FORMALIZED MATHEMATICS
Volume 10, Number **3**, 2002
University of Białystok

On the Decomposition of a Simple Closed Curve into Two Arcs

Andrzej Trybulec[1]
University of Białystok

Yatsuka Nakamura
Shinshu University
Nagano

Summary. The purpose of the paper is to prove lemmas needed for the Jordan curve theorem. The main result is that the decomposition of a simple closed curve into two arcs with the ends p_1, p_2 is unique in the sense that every arc on the curve with the same ends must be equal to one of them.

MML Identifier: `JORDAN16`.

The articles [25], [24], [26], [14], [27], [2], [4], [8], [3], [22], [17], [21], [7], [6], [20], [1], [23], [15], [9], [5], [10], [19], [18], [11], [13], [12], and [16] provide the terminology and notation for this paper.

One can prove the following proposition

(1) Let S_1 be a finite non empty subset of $\mathbb{R}$ and e be a real number. If for every real number r such that $r \in S_1$ holds $r < e$, then $\max S_1 < e$.

For simplicity, we use the following convention: C is a simple closed curve, A, A_1, A_2 are subsets of $\mathcal{E}^2_{\rm T}$, p, p_1, p_2, q, q_1, q_2 are points of $\mathcal{E}^2_{\rm T}$, and n is a natural number.

Let us consider n. Note that there exists a subset of $\mathcal{E}^n_{\rm T}$ which is trivial.

We now state a number of propositions:

(2) For all sets a, b, c, X such that $a \in X$ and $b \in X$ and $c \in X$ holds $\{a, b, c\} \subseteq X$.

(3) $\emptyset_{\mathcal{E}^n_{\rm T}}$ is Bounded.

(4) $\mathrm{LowerArc}\, C \neq \mathrm{UpperArc}\, C$.

(5) $\mathrm{Segment}(A, p_1, p_2, q_1, q_2) \subseteq A$.

[1]This work has been partially supported by CALCULEMUS grant HPRN-CT-2000-00102. The work has been done while the author visited Shinshu University.

ISSN 1426–2630

(6) Let T be a non empty topological space and A, B be subsets of the carrier of T. If $A \subseteq B$, then $T \restriction A$ is a subspace of $T \restriction B$.

(7) If A is an arc from p_1 to p_2 and $q \in A$, then $q \in \mathrm{LSegment}(A, p_1, p_2, q)$.

(8) If A is an arc from p_1 to p_2 and $q \in A$, then $q \in \mathrm{RSegment}(A, p_1, p_2, q)$.

(9) If A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2, then $q_1 \in \mathrm{Segment}(A, p_1, p_2, q_1, q_2)$ and $q_2 \in \mathrm{Segment}(A, p_1, p_2, q_1, q_2)$.

(10) $\mathrm{Segment}(p, q, C) \subseteq C$.

(11) If $p \in C$ and $q \in C$, then $\mathrm{LE}(p, q, C)$ or $\mathrm{LE}(q, p, C)$.

(12) Let X, Y be non empty topological spaces, Y_0 be a non empty subspace of Y, f be a map from X into Y, and g be a map from X into Y_0. If $f = g$ and f is continuous, then g is continuous.

(13) Let S, T be non empty topological spaces, S_0 be a non empty subspace of S, T_0 be a non empty subspace of T, and f be a map from S into T. Suppose f is a homeomorphism. Let g be a map from S_0 into T_0. If $g = f \restriction S_0$ and g is onto, then g is a homeomorphism.

(14) Let P_1, P_2, P_3 be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and P_3 is an arc from p_1 to p_2 and $P_2 \cap P_3 = \{p_1, p_2\}$ and $P_1 \subseteq P_2 \cup P_3$. Then $P_1 = P_2$ or $P_1 = P_3$.

