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Euler Circuits and Paths[1]

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Summary. We prove the Euler theorem on existence of Euler circuits and paths in multigraphs.

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The notation and terminology used in this paper are introduced in the following papers: [19], [23], [13], [10], [22], [24], [6], [9], [7], [4], [8], [2], [20], [12], [3], [5], [21], [1], [14], [15], [11], [16], [17], and [18].

1. Preliminaries

Let D be a set, let T be a non empty set of finite sequences of D, and let S be a non empty subset of T. We see that the element of S is a finite sequence of elements of D.

Let i, j be even integers. One can verify that $i - j$ is even.

We now state two propositions:

(1) For all integers i, j holds i is even iff j is even iff $i - j$ is even.

(2) Let p be a finite sequence and m, n, a be natural numbers. Suppose $a \in \operatorname{dom}\langle p(m), \ldots, p(n)\rangle$. Then there exists a natural number k such that $k \in \operatorname{dom} p$ and $p(k) = \langle p(m), \ldots, p(n)\rangle(a)$ and $k + 1 = m + a$ and $m \leqslant k$ and $k \leqslant n$.

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Let G be a graph. A vertex of G is an element of the vertices of G.

For simplicity, we follow the rules: G denotes a graph, v, v_1, v_2 denote vertices of G, c, c_1, c_2 denote chains of G, p, p_1, p_2 denote paths of G, v_3, v_4, v_5 denote finite sequences of elements of the vertices of G, e, X denote sets, and n, m denote natural numbers.

One can prove the following propositions:

(3) If v_3 is vertex sequence of c, then v_3 is non empty.

(4) If c is cyclic and v_3 is vertex sequence of c, then $v_3(1) = v_3(\operatorname{len} v_3)$.

(5) If $n \in \operatorname{dom} p$ and $m \in \operatorname{dom} p$ and $n \neq m$, then $p(n) \neq p(m)$.

(6) ε is a path of G.

(7) If $e \in$ the edges of G, then $\langle e \rangle$ is a path of G.

(8) $\langle p(m), \ldots, p(n) \rangle$ is a path of G.

(9) Suppose $\operatorname{rng} p_1$ misses $\operatorname{rng} p_2$ and v_4 is vertex sequence of p_1 and v_5 is vertex sequence of p_2 and $v_4(\operatorname{len} v_4) = v_5(1)$. Then $p_1 \frown p_2$ is a path of G.

(10) p is one-to-one.

(11) If $c_1 \frown c_2$ is a path of G, then $\operatorname{rng} c_1$ misses $\operatorname{rng} c_2$.

(12) If $c = \varepsilon$, then c is cyclic.

Let G be a graph. Observe that there exists a path of G which is cyclic.

Next we state several propositions:

(13) For every cyclic path p of G holds $\langle p(m+1), \ldots, p(\operatorname{len} p) \rangle \frown \langle p(1), \ldots, p(m) \rangle$ is a cyclic path of G.

(14) If $m+1 \in \operatorname{dom} p$, then $\operatorname{len}(\langle p(m+1), \ldots, p(\operatorname{len} p) \rangle \frown \langle p(1), \ldots, p(m) \rangle) = \operatorname{len} p$ and $\operatorname{rng}(\langle p(m+1), \ldots, p(\operatorname{len} p) \rangle \frown \langle p(1), \ldots, p(m) \rangle) = \operatorname{rng} p$ and $(\langle p(m+1), \ldots, p(\operatorname{len} p) \rangle \frown \langle p(1), \ldots, p(m) \rangle)(1) = p(m+1)$.

(15) For every cyclic path p of G such that $n \in \operatorname{dom} p$ there exists a cyclic path p' of G such that $p'(1) = p(n)$ and $\operatorname{len} p' = \operatorname{len} p$ and $\operatorname{rng} p' = \operatorname{rng} p$.

(16) Let s, t be vertices of G. Suppose $s =$ (the source of G)(e) and $t =$ (the target of G)(e). Then $\langle t, s \rangle$ is vertex sequence of $\langle e \rangle$.

