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Real Linear-Metric Space and Isometric Functions

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The notation and terminology used in this paper are introduced in the following papers: [11], [6], [2], [13], [3], [9], [12], [8], [1], [10], [7], [16], [14], [4], [15], and [5].

1. Convex and Internal Metric Spaces

Let V be a non empty metric structure. We say that V is convex if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let x, y be elements of the carrier of V and r be a real number. Suppose $0 \leqslant r$ and $r \leqslant 1$. Then there exists an element z of the carrier of V such that $\rho(x,z) = r \cdot \rho(x,y)$ and $\rho(z,y) = (1-r) \cdot \rho(x,y)$.

Let V be a non empty metric structure. We say that V is internal if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let x, y be elements of the carrier of V and p, q be real numbers. Suppose $p > 0$ and $q > 0$. Then there exists a finite sequence f of elements of the carrier of V such that

(i) $\pi_1 f = x$,

(ii) $\pi_{\mathrm{len}\, f} f = y$,

(iii) for every natural number i such that $1 \leqslant i$ and $i \leqslant \mathrm{len}\, f - 1$ holds $\rho(\pi_i f, \pi_{i+1} f) < p$, and

(iv) for every finite sequence F of elements of $\mathbb{R}$ such that $\mathrm{len}\, F = \mathrm{len}\, f - 1$ and for every natural number i such that $1 \leqslant i$ and $i \leqslant \mathrm{len}\, F$ holds $\pi_i F = \rho(\pi_i f, \pi_{i+1} f)$ holds $|\rho(x,y) - \sum F| < q$.

One can prove the following proposition

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(1) Let V be a non empty metric space. Suppose V is convex. Let x, y be elements of the carrier of V and p be a real number. Suppose $p > 0$. Then there exists a finite sequence f of elements of the carrier of V such that

(i) $\pi_1 f = x$,

(ii) $\pi_{\mathrm{len}\, f} f = y$,

(iii) for every natural number i such that $1 \leqslant i$ and $i \leqslant \mathrm{len}\, f - 1$ holds $\rho(\pi_i f, \pi_{i+1} f) < p$, and

(iv) for every finite sequence F of elements of $\mathbb{R}$ such that $\mathrm{len}\, F = \mathrm{len}\, f - 1$ and for every natural number i such that $1 \leqslant i$ and $i \leqslant \mathrm{len}\, F$ holds $\pi_i F = \rho(\pi_i f, \pi_{i+1} f)$ holds $\rho(x, y) = \sum F$.

Let us observe that every non empty metric space which is convex is also internal.

One can verify that there exists a non empty metric space which is convex.

A Geometry is a Reflexive discernible symmetric triangle internal non empty metric structure.

2. Isometric Functions

Let V be a non empty metric structure and let f be a map from V into V. We say that f is isometric if and only if:

(Def. 3) $\mathrm{rng}\, f =$ the carrier of V and for all elements x, y of the carrier of V holds $\rho(x, y) = \rho(f(x), f(y))$.

Let V be a non empty metric structure. The functor $\mathrm{ISOM}\, V$ yields a set and is defined as follows:

(Def. 4) For every set x holds $x \in \mathrm{ISOM}\, V$ iff there exists a map f from V into V such that $f = x$ and f is isometric.

Let V be a non empty metric structure. Then $\mathrm{ISOM}\, V$ is a subset of (the carrier of V)$^{\text{the carrier of } V}$.

One can prove the following proposition

(2) Let V be a discernible Reflexive non empty metric structure and f be a map from V into V. If f is isometric, then f is one-to-one.

Let V be a discernible Reflexive non empty metric structure. One can check that every map from V into V which is isometric is also one-to-one.

Let V be a non empty metric structure. Observe that there exists a map from V into V which is isometric.

The following three propositions are true:

(3) Let V be a discernible Reflexive non empty metric structure and f be an isometric map from V into V. Then f^{-1} is isometric.

(4) For every non empty metric structure V and for all isometric maps f, g from V into V holds $f \cdot g$ is isometric.

(5) For every non empty metric structure V holds id_V is isometric.

Let V be a non empty metric structure. Note that ISOM V is non empty.

3. Real Linear-Metric Spaces

We introduce RLSMetrStruct which are extensions of RLS structure and metric structure and are systems

⟨ a carrier, a distance, a zero, an addition, an external multiplication ⟩,

where the carrier is a set, the distance is a function from [: the carrier, the carrier :] into $\mathbb{R}$, the zero is an element of the carrier, the addition is a binary operation on the carrier, and the external multiplication is a function from [: $\mathbb{R}$, the carrier :] into the carrier.

