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Lattice of Substitutions is a Heyting Algebra

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The terminology and notation used in this paper have been introduced in the following articles: [2], [15], [1], [7], [13], [9], [3], [4], [10], [18], [5], [16], [17], [11], [14], [8], [12], and [6].

1. Preliminaries

We adopt the following convention: V, C, x are sets and A, B are elements of SubstitutionSet(V, C).

Let a, b be sets. Note that $\{\langle a, b\rangle\}$ is function-like and relation-like.

Let A, B be sets. Observe that $A \dot{\rightarrow} B$ is functional.

Next we state several propositions:

(1) For all non empty sets V, C there exists an element f of $V \dot{\rightarrow} C$ such that $f \neq \emptyset$.

(2) For all sets a, b such that $b \in$ SubstitutionSet(V, C) and $a \in b$ holds a is a finite function.

(3) For every element f of $V \dot{\rightarrow} C$ and for every set g such that $g \subseteq f$ holds $g \in V \dot{\rightarrow} C$.

(4) $V \dot{\rightarrow} C \subseteq 2^{[: V, C :]}$.

(5) If V is finite and C is finite, then $V \dot{\rightarrow} C$ is finite.

One can check that there exists a set which is functional, finite, and non empty.

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2. Some Properties of Sets of Substitutions

One can prove the following four propositions:

(6) For every finite element a of $V \dot{\rightarrow} C$ holds $\{a\} \in \mathrm{SubstitutionSet}(V, C)$.

(7) If $A \frown B = A$, then for every set a such that $a \in A$ there exists a set b such that $b \in B$ and $b \subseteq a$.

(8) If $\mu(A \frown B) = A$, then for every set a such that $a \in A$ there exists a set b such that $b \in B$ and $b \subseteq a$.

(9) If for every set a such that $a \in A$ there exists a set b such that $b \in B$ and $b \subseteq a$, then $\mu(A \frown B) = A$.

Let V be a set, let C be a finite set, and let A be an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$. The functor $\mathrm{Involved}\, A$ is defined by:

(Def. 1) $x \in \mathrm{Involved}\, A$ iff there exists a finite function f such that $f \in A$ and $x \in \mathrm{dom}\, f$.

In the sequel C denotes a finite set.

The following propositions are true:

(10) For every set V and for every finite set C and for every element A of $\mathrm{Fin}(V \dot{\rightarrow} C)$ holds $\mathrm{Involved}\, A \subseteq V$.

(11) For every set V and for every finite set C and for every element A of $\mathrm{Fin}(V \dot{\rightarrow} C)$ such that $A = \emptyset$ holds $\mathrm{Involved}\, A = \emptyset$.

(12) For every set V and for every finite set C and for every element A of $\mathrm{Fin}(V \dot{\rightarrow} C)$ holds $\mathrm{Involved}\, A$ is finite.

(13) For every finite set C and for every element A of $\mathrm{Fin}(\emptyset \dot{\rightarrow} C)$ holds $\mathrm{Involved}\, A = \emptyset$.

Let V be a set, let C be a finite set, and let A be an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$. The functor $-A$ yielding an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$ is defined as follows:

(Def. 2) $-A = \{f; f$ ranges over elements of $\mathrm{Involved}\, A \dot{\rightarrow} C : \bigwedge_{g\,:\,\text{element of } V \dot{\rightarrow} C} (g \in A \Rightarrow f \not\approx g)\}$.

One can prove the following propositions:

(14) $A \frown -A = \emptyset$.

(15) If $A = \emptyset$, then $-A = \{\emptyset\}$.

(16) If $A = \{\emptyset\}$, then $-A = \emptyset$.

(17) For every set V and for every finite set C and for every element A of $\mathrm{SubstitutionSet}(V, C)$ holds $\mu(A \frown -A) = \bot_{\mathrm{SubstLatt}(V,C)}$.

(18) For every non empty set V and for every finite non empty set C and for every element A of $\mathrm{SubstitutionSet}(V, C)$ such that $A = \emptyset$ holds $\mu(-A) = \top_{\mathrm{SubstLatt}(V,C)}$.

(19) Let V be a set, C be a finite set, A be an element of SubstitutionSet(V, C), a be an element of $V \dot{\rightarrow} C$, and B be an element of SubstitutionSet(V, C). Suppose $B = \{a\}$. If $A \frown B = \emptyset$, then there exists a finite set b such that $b \in -A$ and $b \subseteq a$.

Let V be a set, let C be a finite set, and let A, B be elements of $\mathrm{Fin}(V \dot{\rightarrow} C)$. The functor $A \rightarrowtail B$ yielding an element of $\mathrm{Fin}(V \dot{\rightarrow} C)$ is defined as follows:

(Def. 3) $A \rightarrowtail B = (V \dot{\rightarrow} C) \cap \{\bigcup\{f(i) \setminus i; i$ ranges over elements of $V \dot{\rightarrow} C$: $i \in A\}; f$ ranges over elements of $A \dot{\rightarrow} B : \mathrm{dom}\, f = A\}$.