(17) Suppose $e \in$ the edges of G and v_3 is vertex sequence of c and $v_3(\operatorname{len} v_3) =$ (the source of G)(e). Then

(i) $c \frown \langle e \rangle$ is a chain of G, and

(ii) there exists a finite sequence v_1' of elements of the vertices of G such that $v_1' = v_3 \frown\frown \langle$(the source of G)(e), (the target of G)$(e)\rangle$ and v_1' is vertex sequence of $c \frown \langle e \rangle$ and $v_1'(1) = v_3(1)$ and $v_1'(\operatorname{len} v_1') =$ (the target of G)(e).

(18) Suppose $e \in$ the edges of G and v_3 is vertex sequence of c and $v_3(\operatorname{len} v_3) =$ (the target of G)(e). Then

(i) $c \frown \langle e \rangle$ is a chain of G, and

(ii) there exists a finite sequence v_1' of elements of the vertices of G such that $v_1' = v_3 \frown\frown \langle$(the target of G)(e), (the source of G)$(e)\rangle$ and v_1' is vertex sequence of $c ^\frown \langle e \rangle$ and $v'_1(1) = v_3(1)$ and $v'_1(\operatorname{len} v'_1) = (\text{the source of } G)(e)$.

(19) Suppose v_3 is vertex sequence of c. Let n be a natural number. Suppose $n \in \operatorname{dom} c$. Then

(i) $v_3(n) = (\text{the target of } G)(c(n))$ and $v_3(n+1) = (\text{the source of } G)(c(n))$, or

(ii) $v_3(n) = (\text{the source of } G)(c(n))$ and $v_3(n+1) = (\text{the target of } G)(c(n))$.

(20) If v_3 is vertex sequence of c and $e \in \operatorname{rng} c$, then $(\text{the target of } G)(e) \in \operatorname{rng} v_3$ and $(\text{the source of } G)(e) \in \operatorname{rng} v_3$.

Let G be a graph and let X be a set. Then G-VSet(X) is a subset of the vertices of G.

One can prove the following propositions:

(21) G-VSet$(\emptyset) = \emptyset$.

(22) If $e \in$ the edges of G and $e \in X$, then G-VSet(X) is non empty.

(23) G is connected if and only if for all v_1, v_2 such that $v_1 \neq v_2$ there exist c, v_3 such that c is non empty and v_3 is vertex sequence of c and $v_3(1) = v_1$ and $v_3(\operatorname{len} v_3) = v_2$.

(24) Let G be a connected graph, X be a set, and v be a vertex of G. Suppose X meets the edges of G and $v \notin G$-VSet(X). Then there exists a vertex v' of G and there exists an element e of the edges of G such that $v' \in G$-VSet(X) but $e \notin X$ but $v' = (\text{the target of } G)(e)$ or $v' = (\text{the source of } G)(e)$.

2. Degree of a vertex

Let G be a graph, let v be a vertex of G, and let X be a set. The functor EdgesIn(v, X) yields a subset of the edges of G and is defined as follows:

(Def. 1) For every set e holds $e \in \text{EdgesIn}(v, X)$ iff $e \in$ the edges of G and $e \in X$ and $(\text{the target of } G)(e) = v$.

The functor EdgesOut(v, X) yields a subset of the edges of G and is defined as follows:

(Def. 2) For every set e holds $e \in \text{EdgesOut}(v, X)$ iff $e \in$ the edges of G and $e \in X$ and $(\text{the source of } G)(e) = v$.

Let G be a graph, let v be a vertex of G, and let X be a set. The functor EdgesAt(v, X) yields a subset of the edges of G and is defined as follows:

(Def. 3) $\text{EdgesAt}(v, X) = \text{EdgesIn}(v, X) \cup \text{EdgesOut}(v, X)$.

Let G be a finite graph, let v be a vertex of G, and let X be a set. One can check the following observations:

* EdgesIn(v, X) is finite,

* EdgesOut(v, X) is finite, and
* EdgesAt(v, X) is finite.

Let G be a graph, let v be a vertex of G, and let X be an empty set. One can verify the following observations:

* EdgesIn(v, X) is empty,
* EdgesOut(v, X) is empty, and
* EdgesAt(v, X) is empty.