(15) Let C be a simple closed curve, A_1, A_2 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose A_1 is an arc from p_1 to p_2 and A_2 is an arc from p_1 to p_2 and $A_1 \subseteq C$ and $A_2 \subseteq C$ and $A_1 \neq A_2$. Then $A_1 \cup A_2 = C$ and $A_1 \cap A_2 = \{p_1, p_2\}$.

(16) Let A_1, A_2 be subsets of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If A_1 is an arc from p_1 to p_2 and $A_1 \cap A_2 = \{q_1, q_2\}$, then $A_1 \neq A_2$.

(17) Let C be a simple closed curve, A_1, A_2 be subsets of $\mathcal{E}^2_{\mathrm{T}}$, and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose A_1 is an arc from p_1 to p_2 and A_2 is an arc from p_1 to p_2 and $A_1 \subseteq C$ and $A_2 \subseteq C$ and $A_1 \cap A_2 = \{p_1, p_2\}$. Then $A_1 \cup A_2 = C$.

(18) Suppose $A_1 \subseteq C$ and $A_2 \subseteq C$ and $A_1 \neq A_2$ and A_1 is an arc from p_1 to p_2 and A_2 is an arc from p_1 to p_2. Let given A. If A is an arc from p_1 to p_2 and $A \subseteq C$, then $A = A_1$ or $A = A_2$.

(19) Let C be a simple closed curve and A be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. If A is an arc from W-min C to E-max C and $A \subseteq C$, then $A = \mathrm{LowerArc}\, C$ or $A = \mathrm{UpperArc}\, C$.

(20) Suppose A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2. Then there exists a map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction A$ and there exist real numbers s_1, s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $g(s_2) = q_2$ and $0 \leqslant s_1$ and $s_1 \leqslant s_2$ and $s_2 \leqslant 1$.

(21) Suppose A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2 and $q_1 \neq q_2$. Then there exists a map g from $\mathbb{I}$ into $(\mathcal{E}^2_{\mathrm{T}}) \restriction A$ and there exist real numbers

s_1, s_2 such that g is a homeomorphism and $g(0) = p_1$ and $g(1) = p_2$ and $g(s_1) = q_1$ and $g(s_2) = q_2$ and $0 \leqslant s_1$ and $s_1 < s_2$ and $s_2 \leqslant 1$.

(22) If A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2, then $\mathrm{Segment}(A, p_1, p_2, q_1, q_2)$ is non empty.

(23) If $p \in C$, then $p \in \mathrm{Segment}(p, \text{W-min}\, C, C)$ and $\text{W-min}\, C \in \mathrm{Segment}(p, \text{W-min}\, C, C)$.

Let f be a partial function from $\mathbb{R}$ to $\mathbb{R}$. We say that f is continuous if and only if:

(Def. 1) f is continuous on $\mathrm{dom}\, f$.

Let f be a function from $\mathbb{R}$ into $\mathbb{R}$. Let us observe that f is continuous if and only if:

(Def. 2) f is continuous on $\mathbb{R}$.

Let a, b be real numbers. The functor $\mathrm{AffineMap}(a, b)$ yielding a function from $\mathbb{R}$ into $\mathbb{R}$ is defined by:

(Def. 3) For every real number x holds $(\mathrm{AffineMap}(a, b))(x) = a \cdot x + b$.

Let a, b be real numbers. Observe that $\mathrm{AffineMap}(a, b)$ is continuous.

Let us mention that there exists a function from $\mathbb{R}$ into $\mathbb{R}$ which is continuous.

We now state a number of propositions:

(24) Let f, g be continuous partial functions from $\mathbb{R}$ to $\mathbb{R}$. Then $g \cdot f$ is a continuous partial function from $\mathbb{R}$ to $\mathbb{R}$.

(25) For all real numbers a, b holds $(\mathrm{AffineMap}(a, b))(0) = b$.

(26) For all real numbers a, b holds $(\mathrm{AffineMap}(a, b))(1) = a + b$.

(27) For all real numbers a, b such that $a \neq 0$ holds $\mathrm{AffineMap}(a, b)$ is one-to-one.

