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Directed Sets, Nets, Ideals, Filters, and Maps [1]

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Summary. Notation and facts necessary to start with the formalization of continuous lattices according to [8] are introduced. The article contains among other things, the definition of directed and filtered subsets of a poset (see 1.1 in [8, p. 2]), the definition of nets on the poset (see 1.2 in [8, p. 2]), the definition of ideals and filters and the definition of maps preserving arbitrary and directed sups and arbitrary and filtered infs (1.9 also in [8, p. 4]). The concepts of semilattices, sup-semiletices and poset lattices (1.8 in [8, p. 4]) are also introduced. A number of facts concerning the above notion and including remarks 1.4, 1.5, and 1.10 from [8, pp. 3–5] is presented.

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The notation and terminology used in this paper are introduced in the following papers: [13], [15], [16], [18], [17], [7], [5], [6], [11], [4], [10], [19], [3], [2], [12], [1], [14], and [9].

1. Directed subsets

Let L be a relational structure and let X be a subset of L. We say that X is directed if and only if:

(Def. 1) For all elements x, y of L such that $x \in X$ and $y \in X$ there exists an element z of L such that $z \in X$ and $x \leq z$ and $y \leq z$.

We say that X is filtered if and only if:

(Def. 2) For all elements x, y of L such that $x \in X$ and $y \in X$ there exists an element z of L such that $z \in X$ and $z \leq x$ and $z \leq y$.

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Next we state two propositions:

(1) Let L be a non empty transitive relational structure and X be a subset of L. Then X is non empty directed if and only if for every finite subset Y of X there exists an element x of L such that $x \in X$ and $x \geq Y$.

(2) Let L be a non empty transitive relational structure and X be a subset of L. Then X is non empty filtered if and only if for every finite subset Y of X there exists an element x of L such that $x \in X$ and $x \leq Y$.

Let L be a relational structure. One can verify that $\emptyset_L$ is directed and filtered.

Let L be a relational structure. Observe that there exists a subset of L which is directed and filtered.

One can prove the following three propositions:

(3) Let L_1, L_2 be relational structures. Suppose the relational structure of L_1 = the relational structure of L_2. Let X_1 be a subset of L_1 and X_2 be a subset of L_2. If $X_1 = X_2$ and X_1 is directed, then X_2 is directed.

(4) Let L_1, L_2 be relational structures. Suppose the relational structure of L_1 = the relational structure of L_2. Let X_1 be a subset of L_1 and X_2 be a subset of L_2. If $X_1 = X_2$ and X_1 is filtered, then X_2 is filtered.

(5) For every non empty reflexive relational structure L and for every element x of L holds $\{x\}$ is directed and filtered.

Let L be a non empty reflexive relational structure. Note that there exists a subset of L which is directed, filtered, non empty, and finite.

Let L be a relational structure with l.u.b.'s. Note that Ω_L is directed.

Let L be an upper-bounded non empty relational structure. Observe that Ω_L is directed.

Let L be a relational structure with g.l.b.'s. One can check that Ω_L is filtered.

Let L be a lower-bounded non empty relational structure. Note that Ω_L is filtered.

Let L be a non empty relational structure and let S be a relational substructure of L. We say that S is filtered-infs-inheriting if and only if:

(Def. 3) For every filtered subset X of S such that $X \neq \emptyset$ and $\inf X$ exists in L holds $\bigsqcap_L X \in$ the carrier of S.

We say that S is directed-sups-inheriting if and only if:

(Def. 4) For every directed subset X of S such that $X \neq \emptyset$ and $\sup X$ exists in L holds $\bigsqcup_L X \in$ the carrier of S.

Let L be a non empty relational structure. Observe that every relational substructure of L which is infs-inheriting is also filtered-infs-inheriting and every relational substructure of L which is sups-inheriting is also directed-sups-inheriting.

Let L be a non empty relational structure. Observe that there exists a relational substructure of L which is infs-inheriting, sups-inheriting, non empty, full, and strict.

We now state two propositions:

(6) Let L be a non empty transitive relational structure, S be a filtered-infs-inheriting non empty full relational substructure of L, and X be a filtered subset of S. Suppose $X \neq \emptyset$ and inf X exists in L. Then inf X exists in S and $\bigsqcap_S X = \bigsqcap_L X$.

(7) Let L be a non empty transitive relational structure, S be a directed-sups-inheriting non empty full relational substructure of L, and X be a directed subset of S. Suppose $X \neq \emptyset$ and sup X exists in L. Then sup X exists in S and $\bigsqcup_S X = \bigsqcup_L X$.

2. Nets

Let L_1, L_2 be non empty 1-sorted structures, let f be a map from L_1 into L_2, and let x be an element of L_1. Then $f(x)$ is an element of L_2.

