ misses $\mathrm{dom}\, R$ holds P misses R.

(9) For all sets X, Y and for all binary relations P, R such that X misses Y holds $P{\restriction}X$ misses $R{\restriction}Y$.

(10) For all functions f, g, h such that $f \subseteq h$ and $g \subseteq h$ and f misses g holds $\mathrm{dom}\, f$ misses $\mathrm{dom}\, g$.

(11) For every set Y and for every binary relation R holds $Y{\restriction}R \subseteq R{\restriction}R^{-1}(Y)$.

(12) For every set Y and for every function f holds $Y{\restriction}f = f{\restriction}f^{-1}(Y)$.

2. Markings on Petri Nets

Let P be a set. A function is called a marking of P if:

(Def. 1) $\mathrm{dom}\,\mathrm{it} = P$ and $\mathrm{rng}\,\mathrm{it} \subseteq \mathbb{N}$.

We adopt the following convention: P, p, x denote sets, m_1, m_2, m_3, m_4, m denote markings of P, and i, j, j_1, k denote natural numbers.

Let P be a set, let m be a marking of P, and let p be a set. Then $m(p)$ is a natural number. We introduce the m multitude of p as a synonym of $m(p)$.

The scheme *MarkingLambda* deals with a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a natural number, and states that:

> There exists a marking m of $\mathcal{A}$ such that for every p such that $p \in \mathcal{A}$ holds the m multitude of $p = \mathcal{F}(p)$

for all values of the parameters.

Let us consider P, m_1, m_2. Let us observe that $m_1 = m_2$ if and only if:

(Def. 2) For every p such that $p \in P$ holds the m_1 multitude of $p =$ the m_2 multitude of p.

Let us consider P. The functor $\{\}_P$ yielding a marking of P is defined by:

(Def. 3) $\{\}_P = P \longmapsto 0$.

Let P be a set and let m_1, m_2 be markings of P. The predicate $m_1 \subseteq m_2$ is defined by:

(Def. 4) For every set p such that $p \in P$ holds the m_1 multitude of $p \leqslant$ the m_2 multitude of p.

Let us note that the predicate $m_1 \subseteq m_2$ is reflexive.

We now state two propositions:

(13) $\{\}_P \subseteq m$.

(14) If $m_1 \subseteq m_2$ and $m_2 \subseteq m_3$, then $m_1 \subseteq m_3$.

Let P be a set and let m_1, m_2 be markings of P. The functor $m_1 + m_2$ yields a marking of P and is defined as follows:

(Def. 5) For every set p such that $p \in P$ holds the $m_1 + m_2$ multitude of $p =$ (the m_1 multitude of p) $+$ (the m_2 multitude of p).

Let us notice that the functor $m_1 + m_2$ is commutative.

The following proposition is true

(15) $m + \{\}_P = m$.

Let P be a set and let m_1, m_2 be markings of P. Let us assume that $m_2 \subseteq m_1$. The functor $m_1 - m_2$ yielding a marking of P is defined by:

(Def. 6) For every set p such that $p \in P$ holds the $m_1 - m_2$ multitude of $p =$ (the m_1 multitude of p) $-$ (the m_2 multitude of p).

One can prove the following propositions:

(16) $m_1 \subseteq m_1 + m_2$.

(17) $m - \{\}_P = m$.

(18) If $m_1 \subseteq m_2$ and $m_2 \subseteq m_3$, then $m_3 - m_2 \subseteq m_3 - m_1$.

(19) $(m_1 + m_2) - m_2 = m_1$.

(20) If $m \subseteq m_1$ and $m_1 \subseteq m_2$, then $m_1 - m \subseteq m_2 - m$.

(21) If $m_1 \subseteq m_2$, then $(m_2 + m_3) - m_1 = (m_2 - m_1) + m_3$.

(22) If $m_1 \subseteq m_2$ and $m_2 \subseteq m_1$, then $m_1 = m_2$.

(23) $(m_1 + m_2) + m_3 = m_1 + (m_2 + m_3)$.

(24) If $m_1 \subseteq m_2$ and $m_3 \subseteq m_4$, then $m_1 + m_3 \subseteq m_2 + m_4$.

