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The Field of Complex Numbers

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Summary. This article contains the definition and many facts about the field of complex numbers.

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The articles [4], [1], [2], [5], [6], and [3] provide the terminology and notation for this paper.

The following propositions are true:

(1) $1_{\mathbb{C}} \neq 0_{\mathbb{C}}$.

(2) For all elements x_1, y_1, x_2, y_2 of $\mathbb{R}$ holds $(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i$.

The strict double loop structure $\mathbb{C}_{\rm F}$ is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of $\mathbb{C}_{\rm F} = \mathbb{C}$,

(ii) the addition of $\mathbb{C}_{\rm F} = +_{\mathbb{C}}$,

(iii) the multiplication of $\mathbb{C}_{\rm F} = \cdot_{\mathbb{C}}$,

(iv) the unity of $\mathbb{C}_{\rm F} = 1_{\mathbb{C}}$, and

(v) the zero of $\mathbb{C}_{\rm F} = 0_{\mathbb{C}}$.

Let us observe that $\mathbb{C}_{\rm F}$ is non empty.

Let us observe that $\mathbb{C}_{\rm F}$ is add-associative right zeroed right complementable Abelian commutative associative left unital right unital distributive field-like and non degenerated.

We now state several propositions:

(3) For all elements x_1, y_1 of the carrier of $\mathbb{C}_{\rm F}$ and for all elements x_2, y_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $y_1 = y_2$ holds $x_1 + y_1 = x_2 + y_2$.

(4) For every element x_1 of the carrier of $\mathbb{C}_{\rm F}$ and for every element x_2 of $\mathbb{C}$ such that $x_1 = x_2$ holds $-x_1 = -x_2$.

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(5) For all elements x_1, y_1 of the carrier of $\mathbb{C}_{\rm F}$ and for all elements x_2, y_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $y_1 = y_2$ holds $x_1 - y_1 = x_2 - y_2$.

(6) For all elements x_1, y_1 of the carrier of $\mathbb{C}_{\rm F}$ and for all elements x_2, y_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $y_1 = y_2$ holds $x_1 \cdot y_1 = x_2 \cdot y_2$.

(7) For every element x_1 of the carrier of $\mathbb{C}_{\rm F}$ and for every element x_2 of $\mathbb{C}$ such that $x_1 = x_2$ and $x_1 \neq 0_{\mathbb{C}_{\rm F}}$ holds ${x_1}^{-1} = {x_2}^{-1}$.

(8) Let x_1, y_1 be elements of the carrier of $\mathbb{C}_{\rm F}$ and x_2, y_2 be elements of $\mathbb{C}$. If $x_1 = x_2$ and $y_1 = y_2$ and $y_1 \neq 0_{\mathbb{C}_{\rm F}}$, then $\frac{x_1}{y_1} = \frac{x_2}{y_2}$.

(9) $0_{\mathbb{C}_{\rm F}} = 0_{\mathbb{C}}$.

(10) $\mathbf{1}_{\mathbb{C}_{\rm F}} = 1_{\mathbb{C}}$.

(11) $\mathbf{1}_{\mathbb{C}_{\rm F}} + \mathbf{1}_{\mathbb{C}_{\rm F}} \neq 0_{\mathbb{C}_{\rm F}}$.

Let z be an element of the carrier of $\mathbb{C}_{\rm F}$. The functor z^* yielding an element of $\mathbb{C}_{\rm F}$ is defined by:

(Def. 2) There exists an element z' of $\mathbb{C}$ such that $z = z'$ and $z^* = z'^*$.

Let z be an element of the carrier of $\mathbb{C}_{\rm F}$. The functor $|z|$ yielding an element of $\mathbb{R}$ is defined by:

(Def. 3) There exists an element z' of $\mathbb{C}$ such that $z = z'$ and $|z| = |z'|$.

We now state the proposition

(12) For every element x_1 of the carrier of $\mathbb{C}_{\rm F}$ and for every element x_2 of $\mathbb{C}$ such that $x_1 = x_2$ holds ${x_1}^* = {x_2}^*$.

In the sequel z, z_1, z_2, z_3, z_4 denote elements of the carrier of $\mathbb{C}_{\rm F}$.