Let G be a graph and let v be a vertex of G. The functor EdgesIn v yields a subset of the edges of G and is defined as follows:

(Def. 4) EdgesIn v = EdgesIn$(v,$ the edges of $G)$.

The functor EdgesOut v yields a subset of the edges of G and is defined by:

(Def. 5) EdgesOut v = EdgesOut$(v,$ the edges of $G)$.

One can prove the following propositions:

(25) EdgesIn$(v, X) \subseteq$ EdgesIn v.

(26) EdgesOut$(v, X) \subseteq$ EdgesOut v.

Let G be a finite graph and let v be a vertex of G. Note that EdgesIn v is finite and EdgesOut v is finite.

For simplicity, we follow the rules: G denotes a finite graph, v denotes a vertex of G, c denotes a chain of G, v_3 denotes a finite sequence of elements of the vertices of G, and X_1, X_2 denote sets.

One can prove the following two propositions:

(27) card EdgesIn v = EdgIn(v).

(28) card EdgesOut v = EdgOut(v).

Let G be a finite graph, let v be a vertex of G, and let X be a set. The functor Degree(v, X) yields a natural number and is defined as follows:

(Def. 6) Degree(v, X) = card EdgesIn(v, X) + card EdgesOut(v, X).

The following propositions are true:

(29) The degree of v = Degree$(v,$ the edges of $G)$.

(30) If Degree$(v, X) \neq 0$, then EdgesAt(v, X) is non empty.

(31) Suppose $e \in$ the edges of G but $e \notin X$ but v = (the target of $G)(e)$ or v = (the source of $G)(e)$. Then the degree of $v \neq$ Degree(v, X).

(32) If $X_2 \subseteq X_1$, then card EdgesIn$(v, X_1 \setminus X_2)$ = card EdgesIn(v, X_1) − card EdgesIn(v, X_2).

(33) If $X_2 \subseteq X_1$, then card EdgesOut$(v, X_1 \setminus X_2)$ = card EdgesOut(v, X_1) − card EdgesOut(v, X_2).

(34) If $X_2 \subseteq X_1$, then Degree$(v, X_1 \setminus X_2)$ = Degree(v, X_1) − Degree(v, X_2).

(35) EdgesIn(v, X) = EdgesIn$(v, X \cap$ the edges of $G)$ and EdgesOut(v, X) = EdgesOut$(v, X \cap$ the edges of $G)$.

(36) $\mathrm{Degree}(v, X) = \mathrm{Degree}(v, X \cap \text{the edges of } G)$.

(37) If c is non empty and v_3 is vertex sequence of c, then $v \in \mathrm{rng}\, v_3$ iff $\mathrm{Degree}(v, \mathrm{rng}\, c) \neq 0$.

(38) For every non empty finite connected graph G and for every vertex v of G holds the degree of $v \neq 0$.

3. Adding an edge to a graph

Let G be a graph and let v_1, v_2 be vertices of G. The functor $\mathrm{AddNewEdge}(v_1, v_2)$ yielding a strict graph is defined by the conditions (Def. 7).

(Def. 7)(i) The vertices of $\mathrm{AddNewEdge}(v_1, v_2) =$ the vertices of G,

(ii) the edges of $\mathrm{AddNewEdge}(v_1, v_2) =$ (the edges of G) $\cup$ {the edges of G},

(iii) the source of $\mathrm{AddNewEdge}(v_1, v_2) =$ (the source of G)$+\cdot$((the edges of G)$\dot{\longmapsto}$(v_1)), and

(iv) the target of $\mathrm{AddNewEdge}(v_1, v_2) =$ (the target of G)$+\cdot$((the edges of G)$\dot{\longmapsto}$(v_2)).

Let G be a finite graph and let v_1, v_2 be vertices of G. Observe that $\mathrm{AddNewEdge}(v_1, v_2)$ is finite.

For simplicity, we adopt the following rules: G is a graph, v, v_1, v_2 are vertices of G, c is a chain of G, p is a path of G, v_3 is a finite sequence of elements of the vertices of G, v' is a vertex of $\mathrm{AddNewEdge}(v_1, v_2)$, p' is a path of $\mathrm{AddNewEdge}(v_1, v_2)$, and v'_1 is a finite sequence of elements of the vertices of $\mathrm{AddNewEdge}(v_1, v_2)$.

