FORMALIZED MATHEMATICS
Volume 9, Number 2, 2001
University of Białystok

Meet Continuous Lattices Revisited[1]

Artur Korniłowicz
University of Białystok

Summary. This work is a continuation of formalization of [10]. Theorems from Chapter III, Section 2, pp. 153–156 are proved.

MML Identifier: `WAYBEL30`.

The articles [25], [20], [8], [9], [1], [23], [18], [24], [19], [26], [22], [6], [3], [7], [14], [4], [17], [15], [16], [2], [11], [12], [13], [21], and [5] provide the terminology and notation for this paper.

The following two propositions are true:

(1) For every set x and for every non empty set D holds $x \cap \bigcup D = \bigcup \{x \cap d : d$ ranges over elements of $D\}$.

(2) Let R be a non empty reflexive transitive relational structure and D be a non empty directed subset of $\langle \mathrm{Ids}(R), \subseteq \rangle$. Then $\bigcup D$ is an ideal of R.

Let R be a non empty reflexive transitive relational structure. Observe that $\langle \mathrm{Ids}(R), \subseteq \rangle$ is up-complete.

We now state two propositions:

(3) Let R be a non empty reflexive transitive relational structure and D be a non empty directed subset of $\langle \mathrm{Ids}(R), \subseteq \rangle$. Then $\sup D = \bigcup D$.

(4) Let R be a semilattice, D be a non empty directed subset of $\langle \mathrm{Ids}(R), \subseteq \rangle$, and x be an element of $\langle \mathrm{Ids}(R), \subseteq \rangle$. Then $\sup(\{x\} \sqcap D) = \bigcup \{x \cap d : d$ ranges over elements of $D\}$.

Let R be a semilattice. Observe that $\langle \mathrm{Ids}(R), \subseteq \rangle$ satisfies MC.

Let R be a non empty trivial relational structure. Note that every topological augmentation of R is trivial.

Next we state three propositions:

[1]This work has been supported by KBN Grant 8 T11C 018 12.

ISSN 1426–2630

(5) Let S be a Scott complete top-lattice, T be a complete lattice, and A be a Scott topological augmentation of T. Suppose the relational structure of S = the relational structure of T. Then the FR-structure of A = the FR-structure of S.

(6) Let N be a Lawson complete top-lattice, T be a complete lattice, and A be a Lawson correct topological augmentation of T. Suppose the relational structure of N = the relational structure of T. Then the FR-structure of A = the FR-structure of N.

(7) Let N be a Lawson complete top-lattice, S be a Scott topological augmentation of N, A be a subset of N, and J be a subset of S. If $A = J$ and J is closed, then A is closed.

Let A be a complete lattice. Observe that $\lambda(A)$ is non empty.

Let S be a Scott complete top-lattice. Observe that $\langle \sigma(S), \subseteq \rangle$ is complete and non trivial.

Let N be a Lawson complete top-lattice. Observe that $\langle \sigma(N), \subseteq \rangle$ is complete and non trivial and $\langle \lambda(N), \subseteq \rangle$ is complete and non trivial.

The following propositions are true:

(8) Let T be a non empty reflexive relational structure. Then $\sigma(T) \subseteq \{W \setminus \uparrow F; W$ ranges over subsets of T, F ranges over subsets of T: $W \in \sigma(T) \ \wedge$ F is finite$\}$.

(9) For every Lawson complete top-lattice N holds $\lambda(N)$ = the topology of N.

(10) For every Lawson complete top-lattice N holds $\sigma(N) \subseteq \lambda(N)$.

(11) Let M, N be complete lattices. Suppose the relational structure of M = the relational structure of N. Then $\lambda(M) = \lambda(N)$.

(12) For every Lawson complete top-lattice N and for every subset X of N holds $X \in \lambda(N)$ iff X is open.

Let us note that every reflexive non empty FR-structure which is trivial and topological space-like is also Scott.

Let us observe that every complete top-lattice which is trivial is also Lawson.

Let us note that there exists a complete top-lattice which is strict, continuous, lower-bounded, meet-continuous, and Scott.

One can verify that there exists a complete top-lattice which is strict, continuous, compact, Hausdorff, and Lawson.

Next we state the proposition

(13) Let N be a meet-continuous lattice and A be a subset of N. If A has the property (S), then $\uparrow A$ has the property (S).

Let N be a meet-continuous lattice and let A be a property(S) subset of N. Note that $\uparrow A$ is property(S).

We now state several propositions:

(14) Let N be a meet-continuous Lawson complete top-lattice, S be a Scott topological augmentation of N, and A be a subset of N. If $A \in \lambda(N)$, then $\uparrow A \in \sigma(S)$.

(15) Let N be a meet-continuous Lawson complete top-lattice, S be a Scott topological augmentation of N, A be a subset of N, and J be a subset of S. If $A = J$, then if A is open, then $\uparrow J$ is open.

