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Full Trees

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The articles [13], [12], [6], [17], [1], [15], [11], [5], [7], [10], [8], [18], [2], [19], [14], [16], [3], [4], and [9] provide the terminology and notation for this paper.

1. Trees and Binary Trees

One can prove the following propositions:

(1) For every set D and for every finite sequence p and for every natural number n such that $p \in D^*$ holds $p \restriction \operatorname{Seg} n \in D^*$.

(2) For every binary tree T holds every element of T is a finite sequence of elements of *Boolean*.

Let T be a binary tree. We see that the element of T is a finite sequence of elements of *Boolean*.

Next we state several propositions:

(3) For every tree T such that $T = \{0,1\}^*$ holds T is binary.

(4) For every tree T such that $T = \{0,1\}^*$ holds $\mathrm{Leaves}(T) = \emptyset$.

(5) Let T be a binary tree, n be a natural number, and t be an element of T. If $t \in T\text{-level}(n)$, then t is a tuple of n and *Boolean*.

(6) For every tree T such that for every element t of T holds $\operatorname{succ} t = \{t \frown \langle 0 \rangle, t \frown \langle 1 \rangle\}$ holds $\mathrm{Leaves}(T) = \emptyset$.

(7) For every tree T such that for every element t of T holds $\operatorname{succ} t = \{t \frown \langle 0 \rangle, t \frown \langle 1 \rangle\}$ holds T is binary.

(8) For every tree T holds $T = \{0,1\}^*$ iff for every element t of T holds $\operatorname{succ} t = \{t \frown \langle 0 \rangle, t \frown \langle 1 \rangle\}$.

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In this article we present several logical schemes. The scheme *DecoratedBinTreeEx* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

> There exists a binary tree D decorated with elements of $\mathcal{A}$ such that $\operatorname{dom} D = \{0,1\}^*$ and $D(\varepsilon) = \mathcal{B}$ and for every node x of D holds $\mathcal{P}[D(x), D(x \frown \langle 0 \rangle), D(x \frown \langle 1 \rangle)]$

provided the following requirement is met:

- For every element a of $\mathcal{A}$ there exist elements b, c of $\mathcal{A}$ such that $\mathcal{P}[a,b,c]$.

The scheme *DecoratedBinTreeEx1* deals with a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and two binary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

> There exists a binary tree D decorated with elements of $\mathcal{A}$ such that $\operatorname{dom} D = \{0,1\}^*$ and $D(\varepsilon) = \mathcal{B}$ and for every node x of D holds $\mathcal{P}[D(x), D(x \frown \langle 0 \rangle)]$ and $\mathcal{Q}[D(x), D(x \frown \langle 1 \rangle)]$

provided the following requirements are met:

- For every element a of $\mathcal{A}$ there exists an element b of $\mathcal{A}$ such that $\mathcal{P}[a,b]$, and
- For every element a of $\mathcal{A}$ there exists an element b of $\mathcal{A}$ such that $\mathcal{Q}[a,b]$.

Let T be a binary tree and let n be a non empty natural number. The functor NumberOnLevel(n,T) yields a function from T-level(n) into $\mathbb{N}$ and is defined as follows:

(Def. 1) For every element t of T such that $t \in T$-level(n) and for every tuple F of n and *Boolean* such that $F = \operatorname{Rev}(t)$ holds $(\text{NumberOnLevel}(n,T))(t) = \text{Absval}(F) + 1$.

Let T be a binary tree and let n be a non empty natural number. Note that NumberOnLevel(n,T) is one-to-one.

2. Full Trees

Let T be a tree. We say that T is full if and only if:

(Def. 2) $T = \{0,1\}^*$.

We now state three propositions:

(9) $\{0,1\}^*$ is a tree.

(10) For every tree T such that $T = \{0,1\}^*$ and for every natural number n holds $\langle \underbrace{0,\ldots,0}_{n} \rangle \in T$-level$(n)$.

(11) Let T be a tree. Suppose $T = \{0,1\}^*$. Let n be a non empty natural number and y be a tuple of n and *Boolean*. Then $y \in T$-level(n).

Let T be a binary tree and let n be a natural number. Observe that T-level(n) is finite.

One can check that every tree which is full is also binary.

One can verify that there exists a tree which is full.

One can prove the following proposition

(12) For every full tree T and for every non empty natural number n holds Seg (the n-th power of 2) $\subseteq$ rng NumberOnLevel(n, T).

Let T be a full tree and let n be a non empty natural number. The functor FinSeqLevel(n, T) yielding a finite sequence of elements of T-level(n) is defined by:

(Def. 3) FinSeqLevel(n, T) = (NumberOnLevel(n, T))$^{-1}$.

Let T be a full tree and let n be a non empty natural number. Note that FinSeqLevel(n, T) is one-to-one.

Next we state a number of propositions:

(13) For every full tree T and for every non empty natural number n holds (NumberOnLevel(n, T))($\langle\underbrace{0, \ldots, 0}_{n}\rangle$) = 1.

(14) Let T be a full tree, n be a non empty natural number, and y be a tuple of n and *Boolean*. If $y = \langle\underbrace{0, \ldots, 0}_{n}\rangle$, then (NumberOnLevel($n, T$))($\neg y$) = the n-th power of 2.

(15) For every full tree T and for every non empty natural number n holds (FinSeqLevel(n, T))(1) = $\langle\underbrace{0, \ldots, 0}_{n}\rangle$.

(16) Let T be a full tree, n be a non empty natural number, and y be a tuple of n and *Boolean*. If $y = \langle\underbrace{0, \ldots, 0}_{n}\rangle$, then (FinSeqLevel($n, T$))(the n-th power of 2) = $\neg y$.

(17) Let T be a full tree, n be a non empty natural number, and i be a natural number. If $i \in$ Seg (the n-th power of 2), then (FinSeqLevel(n, T))(i) = Rev(n-BinarySequence($i -' 1$)).

(18) For every full tree T and for every natural number n holds $\overline{\overline{T\text{-level}(n)}}$ = the n-th power of 2.

(19) For every full tree T and for every non empty natural number n holds len FinSeqLevel(n, T) = the n-th power of 2.

(20) For every full tree T and for every non empty natural number n holds dom FinSeqLevel(n, T) = Seg (the n-th power of 2).

(21) For every full tree T and for every non empty natural number n holds rng FinSeqLevel(n, T) = T-level(n).

(22) For every full tree T holds (FinSeqLevel($1, T$))(1) = $\langle 0\rangle$.

(23) For every full tree T holds $(\mathrm{FinSeqLevel}(1, T))(2) = \langle 1 \rangle$.

(24) Let T be a full tree and n, i be non empty natural numbers. Suppose $i \leqslant$ the $(n+1)$-th power of 2. Let F be a tuple of n and *Boolean*. If $F = (\mathrm{FinSeqLevel}(n, T))((i+1) \div 2)$, then $(\mathrm{FinSeqLevel}(n+1, T))(i) = F \frown \langle (i+1) \bmod 2 \rangle$.

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