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Property of Complex Functions

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Summary. This article introduces properties of complex function, calculations of them, boundedness and constant.

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The articles [11], [2], [1], [9], [3], [4], [5], [12], [6], [7], [10], and [8] provide the terminology and notation for this paper.

1. Definitions of Complex Functions

For simplicity, we adopt the following convention: X, Y are sets, C is a non empty set, c is an element of C, f, f_1, f_2, f_3, g, g_1 are partial functions from C to $\mathbb{C}$, p is a real number, and r, q are elements of $\mathbb{C}$.

A Complex is an element of $\mathbb{C}$.

Let us consider C, f_1, f_2. The functor $\frac{f_1}{f_2}$ yields a partial function from C to $\mathbb{C}$ and is defined as follows:

(Def. 1) $\mathrm{dom}(\frac{f_1}{f_2}) = \mathrm{dom}\, f_1 \cap (\mathrm{dom}\, f_2 \setminus {f_2}^{-1}(\{0_{\mathbb{C}}\}))$ and for every c such that $c \in \mathrm{dom}(\frac{f_1}{f_2})$ holds $(\frac{f_1}{f_2})_c = (f_1)_c \cdot ((f_2)_c)^{-1}$.

Let us consider C, f. The functor $\frac{1}{f}$ yields a partial function from C to $\mathbb{C}$ and is defined by:

(Def. 2) $\mathrm{dom}(\frac{1}{f}) = \mathrm{dom}\, f \setminus f^{-1}(\{0_{\mathbb{C}}\})$ and for every c such that $c \in \mathrm{dom}(\frac{1}{f})$ holds $(\frac{1}{f})_c = (f_c)^{-1}$.

Next we state a number of propositions:

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(3)[1] $\mathrm{dom}(f_1+f_2)=\mathrm{dom}\, f_1\cap\mathrm{dom}\, f_2$ and for every c such that $c\in\mathrm{dom}(f_1+f_2)$ holds $(f_1+f_2)_c=(f_1)_c+(f_2)_c$.

(4) $\mathrm{dom}(f_1-f_2)=\mathrm{dom}\, f_1\cap\mathrm{dom}\, f_2$ and for every c such that $c\in\mathrm{dom}(f_1-f_2)$ holds $(f_1-f_2)_c=(f_1)_c-(f_2)_c$.

(5) $\mathrm{dom}(f_1\, f_2)=\mathrm{dom}\, f_1\cap\mathrm{dom}\, f_2$ and for every c such that $c\in\mathrm{dom}(f_1\, f_2)$ holds $(f_1\, f_2)_c=(f_1)_c\cdot(f_2)_c$.

(6) $\mathrm{dom}(\frac{f_1}{f_2})=\mathrm{dom}\, f_1\cap(\mathrm{dom}\, f_2\setminus {f_2}^{-1}(\{0_{\mathbb{C}}\}))$ and for every c such that $c\in\mathrm{dom}(\frac{f_1}{f_2})$ holds $(\frac{f_1}{f_2})_c=(f_1)_c\cdot((f_2)_c)^{-1}$.

(7) $\mathrm{dom}(r\, f)=\mathrm{dom}\, f$ and for every c such that $c\in\mathrm{dom}(r\, f)$ holds $(r\, f)_c=r\cdot f_c$.

(9)[2] $\mathrm{dom}(-f)=\mathrm{dom}\, f$ and for every c such that $c\in\mathrm{dom}(-f)$ holds $(-f)_c=-f_c$.

(10) $\mathrm{dom}(\frac{1}{f})=\mathrm{dom}\, f\setminus f^{-1}(\{0_{\mathbb{C}}\})$ and for every c such that $c\in\mathrm{dom}(\frac{1}{f})$ holds $(\frac{1}{f})_c=(f_c)^{-1}$.

(15)[3] $\mathrm{dom}(\frac{1}{g})\subseteq\mathrm{dom}\, g$ and $\mathrm{dom}\, g\cap(\mathrm{dom}\, g\setminus g^{-1}(\{0_{\mathbb{C}}\}))=\mathrm{dom}\, g\setminus g^{-1}(\{0_{\mathbb{C}}\})$.

(16) $\mathrm{dom}(f_1\, f_2)\setminus(f_1\, f_2)^{-1}(\{0_{\mathbb{C}}\})=(\mathrm{dom}\, f_1\setminus {f_1}^{-1}(\{0_{\mathbb{C}}\}))\cap(\mathrm{dom}\, f_2\setminus {f_2}^{-1}(\{0_{\mathbb{C}}\}))$.

(17) If $c\in\mathrm{dom}(\frac{1}{f})$, then $f_c\neq 0_{\mathbb{C}}$.

(18) $(\frac{1}{f})^{-1}(\{0_{\mathbb{C}}\})=\emptyset$.

(19) $|f|^{-1}(\{0\})=f^{-1}(\{0_{\mathbb{C}}\})$ and $(-f)^{-1}(\{0_{\mathbb{C}}\})=f^{-1}(\{0_{\mathbb{C}}\})$.