(25) If $m_1 \subseteq m_2$, then $m_2 - m_1 \subseteq m_2$.

(26) If $m_1 \subseteq m_2$ and $m_3 \subseteq m_4$ and $m_4 \subseteq m_1$, then $m_1 - m_4 \subseteq m_2 - m_3$.

(27) If $m_1 \subseteq m_2$, then $m_2 = (m_2 - m_1) + m_1$.

(28) $(m_1 + m_2) - m_1 = m_2$.

(29) If $m_2 + m_3 \subseteq m_1$, then $m_1 - m_2 - m_3 = m_1 - (m_2 + m_3)$.

(30) If $m_3 \subseteq m_2$ and $m_2 \subseteq m_1$, then $m_1 - (m_2 - m_3) = (m_1 - m_2) + m_3$.

(31) $m \in \mathbb{N}^P$.

(32) If $x \in \mathbb{N}^P$, then x is a marking of P.

3. Transitions and Firing

Let us consider P. Transition of P is defined by:

(Def. 7) There exist m_1, m_2 such that it $= \langle m_1, m_2 \rangle$.

In the sequel t, t_1, t_2 denote transitions of P.

Let us consider P, t. Then $t_{\mathbf{1}}$ is a marking of P. We introduce Pre t as a synonym of $t_{\mathbf{1}}$. $t_{\mathbf{2}}$ is a marking of P. We introduce Post t as a synonym of $t_{\mathbf{2}}$.

Let us consider P, m, t. The functor $\mathrm{fire}(t, m)$ yielding a marking of P is defined by:

(Def. 8) $\mathrm{fire}(t, m) = \begin{cases} (m - \mathrm{Pre}\ t) + \mathrm{Post}\ t, & \text{if } \mathrm{Pre}\ t \subseteq m, \\ m, & \text{otherwise.} \end{cases}$

The following proposition is true

(33) If $\mathrm{Pre}\ t_1 + \mathrm{Pre}\ t_2 \subseteq m$, then $\mathrm{fire}(t_2, \mathrm{fire}(t_1, m)) = (m - \mathrm{Pre}\ t_1 - \mathrm{Pre}\ t_2) + \mathrm{Post}\ t_1 + \mathrm{Post}\ t_2$.

Let us consider P, t. The functor fire t yielding a function is defined by:

(Def. 9) $\mathrm{dom}\,\mathrm{fire}\ t = \mathbb{N}^P$ and for every marking m of P holds $(\mathrm{fire}\ t)(m) = \mathrm{fire}(t, m)$.

Next we state two propositions:

(34) $\mathrm{rng}\,\mathrm{fire}\ t \subseteq \mathbb{N}^P$.

(35) $\mathrm{fire}(t_2, \mathrm{fire}(t_1, m)) = (\mathrm{fire}\ t_2 \cdot \mathrm{fire}\ t_1)(m)$.

Let us consider P. A non empty set is called a Petri net over P if:

(Def. 10) For every set x such that $x \in$ it holds x is a transition of P.

In the sequel N denotes a Petri net over P.

Let us consider P, N. We see that the element of N is a transition of P.

In the sequel e, e_1, e_2 denote elements of N.

4. Firing Sequences of Transitions

Let us consider P, N. A firing-sequence of N is an element of N^*.

In the sequel C, C_1, C_2 are firing-sequences of N.

Let us consider P, N, C. The functor fire C yielding a function is defined by the condition (Def. 11).

(Def. 11) There exists a function yielding finite sequence F such that fire $C = \mathrm{compose}_{\mathbb{N}^P} F$ and $\mathrm{len}\,F = \mathrm{len}\,C$ and for every natural number i such that $i \in \mathrm{dom}\,C$ holds $F(i) = \mathrm{fire}\,(C_i$ **qua** element of $N)$.

The following propositions are true:

(36) fire $(\varepsilon_N) = \mathrm{id}_{\mathbb{N}^P}$.

(37) fire $\langle e \rangle =$ fire e.