One can verify that there exists a RLSMetrStruct which is non empty and strict.

Let X be a non empty set, let F be a function from [: X, X :] into $\mathbb{R}$, let O be an element of X, let B be a binary operation on X, and let G be a function from [: $\mathbb{R}$, X :] into X. One can verify that $\langle X, F, O, B, G\rangle$ is non empty.

Let V be a non empty RLSMetrStruct. We say that V is homogeneous if and only if:

(Def. 5) For every real number r and for all elements v, w of the carrier of V holds $\rho(r \cdot v, r \cdot w) = |r| \cdot \rho(v, w)$.

Let V be a non empty RLSMetrStruct. We say that V is translatible if and only if:

(Def. 6) For all elements u, w, v of the carrier of V holds $\rho(v, w) = \rho(v+u, w+u)$.

Let V be a non empty RLSMetrStruct and let v be an element of the carrier of V. The functor Norm v yielding a real number is defined as follows:

(Def. 7) $\mathrm{Norm}\, v = \rho(0_V, v)$.

Let us note that there exists a non empty RLSMetrStruct which is strict, Abelian, add-associative, right zeroed, right complementable, real linear space-like, Reflexive, discernible, symmetric, triangle, homogeneous, and translatible.

A RealLinearMetrSpace is an Abelian add-associative right zeroed right complementable real linear space-like Reflexive discernible symmetric triangle homogeneous translatible non empty RLSMetrStruct.

We now state three propositions:

(6) Let V be a homogeneous Abelian add-associative right zeroed right complementable real linear space-like non empty RLSMetrStruct, r be a real number, and v be an element of the carrier of V. Then $\mathrm{Norm}(r \cdot v) = |r| \cdot \mathrm{Norm}\, v$.

(7) Let V be a translatible Abelian add-associative right zeroed right complementable triangle non empty RLSMetrStruct and v, w be elements of the carrier of V. Then $\mathrm{Norm}(v + w) \leqslant \mathrm{Norm}\, v + \mathrm{Norm}\, w$.

(8) Let V be a translatible add-associative right zeroed right complementable non empty RLSMetrStruct and v, w be elements of the carrier of V. Then $\rho(v, w) = \mathrm{Norm}(w - v)$.

Let n be a natural number. The functor RLMSpace n yielding a strict RealLinearMetrSpace is defined by the conditions (Def. 8).

(Def. 8)(i) The carrier of RLMSpace $n = \mathcal{R}^n$,

(ii) the distance of RLMSpace $n = \rho^n$,

(iii) the zero of RLMSpace $n = \langle \underbrace{0, \ldots, 0}_{n} \rangle$,

(iv) for all elements x, y of $\mathcal{R}^n$ holds (the addition of RLMSpace n)(x, y) $= x + y$, and

(v) for every element x of $\mathcal{R}^n$ and for every element r of $\mathbb{R}$ holds (the external multiplication of RLMSpace n)(r, x) $= r \cdot x$.

Next we state the proposition

(9) For every natural number n and for every isometric map f from RLMSpace n into RLMSpace n holds $\mathrm{rng}\, f = \mathcal{R}^n$.

4. Groups of Isometric Functions

Let n be a natural number. The functor IsomGroup n yielding a strict groupoid is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of IsomGroup n = ISOM RLMSpace n, and

(ii) for all functions f, g such that $f \in$ ISOM RLMSpace n and $g \in$ ISOM RLMSpace n holds (the multiplication of IsomGroup n)(f, g) $= f \cdot g$.

Let n be a natural number. Note that IsomGroup n is non empty.

Let n be a natural number. Note that IsomGroup n is associative and group-like.

The following two propositions are true:

(10) For every natural number n holds $1_{\mathrm{IsomGroup}\, n} = \mathrm{id}_{\mathrm{RLMSpace}\, n}$.

(11) Let n be a natural number, f be an element of IsomGroup n, and g be a map from RLMSpace n into RLMSpace n. If $f = g$, then $f^{-1} = g^{-1}$.

Let n be a natural number and let G be a subgroup of IsomGroup n. The functor SubIsomGroupRel G yielding a binary relation on the carrier of RLMSpace n is defined by the condition (Def. 10).

(Def. 10) Let A, B be elements of RLMSpace n. Then $\langle A, B\rangle \in$ SubIsomGroupRel G if and only if there exists a function f such that $f \in$ the carrier of G and $f(A) = B$.

Let n be a natural number and let G be a subgroup of IsomGroup n. Observe that SubIsomGroupRel G is equivalence relation-like.

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