(20) $\mathrm{dom}(\frac{1}{\frac{1}{f}})=\mathrm{dom}(f\restriction\mathrm{dom}(\frac{1}{f}))$.

(21) If $r\neq 0_{\mathbb{C}}$, then $(r\, f)^{-1}(\{0_{\mathbb{C}}\})=f^{-1}(\{0_{\mathbb{C}}\})$.

2. Basic Properties of Operations

The following propositions are true:

(22) $(f_1+f_2)+f_3=f_1+(f_2+f_3)$.

(23) $(f_1\, f_2)\, f_3=f_1\,(f_2\, f_3)$.

(24) $(f_1+f_2)\, f_3=f_1\, f_3+f_2\, f_3$.

(25) $f_3\,(f_1+f_2)=f_3\, f_1+f_3\, f_2$.

(26) $r\,(f_1\, f_2)=(r\, f_1)\, f_2$.

(27) $r\,(f_1\, f_2)=f_1\,(r\, f_2)$.

(28) $(f_1-f_2)\, f_3=f_1\, f_3-f_2\, f_3$.

[1]The propositions (1) and (2) have been removed.
[2]The proposition (8) has been removed.
[3]The propositions (11)–(14) have been removed.

(29) $f_3\, f_1 - f_3\, f_2 = f_3\,(f_1 - f_2)$.

(30) $r\,(f_1 + f_2) = r\, f_1 + r\, f_2$.

(31) $(r \cdot q)\, f = r\,(q\, f)$.

(32) $r\,(f_1 - f_2) = r\, f_1 - r\, f_2$.

(33) $f_1 - f_2 = (-1_{\mathbb{C}})\,(f_2 - f_1)$.

(34) $f_1 - (f_2 + f_3) = f_1 - f_2 - f_3$.

(35) $1_{\mathbb{C}}\, f = f$.

(36) $f_1 - (f_2 - f_3) = (f_1 - f_2) + f_3$.

(37) $f_1 + (f_2 - f_3) = (f_1 + f_2) - f_3$.

(38) $|f_1\, f_2| = |f_1|\,|f_2|$.

(39) $|r\, f| = |r|\,|f|$.

(40) $-f = (-1_{\mathbb{C}})\, f$.

(41) $--f = f$.

(42) $f_1 - f_2 = f_1 + -f_2$.

(43) $f_1 - -f_2 = f_1 + f_2$.

(44) $\frac{1}{\frac{1}{f}} = f{\restriction}\,\mathrm{dom}(\frac{1}{f})$.

(45) $\frac{1}{f_1\, f_2} = \frac{1}{f_1}\,\frac{1}{f_2}$.

(46) If $r \neq 0_{\mathbb{C}}$, then $\frac{1}{r\, f} = r^{-1}\,\frac{1}{f}$.

(47) $1_{\mathbb{C}} \neq 0_{\mathbb{C}}$.

(48) $(-1_{\mathbb{C}})^{-1} = -1_{\mathbb{C}}$.

(49) $\frac{1}{-f} = (-1_{\mathbb{C}})\,\frac{1}{f}$.

(50) $\frac{1}{|f|} = |\frac{1}{f}|$.

(51) $\frac{f}{g} = f\,\frac{1}{g}$.

(52) $r\,\frac{g}{f} = \frac{r\, g}{f}$.

(53) $\frac{f}{g}\, g = f{\restriction}\,\mathrm{dom}(\frac{1}{g})$.

(54) $\frac{f}{g}\,\frac{f_1}{g_1} = \frac{f\, f_1}{g\, g_1}$.

(55) $\frac{1}{\frac{f_1}{f_2}} = \frac{f_2{\restriction}\,\mathrm{dom}(\frac{1}{f_2})}{f_1}$.

(56) $g\,\frac{f_1}{f_2} = \frac{g\, f_1}{f_2}$.

(57) $\frac{g}{\frac{f_1}{f_2}} = \frac{g\,(f_2{\restriction}\,\mathrm{dom}(\frac{1}{f_2}))}{f_1}$.

(58) $-\frac{f}{g} = \frac{-f}{g}$ and $\frac{f}{-g} = -\frac{f}{g}$.

(59) $\frac{f_1}{f} + \frac{f_2}{f} = \frac{f_1 + f_2}{f}$ and $\frac{f_1}{f} - \frac{f_2}{f} = \frac{f_1 - f_2}{f}$.

(60) $\frac{f_1}{f} + \frac{g_1}{g} = \frac{f_1\, g + g_1\, f}{f\, g}$.

(61) $\frac{\frac{f}{g}}{\frac{f_1}{g_1}} = \frac{f\,(g_1{\restriction}\mathrm{dom}(\frac{1}{g_1}))}{g\,f_1}$.

(62) $\frac{f_1}{f} - \frac{g_1}{g} = \frac{f_1\,g - g_1\,f}{f\,g}$.

(63) $|\frac{f_1}{f_2}| = \frac{|f_1|}{|f_2|}$.