(38) fire $e \cdot \mathrm{id}_{\mathbb{N}^P} =$ fire e.

(39) fire $\langle e_1, e_2 \rangle =$ fire $e_2 \cdot$ fire e_1.

(40) dom fire $C = \mathbb{N}^P$ and rng fire $C \subseteq \mathbb{N}^P$.

(41) fire $(C_1 \frown C_2) =$ fire $C_2 \cdot$ fire C_1.

(42) fire $(C \frown \langle e \rangle) =$ fire $e \cdot$ fire C.

Let us consider P, N, C, m. The functor fire(C, m) yielding a marking of P is defined as follows:

(Def. 12) fire$(C, m) =$ (fire $C)(m)$.

5. Sequential Composition

Let us consider P, N. A process in N is a subset of N^*.

In the sequel R, R_1, R_2, R_3, P_1, P_2 denote processes in N.

One can verify that every function which is finite sequence-like is also finite subsequence-like.

Let us consider P, N, R_1, R_2. The functor R_1 before R_2 yields a process in N and is defined by:

(Def. 13) R_1 before $R_2 = \{C_1 \frown C_2 : C_1 \in R_1 \ \wedge \ C_2 \in R_2\}$.

Let us consider P, N and let R_1, R_2 be non empty processes in N. One can verify that R_1 before R_2 is non empty.

One can prove the following propositions:

(43) $(R_1 \cup R_2)$ before $R = (R_1$ before $R) \cup (R_2$ before $R)$.

(44) R before$(R_1 \cup R_2) = (R$ before $R_1) \cup (R$ before $R_2)$.

(45) $\{C_1\}$ before$\{C_2\} = \{C_1 \frown C_2\}$.

(46) $\{C_1, C_2\}$ before$\{C\} = \{C_1 \frown C, C_2 \frown C\}$.

(47) $\{C\}$ before$\{C_1, C_2\} = \{C \frown C_1, C \frown C_2\}$.

6. Concurrent Composition

Let us consider P, N, R_1, R_2. The functor R_1 concur R_2 yielding a process in N is defined as follows:

(Def. 14) R_1 concur $R_2 = \{C : \bigvee_{q_1, q_2 : \text{finite subsequence}} (C = q_1 \cup q_2 \ \wedge \ q_1$ misses $q_2 \ \wedge \ \mathrm{Seq}\, q_1 \in R_1 \ \wedge \ \mathrm{Seq}\, q_2 \in R_2)\}$.

Let us observe that the functor R_1 concur R_2 is commutative.

Next we state four propositions:

(48) $(R_1 \cup R_2) \operatorname{concur} R = (R_1 \operatorname{concur} R) \cup (R_2 \operatorname{concur} R)$.

(49) $\{\langle e_1 \rangle\} \operatorname{concur} \{\langle e_2 \rangle\} = \{\langle e_1, e_2 \rangle, \langle e_2, e_1 \rangle\}$.

(50) $\{\langle e_1 \rangle, \langle e_2 \rangle\} \operatorname{concur} \{\langle e \rangle\} = \{\langle e_1, e \rangle, \langle e_2, e \rangle, \langle e, e_1 \rangle, \langle e, e_2 \rangle\}$.

(51) $(R_1 \operatorname{before} R_2) \operatorname{before} R_3 = R_1 \operatorname{before}(R_2 \operatorname{before} R_3)$.

Let p be a finite subsequence and let i be a natural number. The functor $\operatorname{Shift}^i p$ yielding a finite subsequence is defined as follows:

(Def. 15) $\operatorname{dom} \operatorname{Shift}^i p = \{i + k; k$ ranges over natural numbers: $k \in \operatorname{dom} p\}$ and for every natural number j such that $j \in \operatorname{dom} p$ holds $(\operatorname{Shift}^i p)(i + j) = p(j)$.

In the sequel q, q_1, q_2 denote finite subsequences.

One can prove the following propositions:

(52) $\operatorname{Shift}^0 q = q$.

(53) $\operatorname{Shift}^{i+j} q = \operatorname{Shift}^i \operatorname{Shift}^j q$.