We now state a number of propositions:

(39)(i) The edges of $G \in$ the edges of $\mathrm{AddNewEdge}(v_1, v_2)$,

(ii) the edges of $G =$ (the edges of $\mathrm{AddNewEdge}(v_1, v_2)$) $\setminus$ {the edges of G},

(iii) (the source of $\mathrm{AddNewEdge}(v_1, v_2)$)(the edges of G) $= v_1$, and

(iv) (the target of $\mathrm{AddNewEdge}(v_1, v_2)$)(the edges of G) $= v_2$.

(40) Suppose $e \in$ the edges of G. Then (the source of $\mathrm{AddNewEdge}(v_1, v_2)$)($e$) $=$ (the source of G)(e) and (the target of $\mathrm{AddNewEdge}(v_1, v_2)$)($e$) $=$ (the target of G)(e).

(41) If $v'_1 = v_3$ and v_3 is vertex sequence of c, then v'_1 is vertex sequence of c.

(42) c is a chain of $\mathrm{AddNewEdge}(v_1, v_2)$.

(43) p is a path of $\mathrm{AddNewEdge}(v_1, v_2)$.

(44) If $v' = v_1$ and $v_1 \neq v_2$, then $\mathrm{EdgesIn}(v', X) = \mathrm{EdgesIn}(v_1, X)$.

(45) If $v' = v_2$ and $v_1 \neq v_2$, then $\mathrm{EdgesOut}(v', X) = \mathrm{EdgesOut}(v_2, X)$.

(46) If $v' = v_1$ and $v_1 \neq v_2$ and the edges of $G \in X$, then $\mathrm{EdgesOut}(v', X) = \mathrm{EdgesOut}(v_1, X) \cup \{\text{the edges of } G\}$ and $\mathrm{EdgesOut}(v_1, X) \cap \{\text{the edges of } G\} = \emptyset$.

(47) If $v' = v_2$ and $v_1 \neq v_2$ and the edges of $G \in X$, then $\mathrm{EdgesIn}(v', X) = \mathrm{EdgesIn}(v_2, X) \cup \{\text{the edges of } G\}$ and $\mathrm{EdgesIn}(v_2, X) \cap \{\text{the edges of } G\} = \emptyset$.

(48) If $v' = v$ and $v \neq v_1$ and $v \neq v_2$, then $\mathrm{EdgesIn}(v', X) = \mathrm{EdgesIn}(v, X)$.

(49) If $v' = v$ and $v \neq v_1$ and $v \neq v_2$, then $\mathrm{EdgesOut}(v', X) = \mathrm{EdgesOut}(v, X)$.

(50) If the edges of $G \notin \mathrm{rng}\, p'$, then p' is a path of G.

(51) If the edges of $G \notin \mathrm{rng}\, p'$ and $v_3 = v'_1$ and v'_1 is vertex sequence of p', then v_3 is vertex sequence of p'.

Let G be a connected graph and let v_1, v_2 be vertices of G. One can check that $\mathrm{AddNewEdge}(v_1, v_2)$ is connected.

For simplicity, we adopt the following rules: G is a finite graph, v, v_1, v_2 are vertices of G, v_3 is a finite sequence of elements of the vertices of G, and v' is a vertex of $\mathrm{AddNewEdge}(v_1, v_2)$.

We now state two propositions:

(52) If $v' = v$ and $v_1 \neq v_2$ and $v = v_1$ or $v = v_2$ and the edges of $G \in X$, then $\mathrm{Degree}(v', X) = \mathrm{Degree}(v, X) + 1$.

(53) If $v' = v$ and $v \neq v_1$ and $v \neq v_2$, then $\mathrm{Degree}(v', X) = \mathrm{Degree}(v, X)$.

4. Some properties of and operations on cycles

The following two propositions are true:

(54) For every cyclic path c of G holds $\mathrm{Degree}(v, \mathrm{rng}\, c)$ is even.