(64) $(f_1+f_2){\restriction}X = f_1{\restriction}X + f_2{\restriction}X$ and $(f_1+f_2){\restriction}X = f_1{\restriction}X + f_2$ and $(f_1+f_2){\restriction}X = f_1 + f_2{\restriction}X$.

(65) $(f_1\,f_2){\restriction}X = (f_1{\restriction}X)\,(f_2{\restriction}X)$ and $(f_1\,f_2){\restriction}X = (f_1{\restriction}X)\,f_2$ and $(f_1\,f_2){\restriction}X = f_1\,(f_2{\restriction}X)$.

(66) $(-f){\restriction}X = -f{\restriction}X$ and $\frac{1}{f}{\restriction}X = \frac{1}{f{\restriction}X}$ and $|f|{\restriction}X = |f{\restriction}X|$.

(67) $(f_1-f_2){\restriction}X = f_1{\restriction}X - f_2{\restriction}X$ and $(f_1-f_2){\restriction}X = f_1{\restriction}X - f_2$ and $(f_1-f_2){\restriction}X = f_1 - f_2{\restriction}X$.

(68) $\frac{f_1}{f_2}{\restriction}X = \frac{f_1{\restriction}X}{f_2{\restriction}X}$ and $\frac{f_1}{f_2}{\restriction}X = \frac{f_1{\restriction}X}{f_2}$ and $\frac{f_1}{f_2}{\restriction}X = \frac{f_1}{f_2{\restriction}X}$.

(69) $(r\,f){\restriction}X = r\,(f{\restriction}X)$.

3. Total Partial Functions from a Domain, to Complex

We now state a number of propositions:

(70)(i) f_1 is total and f_2 is total iff $f_1 + f_2$ is total,

(ii) f_1 is total and f_2 is total iff $f_1 - f_2$ is total, and

(iii) f_1 is total and f_2 is total iff $f_1\,f_2$ is total.

(71) f is total iff $r\,f$ is total.

(72) f is total iff $-f$ is total.

(73) f is total iff $|f|$ is total.

(74) $\frac{1}{f}$ is total iff $f^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$ and f is total.

(75) f_1 is total and ${f_2}^{-1}(\{0_{\mathbb{C}}\}) = \emptyset$ and f_2 is total iff $\frac{f_1}{f_2}$ is total.

(76) If f_1 is total and f_2 is total, then $(f_1+f_2)_c = (f_1)_c + (f_2)_c$ and $(f_1-f_2)_c = (f_1)_c - (f_2)_c$ and $(f_1\,f_2)_c = (f_1)_c \cdot (f_2)_c$.

(77) If f is total, then $(r\,f)_c = r \cdot f_c$.

(78) If f is total, then $(-f)_c = -f_c$ and $|f|(c) = |f_c|$.

(79) If $\frac{1}{f}$ is total, then $(\frac{1}{f})_c = (f_c)^{-1}$.

(80) If f_1 is total and $\frac{1}{f_2}$ is total, then $(\frac{f_1}{f_2})_c = (f_1)_c \cdot ((f_2)_c)^{-1}$.

4. Bounded and Constant Partial Functions from a Domain, to Complex

Let us consider C, f, Y. We say that f is bounded on Y if and only if:

(Def. 3) $|f|$ is bounded on Y.

The following propositions are true:

(81) f is bounded on Y iff there exists a real number p such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $|f_c| \leqslant p$.

(82) If $Y \subseteq X$ and f is bounded on X, then f is bounded on Y.

(83) If $X \cap \operatorname{dom} f = \emptyset$, then f is bounded on X.

(84) If f is bounded on Y, then $r\,f$ is bounded on Y.

(85) $|f|$ is lower bounded on X.

(86) If f is bounded on Y, then $|f|$ is bounded on Y and $-f$ is bounded on Y.

(87) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(88) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1\,f_2$ is bounded on $X \cap Y$ and $f_1 - f_2$ is bounded on $X \cap Y$.

(89) If f is bounded on X and bounded on Y, then f is bounded on $X \cup Y$.

(90) Suppose f_1 is a constant on X and f_2 is a constant on Y. Then $f_1 + f_2$ is a constant on $X \cap Y$ and $f_1 - f_2$ is a constant on $X \cap Y$ and $f_1\,f_2$ is a constant on $X \cap Y$.

(91) If f is a constant on Y, then $q\,f$ is a constant on Y.

(92) If f is a constant on Y, then $|f|$ is a constant on Y and $-f$ is a constant on Y.

(93) If f is a constant on Y, then f is bounded on Y.

(94) If f is a constant on Y, then for every r holds $r\,f$ is bounded on Y and $-f$ is bounded on Y and $|f|$ is bounded on Y.

(95) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(96) Suppose f_1 is bounded on X and f_2 is a constant on Y. Then $f_1 - f_2$ is bounded on $X \cap Y$ and $f_2 - f_1$ is bounded on $X \cap Y$ and $f_1\,f_2$ is bounded on $X \cap Y$.

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