Next we state two propositions:

(20) Let A, B be elements of $\mathrm{Fin}(V \dot{\rightarrow} C)$ and s be a set. Suppose $s \in A \rightarrowtail B$. Then there exists a partial function f from A to B such that $s = \bigcup\{f(i) \setminus i; i$ ranges over elements of $V \dot{\rightarrow} C : i \in A\}$ and $\mathrm{dom}\, f = A$.

(21) For every set V and for every finite set C and for every element A of $\mathrm{Fin}(V \dot{\rightarrow} C)$ such that $A = \emptyset$ holds $A \rightarrowtail A = \{\emptyset\}$.

We adopt the following convention: u, v are elements of the carrier of SubstLatt(V, C), a is an element of $V \dot{\rightarrow} C$, and K, L are elements of SubstitutionSet(V, C).

The following proposition is true

(22) For every set X such that $X \subseteq u$ holds X is an element of the carrier of SubstLatt(V, C).

3. Lattice of Substitutions is Implicative

Let us consider V, C. The functor pseudo_compl(V, C) yielding a unary operation on the carrier of SubstLatt(V, C) is defined as follows:

(Def. 4) For every element u' of SubstitutionSet(V, C) such that $u' = u$ holds $(\text{pseudo\_compl}(V, C))(u) = \mu(-u')$.

The functor StrongImpl(V, C) yielding a binary operation on the carrier of SubstLatt(V, C) is defined by:

(Def. 5) For all elements u', v' of SubstitutionSet(V, C) such that $u' = u$ and $v' = v$ holds $(\text{StrongImpl}(V, C))(u, v) = \mu(u' \rightarrowtail v')$.

Let us consider u. The functor 2^u yielding an element of Fin (the carrier of SubstLatt(V, C)) is defined by:

(Def. 6) $2^u = 2^u$.

The functor $\square \setminus_u \square$ yielding a unary operation on the carrier of SubstLatt(V, C) is defined by:

(Def. 7) $(\square \setminus_u \square)(v) = u \setminus v$.

Let us consider V, C. The functor $\mathrm{Atom}(V,C)$ yielding a function from $V \dot{\to} C$ into the carrier of $\mathrm{SubstLatt}(V,C)$ is defined as follows:

(Def. 8) For every element a of $V \dot{\to} C$ holds $(\mathrm{Atom}(V,C))(a) = \mu\{a\}$.

Next we state a number of propositions:

(23) $\bigsqcup^{\mathrm{f}}_{K} \mathrm{Atom}(V,C) = \mathrm{FinUnion}(K, \mathrm{singleton}_{V \dot{\to} C})$.

(24) For every element u of $\mathrm{SubstitutionSet}(V,C)$ holds $u = \bigsqcup^{\mathrm{f}}_{u} \mathrm{Atom}(V,C)$.

(25) $(\Box \setminus_{u} \Box)(v) \sqsubseteq u$.

(26) For every element a of $V \dot{\to} C$ such that a is finite and for every set c such that $c \in (\mathrm{Atom}(V,C))(a)$ holds $c = a$.

(27) For every element a of $V \dot{\to} C$ such that $K = \{a\}$ and $L = u$ and $L \frown K = \emptyset$ holds $(\mathrm{Atom}(V,C))(a) \sqsubseteq (\mathrm{pseudo\_compl}(V,C))(u)$.

(28) For every finite element a of $V \dot{\to} C$ holds $a \in (\mathrm{Atom}(V,C))(a)$.

(29) Let u, v be elements of $\mathrm{SubstitutionSet}(V,C)$. Suppose that for every set c such that $c \in u$ there exists a set b such that $b \in v$ and $b \subseteq c \cup a$. Then there exists a set b such that $b \in u \rightarrowtail v$ and $b \subseteq a$.

(30) Let a be a finite element of $V \dot{\to} C$. Suppose for every element b of $V \dot{\to} C$ such that $b \in u$ holds $b \approx a$ and $u \sqcap (\mathrm{Atom}(V,C))(a) \sqsubseteq v$. Then $(\mathrm{Atom}(V,C))(a) \sqsubseteq (\mathrm{StrongImpl}(V,C))(u, v)$.

(31) $u \sqcap (\mathrm{pseudo\_compl}(V,C))(u) = \bot_{\mathrm{SubstLatt}(V,C)}$.

(32) $u \sqcap (\mathrm{StrongImpl}(V,C))(u, v) \sqsubseteq v$.

Let us consider V, C. Observe that $\mathrm{SubstLatt}(V,C)$ is implicative.

One can prove the following proposition

(33) $u \Rightarrow v = \bigsqcup^{\mathrm{f}}_{2^u}((\text{the meet operation of } \mathrm{SubstLatt}(V,C))^{\circ}(\mathrm{pseudo\_compl}(V,C), (\mathrm{StrongImpl}(V,C))^{\circ}(\Box \setminus_{u} \Box, v)))$.

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