(54) For every finite sequence p such that $p \neq \emptyset$ holds $\operatorname{dom} \operatorname{Shift}^i p = \{j_1 : i + 1 \leqslant j_1 \ \wedge \ j_1 \leqslant i + \operatorname{len} p\}$.

(55) For every finite subsequence q holds $q = \emptyset$ iff $\operatorname{Shift}^i q = \emptyset$.

(56) Let q be a finite subsequence. Then there exists a finite subsequence s_1 such that $\operatorname{dom} s_1 = \operatorname{dom} q$ and $\operatorname{rng} s_1 = \operatorname{dom} \operatorname{Shift}^i q$ and for every k such that $k \in \operatorname{dom} q$ holds $s_1(k) = i + k$ and s_1 is one-to-one.

(57) For every finite subsequence q holds $\overline{\overline{q}} = \overline{\overline{\operatorname{Shift}^i q}}$.

(58) For every finite sequence p holds $\operatorname{dom} p = \operatorname{dom} \operatorname{Seq} \operatorname{Shift}^i p$.

(59) For every finite sequence p such that $k \in \operatorname{dom} p$ holds $(\operatorname{Sgm} \operatorname{dom} \operatorname{Shift}^i p)(k) = i + k$.

(60) For every finite sequence p such that $k \in \operatorname{dom} p$ holds $(\operatorname{Seq} \operatorname{Shift}^i p)(k) = p(k)$.

(61) For every finite sequence p holds $\operatorname{Seq} \operatorname{Shift}^i p = p$.

In the sequel p_1, p_2 are finite sequences.

One can prove the following propositions:

(62) $\operatorname{dom}(p_1 \cup \operatorname{Shift}^{\operatorname{len} p_1} p_2) = \operatorname{Seg}(\operatorname{len} p_1 + \operatorname{len} p_2)$.

(63) For every finite sequence p_1 and for every finite subsequence p_2 such that $\operatorname{len} p_1 \leqslant i$ holds $\operatorname{dom} p_1$ misses $\operatorname{dom} \operatorname{Shift}^i p_2$.

(64) For all finite sequences p_1, p_2 holds $p_1 {}^\frown p_2 = p_1 \cup \operatorname{Shift}^{\operatorname{len} p_1} p_2$.

(65) For every finite sequence p_1 and for every finite subsequence p_2 such that $i \geqslant \operatorname{len} p_1$ holds p_1 misses $\operatorname{Shift}^i p_2$.

(66) $(R_1 \operatorname{concur} R_2) \operatorname{concur} R_3 = R_1 \operatorname{concur}(R_2 \operatorname{concur} R_3)$.

(67) $R_1 \operatorname{before} R_2 \subseteq R_1 \operatorname{concur} R_2$.

(68) If $R_1 \subseteq P_1$ and $R_2 \subseteq P_2$, then $R_1 \operatorname{before} R_2 \subseteq P_1 \operatorname{before} P_2$.

(69) If $R_1 \subseteq P_1$ and $R_2 \subseteq P_2$, then $R_1 \operatorname{concur} R_2 \subseteq P_1 \operatorname{concur} P_2$.

(70) For all finite subsequences p, q such that $q \subseteq p$ holds $\operatorname{Shift}^i q \subseteq \operatorname{Shift}^i p$.

(71) For all finite sequences p_1, p_2 holds $\mathrm{Shift}^{\mathrm{len}\,p_1}\,p_2 \subseteq p_1 \frown p_2$.

(72) If $\mathrm{dom}\,q_1$ misses $\mathrm{dom}\,q_2$, then $\mathrm{dom}\,\mathrm{Shift}^i\,q_1$ misses $\mathrm{dom}\,\mathrm{Shift}^i\,q_2$.

(73) For all finite subsequences q, q_1, q_2 such that $q = q_1 \cup q_2$ and q_1 misses q_2 holds $\mathrm{Shift}^i\,q_1 \cup \mathrm{Shift}^i\,q_2 = \mathrm{Shift}^i\,q$.

(74) For every finite subsequence q holds $\mathrm{dom}\,\mathrm{Seq}\,q = \mathrm{dom}\,\mathrm{Seq}\,\mathrm{Shift}^i\,q$.