One can prove the following propositions:

(13) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$.

(14) (The zero of $\mathbb{C}_{\rm F}$) $+ z = z$ and $z +$ the zero of $\mathbb{C}_{\rm F} = z$.

(15) $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$.

(16) $z \cdot (z_1 + z_2) = z \cdot z_1 + z \cdot z_2$ and $(z_1 + z_2) \cdot z = z_1 \cdot z + z_2 \cdot z$.

(17) (The zero of $\mathbb{C}_{\rm F}$) $\cdot z =$ the zero of $\mathbb{C}_{\rm F}$ and $z \cdot$ the zero of $\mathbb{C}_{\rm F} =$ the zero of $\mathbb{C}_{\rm F}$.

(18) (The unity of $\mathbb{C}_{\rm F}$) $\cdot z = z$ and $z \cdot$ the unity of $\mathbb{C}_{\rm F} = z$.

(19) $-$the zero of $\mathbb{C}_{\rm F} =$ the zero of $\mathbb{C}_{\rm F}$.

(20) If $-z =$ the zero of $\mathbb{C}_{\rm F}$, then $z =$ the zero of $\mathbb{C}_{\rm F}$.

(21) $z + -z =$ the zero of $\mathbb{C}_{\rm F}$ and $-z + z =$ the zero of $\mathbb{C}_{\rm F}$.

(22) If $z_1 + z_2 =$ the zero of $\mathbb{C}_{\rm F}$, then $z_2 = -z_1$ and $z_1 = -z_2$.

(23) $--z = z$.

(24) If $-z_1 = -z_2$, then $z_1 = z_2$.

(25) If $z_1 + z = z_2 + z$ or $z_1 + z = z + z_2$, then $z_1 = z_2$.

(26) $-(z_1 + z_2) = -z_1 + -z_2$.

(27) $(-z_1) \cdot z_2 = -z_1 \cdot z_2$ and $z_1 \cdot -z_2 = -z_1 \cdot z_2$.

(28) $(-z_1) \cdot -z_2 = z_1 \cdot z_2$.

(29) $-z = (-\text{the unity of } \mathbb{C}_{\mathrm{F}}) \cdot z$.

(30) $z_1 - z_2 = z_1 + -z_2$.

(31) If $z_1 - z_2 =$ the zero of $\mathbb{C}_{\mathrm{F}}$, then $z_1 = z_2$.

(32) $z - z =$ the zero of $\mathbb{C}_{\mathrm{F}}$.

(33) $z -$ the zero of $\mathbb{C}_{\mathrm{F}} = z$.

(34) (The zero of $\mathbb{C}_{\mathrm{F}}$) $- z = -z$.

(35) $z_1 - -z_2 = z_1 + z_2$.

(36) $-(z_1 - z_2) = -z_1 + z_2$.

(37) $-(z_1 - z_2) = z_2 - z_1$.

(38) $z_1 + (z_2 - z_3) = (z_1 + z_2) - z_3$.

(39) $z_1 - (z_2 - z_3) = (z_1 - z_2) + z_3$.

(40) $z_1 - z_2 - z_3 = z_1 - (z_2 + z_3)$.

(41) $z_1 = (z_1 + z) - z$.

(42) $z_1 = (z_1 - z) + z$.

(43) $z \cdot (z_1 - z_2) = z \cdot z_1 - z \cdot z_2$ and $(z_1 - z_2) \cdot z = z_1 \cdot z - z_2 \cdot z$.

(44) If $z \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$, then $z \cdot z^{-1} =$ the unity of $\mathbb{C}_{\mathrm{F}}$ and $z^{-1} \cdot z =$ the unity of $\mathbb{C}_{\mathrm{F}}$.

(45) If $z_1 \cdot z_2 =$ the zero of $\mathbb{C}_{\mathrm{F}}$, then $z_1 =$ the zero of $\mathbb{C}_{\mathrm{F}}$ or $z_2 =$ the zero of $\mathbb{C}_{\mathrm{F}}$.

(46) If $z \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$, then $z^{-1} \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$.

