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Noetherian Lattices

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Summary. In this article we define noetherian and co-noetherian lattices and show how some properties concerning upper and lower neighbours, irreducibility and density can be improved when restricted to these kinds of lattices. In addition we define atomic lattices.

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The notation and terminology used here are introduced in the following papers: [18], [13], [17], [14], [19], [7], [1], [8], [6], [20], [3], [9], [2], [10], [15], [16], [5], [11], [4], and [12].

Let us observe that there exists a lattice which is finite.

Let us mention that every lattice which is finite is also complete.

Let L be a lattice and let D be a subset of the carrier of L. The functor $D^{\cdot}$ yields a subset of $\mathrm{Poset}(L)$ and is defined by:

(Def. 1) $D^{\cdot} = \{d^{\cdot}; d$ ranges over elements of the carrier of L: $d \in D\}$.

Let L be a lattice and let D be a subset of the carrier of $\mathrm{Poset}(L)$. The functor ${}^{\cdot}D$ yielding a subset of the carrier of L is defined by:

(Def. 2) ${}^{\cdot}D = \{{}^{\cdot}d; d$ ranges over elements of $\mathrm{Poset}(L)$: $d \in D\}$.

Let L be a finite lattice. Note that $\mathrm{Poset}(L)$ is well founded.

Let L be a lattice. We say that L is noetherian if and only if:

(Def. 3) $\mathrm{Poset}(L)$ is well founded.

We say that L is co-noetherian if and only if:

(Def. 4) $\mathrm{Poset}(L)^{\smile}$ is well founded.

One can verify the following observations:

* there exists a lattice which is noetherian and upper-bounded,
* there exists a lattice which is noetherian and lower-bounded, and
* there exists a lattice which is noetherian and complete.

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One can verify the following observations:

* there exists a lattice which is co-noetherian and upper-bounded,
* there exists a lattice which is co-noetherian and lower-bounded, and
* there exists a lattice which is co-noetherian and complete.

Next we state the proposition

(1) For every lattice L holds L is noetherian iff L° is co-noetherian.

One can check that every lattice which is finite is also noetherian and every lattice which is finite is also co-noetherian.

Let L be a lattice and let a, b be elements of the carrier of L. We say that a is-upper-neighbour-of b if and only if:

(Def. 5) $a \neq b$ and $b \sqsubseteq a$ and for every element c of the carrier of L such that $b \sqsubseteq c$ and $c \sqsubseteq a$ holds $c = a$ or $c = b$.

We introduce b is-lower-neighbour-of a as a synonym of a is-upper-neighbour-of b.

We now state several propositions:

(2) Let L be a lattice, a be an element of the carrier of L, and b, c be elements of the carrier of L such that $b \neq c$. Then

(i) if b is-upper-neighbour-of a and c is-upper-neighbour-of a, then $a = c \sqcap b$, and

(ii) if b is-lower-neighbour-of a and c is-lower-neighbour-of a, then $a = c \sqcup b$.

(3) Let L be a noetherian lattice, a be an element of the carrier of L, and d be an element of the carrier of L. Suppose $a \sqsubseteq d$ and $a \neq d$. Then there exists an element c of the carrier of L such that $c \sqsubseteq d$ and c is-upper-neighbour-of a.

(4) Let L be a co-noetherian lattice, a be an element of the carrier of L, and d be an element of the carrier of L. Suppose $d \sqsubseteq a$ and $a \neq d$. Then there exists an element c of the carrier of L such that $d \sqsubseteq c$ and c is-lower-neighbour-of a.

(5) Let L be an upper-bounded lattice. Then it is not true that there exists an element b of the carrier of L such that b is-upper-neighbour-of $\top_L$.

(6) Let L be a noetherian upper-bounded lattice and a be an element of the carrier of L. Then $a = \top_L$ if and only if it is not true that there exists an element b of the carrier of L such that b is-upper-neighbour-of a.

(7) Let L be a lower-bounded lattice. Then it is not true that there exists an element b of the carrier of L such that b is-lower-neighbour-of $\perp_L$.

(8) Let L be a co-noetherian lower-bounded lattice and a be an element of the carrier of L. Then $a = \perp_L$ if and only if it is not true that there exists an element b of the carrier of L such that b is-lower-neighbour-of a.

Let L be a complete lattice and let a be an element of the carrier of L. The functor a^* yielding an element of the carrier of L is defined by:

(Def. 6) $a^* = \bigsqcap_L \{d; d$ ranges over elements of the carrier of L: $a \sqsubseteq d \ \wedge \ d \neq a\}$.

The functor ${}^*a$ yields an element of the carrier of L and is defined as follows:

(Def. 7) ${}^*a = \bigsqcup_L \{d; d$ ranges over elements of the carrier of L: $d \sqsubseteq a \ \wedge \ d \neq a\}$.

Let L be a complete lattice and let a be an element of the carrier of L. We say that a is completely-meet-irreducible if and only if:

(Def. 8) $a^* \neq a$.

We say that a is completely-join-irreducible if and only if:

(Def. 9) ${}^*a \neq a$.

The following propositions are true:

(9) For every complete lattice L and for every element a of the carrier of L holds $a \sqsubseteq a^*$ and ${}^*a \sqsubseteq a$.