(16) Let N be a meet-continuous Lawson complete top-lattice, S be a Scott topological augmentation of N, x be a point of S, y be a point of N, and J be a basis of y. If $x = y$, then $\{\uparrow A; A$ ranges over subsets of N: $A \in J\}$ is a basis of x.

(17) Let N be a meet-continuous Lawson complete top-lattice, S be a Scott topological augmentation of N, X be an upper subset of N, and Y be a subset of S. If $X = Y$, then $\operatorname{Int} X = \operatorname{Int} Y$.

(18) Let N be a meet-continuous Lawson complete top-lattice, S be a Scott topological augmentation of N, X be a lower subset of N, and Y be a subset of S. If $X = Y$, then $\overline{X} = \overline{Y}$.

(19) Let M, N be complete lattices, L_1 be a Lawson correct topological augmentation of M, and L_2 be a Lawson correct topological augmentation of N. Suppose $\langle \sigma(N), \subseteq \rangle$ is continuous. Then the topology of $[\!: L_1$, (L_2 **qua** topological space) $:\!] = \lambda([\!: M, N :\!])$.

(20) Let M, N be complete lattices, P be a Lawson correct topological augmentation of $[\!: M, N :\!]$, Q be a Lawson correct topological augmentation of M, and R be a Lawson correct topological augmentation of N. Suppose $\langle \sigma(N), \subseteq \rangle$ is continuous. Then the topological structure of $P = [\!: Q$, (R **qua** topological space) $:\!]$.

(21) For every meet-continuous Lawson complete top-lattice N and for every element x of N holds $x \sqcap \square$ is continuous.

Let N be a meet-continuous Lawson complete top-lattice and let x be an element of N. Observe that $x \sqcap \square$ is continuous.

One can prove the following propositions:

(22) For every meet-continuous Lawson complete top-lattice N such that $\langle \sigma(N), \subseteq \rangle$ is continuous holds N satisfies conditions of topological semilattice.

(23) Let N be a meet-continuous Lawson complete top-lattice. Suppose $\langle \sigma(N), \subseteq \rangle$ is continuous. Then N is Hausdorff if and only if for every subset X of $[\!: N$, (N **qua** topological space) $:\!]$ such that $X =$ the internal relation of N holds X is closed.

Let N be a non empty reflexive relational structure and let X be a subset of the carrier of N. The functor X^0 yields a subset of N and is defined by:

(Def. 1) $X^0 = \{u; u$ ranges over elements of N: $\bigwedge_{D\text{: non empty directed subset of } N} (u \leqslant$

$\sup D \Rightarrow X \cap D \neq \emptyset)\}$.

Let N be a non empty reflexive antisymmetric relational structure and let X be an empty subset of the carrier of N. One can check that X^0 is empty.

One can prove the following propositions:

(24) For every non empty reflexive relational structure N and for all subsets A, J of N such that $A \subseteq J$ holds $A^0 \subseteq J^0$.

(25) For every non empty reflexive relational structure N and for every element x of N holds $\uparrow x^0 = \Uparrow x$.

(26) For every Scott top-lattice N and for every upper subset X of N holds $\operatorname{Int} X \subseteq X^0$.

(27) For every non empty reflexive relational structure N and for all subsets X, Y of N holds $X^0 \cup Y^0 \subseteq X \cup Y^0$.

(28) For every meet-continuous lattice N and for all upper subsets X, Y of N holds $X^0 \cup Y^0 = X \cup Y^0$.

(29) Let S be a meet-continuous Scott top-lattice and F be a finite subset of S. Then $\operatorname{Int}{\uparrow}F \subseteq \bigcup\{\Uparrow x; x$ ranges over elements of S: $x \in F\}$.

(30) Let N be a Lawson complete top-lattice. Then N is continuous if and only if N is meet-continuous and Hausdorff.

Let us note that every complete top-lattice which is continuous and Lawson is also Hausdorff and every complete top-lattice which is meet-continuous, Lawson, and Hausdorff is also continuous.

Let N be a non empty FR-structure. We say that N has small semilattices if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let x be a point of N. Then there exists a generalized basis J of x such that for every subset A of N if $A \in J$, then $\operatorname{sub}(A)$ is meet-inheriting.

We say that N has compact semilattices if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let x be a point of N. Then there exists a generalized basis J of x such that for every subset A of N if $A \in J$, then $\operatorname{sub}(A)$ is meet-inheriting and A is compact.

We say that N has open semilattices if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let x be a point of N. Then there exists a basis J of x such that for every subset A of N if $A \in J$, then $\operatorname{sub}(A)$ is meet-inheriting.

One can verify the following observations:

* every non empty topological space-like FR-structure which has open semilattices has also small semilattices,
* every non empty topological space-like FR-structure which has compact semilattices has also small semilattices,

* every non empty FR-structure which is anti-discrete has small semilattices and open semilattices, and
* every non empty FR-structure which is reflexive, trivial, and topological space-like has compact semilattices.