(75) For every finite subsequence q such that $k \in \mathrm{dom}\,\mathrm{Seq}\,q$ there exists j such that $j = (\mathrm{Sgm}\,\mathrm{dom}\,q)(k)$ and $(\mathrm{Sgm}\,\mathrm{dom}\,\mathrm{Shift}^i\,q)(k) = i + j$.

(76) For every finite subsequence q such that $k \in \mathrm{dom}\,\mathrm{Seq}\,q$ holds $(\mathrm{Seq}\,\mathrm{Shift}^i\,q)(k) = (\mathrm{Seq}\,q)(k)$.

(77) For every finite subsequence q holds $\mathrm{Seq}\,q = \mathrm{Seq}\,\mathrm{Shift}^i\,q$.

(78) For every finite subsequence q such that $\mathrm{dom}\,q \subseteq \mathrm{Seg}\,k$ holds $\mathrm{dom}\,\mathrm{Shift}^i\,q \subseteq \mathrm{Seg}(i + k)$.

(79) Let p be a finite sequence and q_1, q_2 be finite subsequences. If $q_1 \subseteq p$, then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \mathrm{Shift}^{\mathrm{len}\,p}\,q_2$.

(80) Let p_1, p_2 be finite sequences and q_1, q_2 be finite subsequences. Suppose $q_1 \subseteq p_1$ and $q_2 \subseteq p_2$. Then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \mathrm{Shift}^{\mathrm{len}\,p_1}\,q_2$ and $\mathrm{dom}\,\mathrm{Seq}\,s_1 = \mathrm{Seg}(\mathrm{len}\,\mathrm{Seq}\,q_1 + \mathrm{len}\,\mathrm{Seq}\,q_2)$.

(81) Let p_1, p_2 be finite sequences and q_1, q_2 be finite subsequences. Suppose $q_1 \subseteq p_1$ and $q_2 \subseteq p_2$. Then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \mathrm{Shift}^{\mathrm{len}\,p_1}\,q_2$ and $\mathrm{dom}\,\mathrm{Seq}\,s_1 = \mathrm{Seg}(\mathrm{len}\,\mathrm{Seq}\,q_1 + \mathrm{len}\,\mathrm{Seq}\,q_2)$ and $\mathrm{Seq}\,s_1 = \mathrm{Seq}\,q_1 \cup \mathrm{Shift}^{\mathrm{len}\,\mathrm{Seq}\,q_1}\,\mathrm{Seq}\,q_2$.

(82) Let p_1, p_2 be finite sequences and q_1, q_2 be finite subsequences. Suppose $q_1 \subseteq p_1$ and $q_2 \subseteq p_2$. Then there exists a finite subsequence s_1 such that $s_1 = q_1 \cup \mathrm{Shift}^{\mathrm{len}\,p_1}\,q_2$ and $(\mathrm{Seq}\,q_1) \frown (\mathrm{Seq}\,q_2) = \mathrm{Seq}\,s_1$.

(83) $(R_1\,\mathrm{concur}\,R_2)\,\mathrm{before}(P_1\,\mathrm{concur}\,P_2) \subseteq (R_1\,\mathrm{before}\,P_1)\,\mathrm{concur}(R_2\,\mathrm{before}\,P_2)$.

Let us consider P, N and let R_1, R_2 be non empty processes in N. Note that $R_1\,\mathrm{concur}\,R_2$ is non empty.

7. Neutral Process

Let us consider P and let N be a Petri net over P. The neutral process in N yields a non empty process in N and is defined as follows:

(Def. 16) The neutral process in $N = \{\varepsilon_N\}$.

Let us consider P, let N be a Petri net over P, and let t be an element of N. The elementary process with t yielding a non empty process in N is defined as follows:

(Def. 17) The elementary process with $t = \{\langle t \rangle\}$.

One can prove the following propositions:

(84) (The neutral process in N) $\mathrm{before}\,R = R$.

(85) R before the neutral process in $N = R$.

(86) (The neutral process in N) concur $R = R$.

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