(47) If $z_1 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$ and $z_2 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$ and ${z_1}^{-1} = {z_2}^{-1}$, then $z_1 = z_2$.

(48) If $z_2 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$ and if $z_1 \cdot z_2 =$ the unity of $\mathbb{C}_{\mathrm{F}}$ or $z_2 \cdot z_1 =$ the unity of $\mathbb{C}_{\mathrm{F}}$, then $z_1 = {z_2}^{-1}$.

(49) If $z_2 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$ and if $z_1 \cdot z_2 = z_3$ or $z_2 \cdot z_1 = z_3$, then $z_1 = z_3 \cdot {z_2}^{-1}$ and $z_1 = {z_2}^{-1} \cdot z_3$.

(50) (The unity of $\mathbb{C}_{\mathrm{F}})^{-1} =$ the unity of $\mathbb{C}_{\mathrm{F}}$.

(51) If $z_1 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$ and $z_2 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$, then $(z_1 \cdot z_2)^{-1} = {z_1}^{-1} \cdot {z_2}^{-1}$.

(52) If $z \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$, then $(z^{-1})^{-1} = z$.

(53) If $z \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$, then $(-z)^{-1} = -z^{-1}$.

(54) If $z \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$ and if $z_1 \cdot z = z_2 \cdot z$ or $z_1 \cdot z = z \cdot z_2$, then $z_1 = z_2$.

(55) If $z_1 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$ and $z_2 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$, then ${z_1}^{-1} + {z_2}^{-1} = (z_1 + z_2) \cdot (z_1 \cdot z_2)^{-1}$.

(56) If $z_1 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$ and $z_2 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$, then ${z_1}^{-1} - {z_2}^{-1} = (z_2 - z_1) \cdot (z_1 \cdot z_2)^{-1}$.

(57) If $z_2 \neq$ the zero of $\mathbb{C}_{\mathrm{F}}$, then $\frac{z_1}{z_2} = z_1 \cdot {z_2}^{-1}$.

(58) If $z \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $z^{-1} = \frac{\text{the unity of } \mathbb{C}_\mathrm{F}}{z}$.

(59) $\frac{z}{\text{the unity of } \mathbb{C}_\mathrm{F}} = z$.

(60) If $z \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{z}{z} =$ the unity of $\mathbb{C}_\mathrm{F}$.

(61) If $z \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{\text{the zero of } \mathbb{C}_\mathrm{F}}{z} =$ the zero of $\mathbb{C}_\mathrm{F}$.

(62) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $\frac{z_1}{z_2} =$ the zero of $\mathbb{C}_\mathrm{F}$, then $z_1 =$ the zero of $\mathbb{C}_\mathrm{F}$.

(63) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_4 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{z_1}{z_2} \cdot \frac{z_3}{z_4} = \frac{z_1 \cdot z_3}{z_2 \cdot z_4}$.

(64) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $z \cdot \frac{z_1}{z_2} = \frac{z \cdot z_1}{z_2}$.

(65) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $\frac{z_1}{z_2} =$ the unity of $\mathbb{C}_\mathrm{F}$, then $z_1 = z_2$.

(66) If $z \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $z_1 = \frac{z_1 \cdot z}{z}$.

(67) If $z_1 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $(\frac{z_1}{z_2})^{-1} = \frac{z_2}{z_1}$.

(68) If $z_1 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{{z_1}^{-1}}{{z_2}^{-1}} = \frac{z_2}{z_1}$.

(69) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{z_1}{{z_2}^{-1}} = z_1 \cdot z_2$.

(70) If $z_1 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{{z_1}^{-1}}{z_2} = (z_1 \cdot z_2)^{-1}$.

(71) If $z_1 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then ${z_1}^{-1} \cdot \frac{z}{z_2} = \frac{z}{z_1 \cdot z_2}$.

(72) If $z \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{z_1}{z_2} = \frac{z_1 \cdot z}{z_2 \cdot z}$ and $\frac{z_1}{z_2} = \frac{z \cdot z_1}{z \cdot z_2}$.

(73) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_3 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{z_1}{z_2 \cdot z_3} = \frac{\frac{z_1}{z_2}}{z_3}$.