Let us mention that there exists a top-lattice which is strict, trivial, and lower.

We now state several propositions:

(31) Let N be top-poset with g.l.b.'s satisfying conditions of topological semilattice and C be a subset of N. If $\mathrm{sub}(C)$ is meet-inheriting, then $\mathrm{sub}(\overline{C})$ is meet-inheriting.

(32) Let N be a meet-continuous Lawson complete top-lattice and S be a Scott topological augmentation of N. Then for every point x of S there exists a basis J of x such that for every subset W of S such that $W \in J$ holds W is a filter of S if and only if N has open semilattices.

(33) Let N be a Lawson complete top-lattice, S be a Scott topological augmentation of N, and x be an element of N. Then $\{\inf A; A$ ranges over subsets of S: $x \in A \ \wedge \ A \in \sigma(S)\} \subseteq \{\inf J; J$ ranges over subsets of N: $x \in J \ \wedge \ J \in \lambda(N)\}$.

(34) Let N be a meet-continuous Lawson complete top-lattice, S be a Scott topological augmentation of N, and x be an element of N. Then $\{\inf A; A$ ranges over subsets of S: $x \in A \ \wedge \ A \in \sigma(S)\} = \{\inf J; J$ ranges over subsets of N: $x \in J \ \wedge \ J \in \lambda(N)\}$.

(35) Let N be a meet-continuous Lawson complete top-lattice. Then N is continuous if and only if N has open semilattices and $\langle \sigma(N), \subseteq \rangle$ is continuous.

One can check that every Lawson complete top-lattice which is continuous has open semilattices.

Let N be a continuous Lawson complete top-lattice. One can check that $\langle \sigma(N), \subseteq \rangle$ is continuous.

We now state several propositions:

(36) Every continuous Lawson complete top-lattice is compact and Hausdorff and has open semilattices and satisfies conditions of topological semilattice.

(37) Every Hausdorff Lawson complete top-lattice with open semilattices satisfying conditions of topological semilattice has compact semilattices.

(38) Let N be a meet-continuous Hausdorff Lawson complete top-lattice and x be an element of N. Then $x = \bigsqcup_N \{\inf V; V$ ranges over subsets of N: $x \in V \ \wedge \ V \in \lambda(N)\}$.

(39) Let N be a meet-continuous Lawson complete top-lattice. Then N is continuous if and only if for every element x of N holds $x = \bigsqcup_N \{\inf V; V$

ranges over subsets of N: $x \in V \ \wedge \ V \in \lambda(N)\}$.

(40) Let N be a meet-continuous Lawson complete top-lattice. Then N is algebraic if and only if N has open semilattices and $\langle \sigma(N), \subseteq \rangle$ is algebraic.

Let N be a meet-continuous algebraic Lawson complete top-lattice. Note that $\langle \sigma(N), \subseteq \rangle$ is algebraic.

References

[1] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[2] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[3] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[4] Grzegorz Bancerek. The "way-below" relation. *Formalized Mathematics*, 6(**1**):169–176, 1997.
[5] Grzegorz Bancerek. Bases and refinements of topologies. *Formalized Mathematics*, 7(**1**):35–43, 1998.
[6] Grzegorz Bancerek. The Lawson topology. *Formalized Mathematics*, 7(**2**):163–168, 1998.
[7] Czesław Byliński. Galois connections. *Formalized Mathematics*, 6(**1**):131–143, 1997.
[8] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[9] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[10] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.
[11] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Formalized Mathematics*, 6(**1**):117–121, 1997.
[12] Artur Korniłowicz. Cartesian products of relations and relational structures. *Formalized Mathematics*, 6(**1**):145–152, 1997.
[13] Artur Korniłowicz. Definitions and properties of the join and meet of subsets. *Formalized Mathematics*, 6(**1**):153–158, 1997.
[14] Artur Korniłowicz. Meet–continuous lattices. *Formalized Mathematics*, 6(**1**):159–167, 1997.
[15] Artur Korniłowicz. On the topological properties of meet-continuous lattices. *Formalized Mathematics*, 6(**2**):269–277, 1997.
[16] Artur Korniłowicz. Introduction to meet-continuous topological lattices. *Formalized Mathematics*, 7(**2**):279–283, 1998.
[17] Robert Milewski. Algebraic lattices. *Formalized Mathematics*, 6(**2**):249–254, 1997.
[18] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[19] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[20] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[21] Andrzej Trybulec. Baire spaces, Sober spaces. *Formalized Mathematics*, 6(**2**):289–294, 1997.
[22] Andrzej Trybulec. Scott topology. *Formalized Mathematics*, 6(**2**):311–319, 1997.
[23] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[25] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[26] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

Received January 6, 2000