(10) For every complete lattice L holds $(\top_L)^* = \top_L$ and $(\top_L)^{\cdot}$ is meet-irreducible.

(11) For every complete lattice L holds ${}^*(\bot_L) = \bot_L$ and $(\bot_L)^{\cdot}$ is join-irreducible.

(12) Let L be a complete lattice and a be an element of the carrier of L. Suppose a is completely-meet-irreducible. Then

(i) a^* is-upper-neighbour-of a, and

(ii) for every element c of the carrier of L such that c is-upper-neighbour-of a holds $c = a^*$.

(13) Let L be a complete lattice and a be an element of the carrier of L. Suppose a is completely-join-irreducible. Then

(i) ${}^*a$ is-lower-neighbour-of a, and

(ii) for every element c of the carrier of L such that c is-lower-neighbour-of a holds $c = {}^*a$.

(14) Let L be a noetherian complete lattice and a be an element of the carrier of L. Suppose $a \neq \top_L$. Then a is completely-meet-irreducible if and only if there exists an element b of the carrier of L such that b is-upper-neighbour-of a and for every element c of the carrier of L such that c is-upper-neighbour-of a holds $c = b$.

(15) Let L be a co-noetherian complete lattice and a be an element of the carrier of L. Suppose $a \neq \bot_L$. Then a is completely-join-irreducible if and only if there exists an element b of the carrier of L such that b is-lower-neighbour-of a and for every element c of the carrier of L such that c is-lower-neighbour-of a holds $c = b$.

(16) Let L be a complete lattice and a be an element of the carrier of L. If a is completely-meet-irreducible, then $a^{\cdot}$ is meet-irreducible.

(17) Let L be a complete noetherian lattice and a be an element of the carrier of L. Suppose $a \neq \top_L$. Then a is completely-meet-irreducible if and only if $a^{\cdot}$ is meet-irreducible.

(18) Let L be a complete lattice and a be an element of the carrier of L. If a is completely-join-irreducible, then $a^{\cdot}$ is join-irreducible.

(19) Let L be a complete co-noetherian lattice and a be an element of the carrier of L. Suppose $a \neq \bot_L$. Then a is completely-join-irreducible if and only if $a^{\cdot}$ is join-irreducible.

(20) Let L be a finite lattice and a be an element of the carrier of L such that $a \neq \bot_L$ and $a \neq \top_L$. Then

(i) a is completely-meet-irreducible iff $a^{\cdot}$ is meet-irreducible, and

(ii) a is completely-join-irreducible iff $a^{\cdot}$ is join-irreducible.

Let L be a lattice and let a be an element of the carrier of L. We say that a is atomic if and only if:

(Def. 10) a is-upper-neighbour-of $\bot_L$.

We say that a is co-atomic if and only if:

(Def. 11) a is-lower-neighbour-of $\top_L$.

One can prove the following propositions:

(21) Let L be a complete lattice and a be an element of the carrier of L. If a is atomic, then a is completely-join-irreducible.

(22) Let L be a complete lattice and a be an element of the carrier of L. If a is co-atomic, then a is completely-meet-irreducible.

Let L be a lattice. We say that L is atomic if and only if the condition (Def. 12) is satisfied.

(Def. 12) Let a be an element of the carrier of L. Then there exists a subset X of the carrier of L such that for every element x of the carrier of L such that $x \in X$ holds x is atomic and $a = \bigsqcup_L X$.

One can verify that there exists a lattice which is atomic and complete.

Let L be a complete lattice and let D be a subset of L. We say that D is supremum-dense if and only if:

(Def. 13) For every element a of the carrier of L there exists a subset D' of D such that $a = \bigsqcup_L D'$.

We say that D is infimum-dense if and only if:

(Def. 14) For every element a of the carrier of L there exists a subset D' of D such that $a = \bigsqcap_L D'$.

One can prove the following propositions:

(23) Let L be a complete lattice and D be a subset of L. Then D is supremum-dense if and only if for every element a of the carrier of L holds $a = \bigsqcup_L \{d; d$ ranges over elements of the carrier of L: $d \in D \ \wedge \ d \sqsubseteq a\}$.

(24) Let L be a complete lattice and D be a subset of L. Then D is infimum-dense if and only if for every element a of the carrier of L holds $a = \bigsqcap_L \{d; d$ ranges over elements of the carrier of L: $d \in D \wedge a \sqsubseteq d\}$.

(25) Let L be a complete lattice and D be a subset of L. Then D is infimum-dense if and only if $D^{\cdot}$ is order-generating.

Let L be a complete lattice. The functor MIRRS L yields a subset of L and is defined by:

(Def. 15) MIRRS $L = \{a; a$ ranges over elements of the carrier of L: a is completely-meet-irreducible$\}$.

The functor JIRRS L yielding a subset of L is defined by:

(Def. 16) JIRRS $L = \{a; a$ ranges over elements of the carrier of L: a is completely-join-irreducible$\}$.

One can prove the following two propositions:

(26) For every complete lattice L and for every subset D of L such that D is supremum-dense holds JIRRS $L \subseteq D$.

(27) For every complete lattice L and for every subset D of L such that D is infimum-dense holds MIRRS $L \subseteq D$.

Let L be a co-noetherian complete lattice. Note that MIRRS L is infimum-dense.

Let L be a noetherian complete lattice. One can check that JIRRS L is supremum-dense.

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