(55) Let c be a path of G. Suppose c is non cyclic and v_3 is vertex sequence of c. Then $\mathrm{Degree}(v, \mathrm{rng}\, c)$ is even if and only if $v \neq v_3(1)$ and $v \neq v_3(\mathrm{len}\, v_3)$.

In the sequel G is a graph, v is a vertex of G, and v_3 is a finite sequence of elements of the vertices of G.

Let G be a graph. The functor G-CycleSet yields a non empty set of finite sequences of the edges of G and is defined as follows:

(Def. 8) For every set x holds $x \in G$-CycleSet iff x is a cyclic path of G.

One can prove the following propositions:

(56) ε is an element of G-CycleSet.

(57) Let c be an element of G-CycleSet. Suppose $v \in G\text{-VSet}(\mathrm{rng}\, c)$. Then $\{c', c'$ ranges over elements of G-CycleSet: $\mathrm{rng}\, c' = \mathrm{rng}\, c \ \wedge \ \bigvee_{v_3} (v_3$ is vertex sequence of $c' \ \wedge \ v_3(1) = v)\}$ is a non empty subset of G-CycleSet.

Let us consider G, v and let c be an element of G-CycleSet. Let us assume that $v \in G$-VSet(rng c). The functor $c^{v}_{\circlearrowleft}$ yields an element of G-CycleSet and is defined as follows:

(Def. 9) $c^{v}_{\circlearrowleft} = \mathrm{choose}(\{c', c'$ ranges over elements of G-CycleSet: $\mathrm{rng}\, c' = \mathrm{rng}\, c \ \wedge \ \bigvee_{v_3} (v_3$ is vertex sequence of $c' \ \wedge \ v_3(1) = v)\})$.

Let G be a graph and let c_1, c_2 be elements of G-CycleSet. Let us assume that G-VSet(rng c_1) meets G-VSet(rng c_2) and rng c_1 misses rng c_2. The functor CatCycles(c_1, c_2) yields an element of G-CycleSet and is defined as follows:

(Def. 10) There exists a vertex v of G such that $v = \mathrm{choose}((G\text{-VSet}(\mathrm{rng}\, c_1)) \cap (G\text{-VSet}(\mathrm{rng}\, c_2)))$ and $\mathrm{CatCycles}(c_1, c_2) = ({c_1}^{v}_{\circlearrowleft}) ^\frown {c_2}^{v}_{\circlearrowleft}$.

The following proposition is true

(58) Let G be a graph and c_1, c_2 be elements of G-CycleSet. Suppose G-VSet(rng c_1) meets G-VSet(rng c_2) but rng c_1 misses rng c_2 but $c_1 \neq \varepsilon$ or $c_2 \neq \varepsilon$. Then CatCycles(c_1, c_2) is non empty.

In the sequel G denotes a finite graph, v denotes a vertex of G, and v_3 denotes a finite sequence of elements of the vertices of G.

Let us consider G, v and let X be a set. Let us assume that $\mathrm{Degree}(v, X) \neq 0$. The functor X-PathSet(v) yielding a non empty set of finite sequences of the edges of G is defined as follows:

(Def. 11) X-PathSet$(v) = \{c, c$ ranges over elements of X^*: c is a path of $G \wedge c$ is non empty $\wedge \ \bigvee_{v_3} (v_3$ is vertex sequence of $c \ \wedge \ v_3(1) = v)\}$.

One can prove the following proposition

(59) For every element p of X-PathSet(v) and for every finite set Y such that $Y =$ the edges of G and $\mathrm{Degree}(v, X) \neq 0$ holds $\mathrm{len}\, p \leqslant \mathrm{card}\, Y$.

Let us consider G, v and let X be a set. Let us assume that for every vertex v_1 of G holds $\mathrm{Degree}(v_1, X)$ is even and $\mathrm{Degree}(v, X) \neq 0$. The functor X-CycleSet$v$ yielding a non empty subset of G-CycleSet is defined as follows:

(Def. 12) X-CycleSet$v = \{c, c$ ranges over elements of G-CycleSet: $\mathrm{rng}\, c \subseteq X \wedge c$ is non empty $\wedge \ \bigvee_{v_3} (v_3$ is vertex sequence of $c \ \wedge \ v_3(1) = v)\}$.