(74) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_3 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{z_1 \cdot z_3}{z_2} = \frac{z_1}{\frac{z_2}{z_3}}$.

(75) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_3 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_4 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{\frac{z_1}{z_2}}{\frac{z_3}{z_4}} = \frac{z_1 \cdot z_4}{z_2 \cdot z_3}$.

(76) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_4 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{z_1}{z_2} + \frac{z_3}{z_4} = \frac{z_1 \cdot z_4 + z_3 \cdot z_2}{z_2 \cdot z_4}$.

(77) If $z \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{z_1}{z} + \frac{z_2}{z} = \frac{z_1 + z_2}{z}$.

(78) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $-\frac{z_1}{z_2} = \frac{-z_1}{z_2}$ and $-\frac{z_1}{z_2} = \frac{z_1}{-z_2}$.

(79) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{z_1}{z_2} = \frac{-z_1}{-z_2}$.

(80) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and $z_4 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{z_1}{z_2} - \frac{z_3}{z_4} = \frac{z_1 \cdot z_4 - z_3 \cdot z_2}{z_2 \cdot z_4}$.

(81) If $z \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{z_1}{z} - \frac{z_2}{z} = \frac{z_1 - z_2}{z}$.

(82) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$ and if $z_1 \cdot z_2 = z_3$ or $z_2 \cdot z_1 = z_3$, then $z_1 = \frac{z_3}{z_2}$.

(83) (the zero of $\mathbb{C}_\mathrm{F})^* =$ the zero of $\mathbb{C}_\mathrm{F}$.

(84) If $z^* =$ the zero of $\mathbb{C}_\mathrm{F}$, then $z =$ the zero of $\mathbb{C}_\mathrm{F}$.

(85) (the unity of $\mathbb{C}_\mathrm{F})^* =$ the unity of $\mathbb{C}_\mathrm{F}$.

(86) $(z^*)^* = z$.

(87) $(z_1 + z_2)^* = {z_1}^* + {z_2}^*$.

(88) $(-z)^* = -z^*$.

(89) $(z_1 - z_2)^* = {z_1}^* - {z_2}^*$.

(90) $(z_1 \cdot z_2)^* = {z_1}^* \cdot {z_2}^*$.

(91) If $z \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $(z^{-1})^* = (z^*)^{-1}$.

(92) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $(\frac{z_1}{z_2})^* = \frac{{z_1}^*}{{z_2}^*}$.

(93) $|$the zero of $\mathbb{C}_\mathrm{F}| = 0$.

(94) If $|z| = 0$, then $z =$ the zero of $\mathbb{C}_\mathrm{F}$.

(95) $0 \leqslant |z|$.

(96) $z \neq$ the zero of $\mathbb{C}_\mathrm{F}$ iff $0 < |z|$.

(97) $|$the unity of $\mathbb{C}_\mathrm{F}| = 1$.

(98) $|-z| = |z|$.

(99) $|z^*| = |z|$.

(100) $|z_1 + z_2| \leqslant |z_1| + |z_2|$.

(101) $|z_1 - z_2| \leqslant |z_1| + |z_2|$.

(102) $|z_1| - |z_2| \leqslant |z_1 + z_2|$.

(103) $|z_1| - |z_2| \leqslant |z_1 - z_2|$.

(104) $|z_1 - z_2| = |z_2 - z_1|$.

(105) $|z_1 - z_2| = 0$ iff $z_1 = z_2$.

(106) $z_1 \neq z_2$ iff $0 < |z_1 - z_2|$.

(107) $|z_1 - z_2| \leqslant |z_1 - z| + |z - z_2|$.

(108) $||z_1| - |z_2|| \leqslant |z_1 - z_2|$.

(109) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$.

(110) If $z \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $|z^{-1}| = |z|^{-1}$.

(111) If $z_2 \neq$ the zero of $\mathbb{C}_\mathrm{F}$, then $\frac{|z_1|}{|z_2|} = |\frac{z_1}{z_2}|$.

(112) $|z \cdot z| = |z \cdot z^*|$.

References

[1] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[2] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[3] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[4] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[5] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[6] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

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