(28) For all real numbers a, b, x, y such that $a > 0$ and $x < y$ holds $(\mathrm{AffineMap}(a, b))(x) < (\mathrm{AffineMap}(a, b))(y)$.

(29) For all real numbers a, b, x, y such that $a < 0$ and $x < y$ holds $(\mathrm{AffineMap}(a, b))(x) > (\mathrm{AffineMap}(a, b))(y)$.

(30) For all real numbers a, b, x, y such that $a \geqslant 0$ and $x \leqslant y$ holds $(\mathrm{AffineMap}(a, b))(x) \leqslant (\mathrm{AffineMap}(a, b))(y)$.

(31) For all real numbers a, b, x, y such that $a \leqslant 0$ and $x \leqslant y$ holds $(\mathrm{AffineMap}(a, b))(x) \geqslant (\mathrm{AffineMap}(a, b))(y)$.

(32) For all real numbers a, b such that $a \neq 0$ holds $\mathrm{rng}\, \mathrm{AffineMap}(a, b) = \mathbb{R}$.

(33) For all real numbers a, b such that $a \neq 0$ holds $(\mathrm{AffineMap}(a, b))^{-1} = \mathrm{AffineMap}(a^{-1}, -\frac{b}{a})$.

(34) For all real numbers a, b such that $a > 0$ holds $(\mathrm{AffineMap}(a, b))^{\circ}[0, 1] = [b, a + b]$.

(35) For every map f from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$ and for all real numbers a, b such that $a \neq 0$ and $f = \mathrm{AffineMap}(a, b)$ holds f is a homeomorphism.

(36) If A is an arc from p_1 to p_2 and LE q_1, q_2, A, p_1, p_2 and $q_1 \neq q_2$, then $\mathrm{Segment}(A, p_1, p_2, q_1, q_2)$ is an arc from q_1 to q_2.

(37) Let p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$ and P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $P \subseteq C$ and P is an arc from p_1 to p_2 and $\text{W-min}\, C \in P$ and $\text{E-max}\, C \in P$. Then $\mathrm{UpperArc}\, C \subseteq P$ or $\mathrm{LowerArc}\, C \subseteq P$.

References

[1] Leszek Borys. Paracompact and metrizable spaces. *Formalized Mathematics*, 2(**4**):481–485, 1991.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[5] Czesław Byliński and Piotr Rudnicki. Bounding boxes for compact sets in $\mathcal{E}^2$. *Formalized Mathematics*, 6(**3**):427–440, 1997.
[6] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[7] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[10] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces - fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[11] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Arcs, line segments and special polygonal arcs. *Formalized Mathematics*, 2(**5**):617–621, 1991.
[12] Agata Darmochwał and Yatsuka Nakamura. The topological space $\mathcal{E}^2_{\mathrm{T}}$. Simple closed curves. *Formalized Mathematics*, 2(**5**):663–664, 1991.
[13] Adam Grabowski and Yatsuka Nakamura. The ordering of points on a curve. Part II. *Formalized Mathematics*, 6(**4**):467–473, 1997.
[14] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[15] Zbigniew Karno. Continuity of mappings over the union of subspaces. *Formalized Mathematics*, 3(**1**):1–16, 1992.
[16] Yatsuka Nakamura. On the dividing function of the simple closed curve into segments. *Formalized Mathematics*, 7(**1**):135–138, 1998.
[17] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Formalized Mathematics*, 5(**2**):167–172, 1996.
[18] Yatsuka Nakamura and Andrzej Trybulec. A decomposition of a simple closed curves and the order of their points. *Formalized Mathematics*, 6(**4**):563–572, 1997.
[19] Yatsuka Nakamura, Andrzej Trybulec, and Czesław Byliński. Bounded domains and unbounded domains. *Formalized Mathematics*, 8(**1**):1–13, 1999.
[20] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.
[21] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[22] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.
[23] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.
[24] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[25] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[26] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[27] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received September 16, 2002