Next we state two propositions:

(60) If $\mathrm{Degree}(v, X) \neq 0$ and for every v holds $\mathrm{Degree}(v, X)$ is even, then for every element c of X-CycleSet$v$ holds c is non empty and $\mathrm{rng}\, c \subseteq X$ and $v \in G$-VSet(rng c).

(61) Let G be a finite connected graph and c be an element of G-CycleSet. Suppose $\mathrm{rng}\, c \neq$ the edges of G and c is non empty. Then $\{v', v'$ ranges over vertices of G: $v' \in G$-VSet(rng c) $\wedge$ the degree of $v' \neq \mathrm{Degree}(v', \mathrm{rng}\, c)\}$ is a non empty subset of the vertices of G.

Let G be a finite connected graph and let c be an element of G-CycleSet. Let us assume that $\mathrm{rng}\, c \neq$ the edges of G and c is non empty. The functor

ExtendCycle c yields an element of G-CycleSet and is defined by the condition (Def. 13).

(Def. 13) There exists an element c' of G-CycleSet and there exists a vertex v of G such that $v = \mathrm{choose}(\{v', v'$ ranges over vertices of G: $v' \in G\text{-VSet}(\mathrm{rng}\, c) \wedge$ the degree of $v' \neq \mathrm{Degree}(v', \mathrm{rng}\, c)\})$ and $c' = \mathrm{choose}(((\text{the edges of } G) \setminus \mathrm{rng}\, c)\text{-CycleSet}v)$ and $\mathrm{ExtendCycle}\, c = \mathrm{CatCycles}(c, c')$.

One can prove the following proposition

(62) Let G be a finite connected graph and c be an element of G-CycleSet. Suppose $\mathrm{rng}\, c \neq$ the edges of G and c is non empty and for every vertex v of G holds the degree of v is even. Then ExtendCycle c is non empty and $\mathrm{card}\,\mathrm{rng}\, c < \mathrm{card}\,\mathrm{rng}\,\mathrm{ExtendCycle}\, c$.

5. Euler circuits and paths

Let G be a graph and let p be a path of G. We say that p is Eulerian if and only if:

(Def. 14) $\mathrm{rng}\, p =$ the edges of G.

We now state three propositions:

(63) Let G be a connected graph, p be a path of G, and v_3 be a finite sequence of elements of the vertices of G. Suppose p is Eulerian and v_3 is vertex sequence of p. Then $\mathrm{rng}\, v_3 =$ the vertices of G.

(64) Let G be a finite connected graph. Then there exists a cyclic path of G which is Eulerian if and only if for every vertex v of G holds the degree of v is even.

(65) Let G be a finite connected graph. Then there exists a path of G which is non cyclic and Eulerian if and only if there exist vertices v_1, v_2 of G such that $v_1 \neq v_2$ and for every vertex v of G holds the degree of v is even iff $v \neq v_1$ and $v \neq v_2$.

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References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Czesław Byliński. A classical first order language. *Formalized Mathematics*, 1(**4**):669–676, 1990.

[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.
[9] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[10] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[11] Czesław Byliński and Piotr Rudnicki. The correspondence between monotonic many sorted signatures and well-founded graphs. Part I. *Formalized Mathematics*, 5(**4**):577–582, 1996.
[12] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[14] Krzysztof Hryniewiecki. Graphs. *Formalized Mathematics*, 2(**3**):365–370, 1991.
[15] Yatsuka Nakamura and Piotr Rudnicki. Vertex sequences induced by chains. *Formalized Mathematics*, 5(**3**):297–304, 1996.
[16] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Formalized Mathematics*, 5(**2**):167–172, 1996.
[17] Piotr Rudnicki and Andrzej Trybulec. Abian's fixed point theorem. *Formalized Mathematics*, 6(**3**):335–338, 1997.
[18] Piotr Rudnicki and Andrzej Trybulec. On same equivalents of well-foundedness. *Formalized Mathematics*, 6(**3**):339–343, 1997.
[19] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[20] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.
[21] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[23] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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