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Conjugate Sequences, Bounded Complex Sequences and Convergent Complex Sequences

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Summary. This article is a continuation of [1].It is divided into five sections. The first one contains a few useful lemmas. In the second part there is a definition of conjugate sequences and proofs of some basic properties of such sequences. The third segment treats of bounded complex sequences,next one contains description of convergent complex sequences. The last and the biggest part of the article contains proofs of main theorems concerning the theory of bounded and convergent complex sequences.

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The notation and terminology used in this paper have been introduced in the following articles: [4], [6], [5], [7], [2], [8], [3], and [1].

1. Preliminaries

We adopt the following convention: n, m denote natural numbers, r, p, g denote elements of $\mathbb{C}$, and s, s', s_1 denote complex sequences.

The following propositions are true:

(1) If $g \neq 0_{\mathbb{C}}$ and $r \neq 0_{\mathbb{C}}$, then $|g^{-1} - r^{-1}| = \frac{|g-r|}{|g| \cdot |r|}$.

(2) For every n there exists a real number r such that $0 < r$ and for every m such that $m \leqslant n$ holds $|s(m)| < r$.

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2. Conjugate sequences

Let us consider s. The functor s^* yields a complex sequence and is defined by:

(Def. 1) For every n holds $s^*(n) = s(n)^*$.

We now state several propositions:

(3) If s is non-zero, then s^* is non-zero.

(4) $(r\,s)^* = r^*\,s^*$.

(5) $(s\,s')^* = s^*\,s'^*$.

(6) If s is non-zero, then $(s^*)^{-1} = (s^{-1})^*$.

(7) If s is non-zero, then $(\frac{s'}{s})^* = \frac{s'^*}{s^*}$.

3. Bounded complex sequences

Let us consider s. We say that s is bounded if and only if:

(Def. 2) There exists a real number r such that for every n holds $|s(n)| < r$.

Let us observe that there exists a complex sequence which is bounded.

Next we state the proposition

(8) s is bounded iff there exists a real number r such that $0 < r$ and for every n holds $|s(n)| < r$.

4. Convergent complex sequences

Let us consider s. We say that s is convergent if and only if:

(Def. 3) There exists g such that for every real number p such that $0 < p$ there exists n such that for every m such that $n \leqslant m$ holds $|s(m) - g| < p$.

Let us consider s. Let us assume that s is convergent. The functor $\lim s$ yields an element of $\mathbb{C}$ and is defined as follows:

(Def. 4) For every real number p such that $0 < p$ there exists n such that for every m such that $n \leqslant m$ holds $|s(m) - \lim s| < p$.

One can prove the following two propositions:

(9) If there exists g such that for every natural number n holds $s(n) = g$, then s is convergent.

(10) For every g such that for every natural number n holds $s(n) = g$ holds $\lim s = g$.

Let us observe that there exists a complex sequence which is convergent.

Let s be a convergent complex sequence. Observe that $|s|$ is convergent.

One can prove the following proposition

(11) If s is convergent, then $\lim |s| = |\lim s|$.

Let s be a convergent complex sequence. Observe that s^* is convergent.

We now state the proposition

(12) If s is convergent, then $\lim(s^*) = (\lim s)^*$.

5. Main theorems

The following propositions are true:

(13) If s is convergent and s' is convergent, then $s + s'$ is convergent.

(14) If s is convergent and s' is convergent, then $\lim(s + s') = \lim s + \lim s'$.

(15) If s is convergent and s' is convergent, then $\lim |s + s'| = |\lim s + \lim s'|$.

(16) If s is convergent and s' is convergent, then $\lim((s + s')^*) = (\lim s)^* + (\lim s')^*$.

(17) If s is convergent, then $r\, s$ is convergent.

(18) If s is convergent, then $\lim(r\, s) = r \cdot \lim s$.

(19) If s is convergent, then $\lim |r\, s| = |r| \cdot |\lim s|$.

(20) If s is convergent, then $\lim((r\, s)^*) = r^* \cdot (\lim s)^*$.

(21) If s is convergent, then $-s$ is convergent.

(22) If s is convergent, then $\lim(-s) = -\lim s$.

(23) If s is convergent, then $\lim |-s| = |\lim s|$.

(24) If s is convergent, then $\lim((-s)^*) = -(\lim s)^*$.

(25) If s is convergent and s' is convergent, then $s - s'$ is convergent.

(26) If s is convergent and s' is convergent, then $\lim(s - s') = \lim s - \lim s'$.

(27) If s is convergent and s' is convergent, then $\lim |s - s'| = |\lim s - \lim s'|$.

(28) If s is convergent and s' is convergent, then $\lim((s - s')^*) = (\lim s)^* - (\lim s')^*$.

Let us mention that every complex sequence which is convergent is also bounded.

Let us note that every complex sequence which is non bounded is also non convergent.

One can prove the following propositions:

(29) If s is a convergent complex sequence and s' is a convergent complex sequence, then $s\, s'$ is convergent.

(30) If s is a convergent complex sequence and s' is a convergent complex sequence, then $\lim(s\, s') = \lim s \cdot \lim s'$.

(31) If s is convergent and s' is convergent, then $\lim |s\, s'| = |\lim s| \cdot |\lim s'|$.

(32) If s is convergent and s' is convergent, then $\lim((s\, s')^*) = (\lim s)^* \cdot (\lim s')^*$.

(33) If s is convergent, then if $\lim s \neq 0_{\mathbb{C}}$, then there exists n such that for every m such that $n \leqslant m$ holds $\frac{|\lim s|}{2} < |s(m)|$.

(34) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then s^{-1} is convergent.

(35) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim(s^{-1}) = (\lim s)^{-1}$.

(36) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim |s^{-1}| = |\lim s|^{-1}$.

(37) If s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim((s^{-1})^{*}) = ((\lim s)^{*})^{-1}$.

(38) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\frac{s'}{s}$ is convergent.

(39) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim(\frac{s'}{s}) = \frac{\lim s'}{\lim s}$.

(40) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim |\frac{s'}{s}| = \frac{|\lim s'|}{|\lim s|}$.

(41) If s' is convergent and s is convergent and $\lim s \neq 0_{\mathbb{C}}$ and s is non-zero, then $\lim((\frac{s'}{s})^{*}) = \frac{(\lim s')^{*}}{(\lim s)^{*}}$.

(42) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $s\,s_1$ is convergent.

(43) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $\lim(s\,s_1) = 0_{\mathbb{C}}$.

(44) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $\lim |s\,s_1| = 0$.

(45) If s is convergent and s_1 is bounded and $\lim s = 0_{\mathbb{C}}$, then $\lim((s\,s_1)^{*}) = 0_{\mathbb{C}}$.

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