Let L_1, L_2 be relational structures and let f be a map from L_1 into L_2. We say that f is antitone if and only if:

(Def. 5) For all elements x, y of L_1 such that $x \leq y$ and for all elements a, b of L_2 such that $a = f(x)$ and $b = f(y)$ holds $a \geq b$.

Let L be a 1-sorted structure. We consider net structures over L as extensions of relational structure as systems

⟨ a carrier, a internal relation, a mapping ⟩,

where the carrier is a set, the internal relation is a binary relation on the carrier, and the mapping is a function from the carrier into the carrier of L.

Let L be a 1-sorted structure, let X be a non empty set, let O be a binary relation on X, and let F be a function from X into the carrier of L. Note that $\langle X, O, F \rangle$ is non empty.

Let N be a relational structure. We say that N is directed if and only if:

(Def. 6) Ω_N is directed.

Let L be a 1-sorted structure. Note that there exists a strict net structure over L which is non empty, reflexive, transitive, antisymmetric, and directed.

Let L be a 1-sorted structure. A prenet over L is a directed non empty net structure over L.

Let L be a 1-sorted structure. A net in L is a transitive prenet over L.

Let L be a non empty 1-sorted structure and let N be a non empty net structure over L. The functor netmap(N, L) yields a map from N into L and is defined by:

(Def. 7) netmap(N, L) = the mapping of N.

Let i be an element of the carrier of N. The functor $N(i)$ yielding an element of L is defined by:

(Def. 8) $N(i) = (\text{the mapping of } N)(i)$.

Let L be a non empty relational structure and let N be a non empty net structure over L. We say that N is monotone if and only if:

(Def. 9) netmap(N, L) is monotone.

We say that N is antitone if and only if:

(Def. 10) netmap(N, L) is antitone.

Let L be a non empty 1-sorted structure, let N be a non empty net structure over L, and let X be a set. We say that N is eventually in X if and only if:

(Def. 11) There exists an element i of N such that for every element j of N such that $i \leq j$ holds $N(j) \in X$.

We say that N is often in X if and only if:

(Def. 12) For every element i of N there exists an element j of N such that $i \leq j$ and $N(j) \in X$.

Next we state three propositions:

(8) Let L be a non empty 1-sorted structure, N be a non empty net structure over L, and X, Y be sets such that $X \subseteq Y$. Then

(i) if N is eventually in X, then N is eventually in Y, and

(ii) if N is often in X, then N is often in Y.

(9) Let L be a non empty 1-sorted structure, N be a non empty net structure over L, and X be a set. Then N is eventually in X if and only if N is not often in (the carrier of L) $\setminus$ (X).

(10) Let L be a non empty 1-sorted structure, N be a non empty net structure over L, and X be a set. Then N is often in X if and only if N is not eventually in (the carrier of L) $\setminus$ (X).

Let L be a non empty relational structure and let N be a non empty net structure over L. We say that N is eventually-directed if and only if:

(Def. 13) For every element i of N holds N is eventually in $\{N(j) : j$ ranges over elements of N, $N(i) \leq N(j)\}$.

We say that N is eventually-filtered if and only if:

(Def. 14) For every element i of N holds N is eventually in $\{N(j) : j$ ranges over elements of N, $N(i) \geq N(j)\}$.

One can prove the following propositions:

(11) Let L be a non empty relational structure and N be a non empty net structure over L. Then N is eventually-directed if and only if for every element i of N there exists an element j of N such that for every element k of N such that $j \leq k$ holds $N(i) \leq N(k)$.

(12) Let L be a non empty relational structure and N be a non empty net structure over L. Then N is eventually-filtered if and only if for every element i of N there exists an element j of N such that for every element k of N such that $j \leq k$ holds $N(i) \geq N(k)$.

Let L be a non empty relational structure. Observe that every prenet over L which is monotone is also eventually-directed and every prenet over L which is antitone is also eventually-filtered.

Let L be a non empty reflexive relational structure. Observe that there exists a prenet over L which is monotone, antitone, and strict.

3. Lower and upper subsets

Let L be a relational structure and let X be a subset of the carrier of L. The functor $\downarrow X$ yielding a subset of L is defined by:

(Def. 15) For every element x of L holds $x \in \downarrow X$ iff there exists an element y of L such that $y \geq x$ and $y \in X$.

The functor $\uparrow X$ yielding a subset of L is defined as follows:

(Def. 16) For every element x of L holds $x \in \uparrow X$ iff there exists an element y of L such that $y \leq x$ and $y \in X$.

One can prove the following three propositions:

(13) Let L_1, L_2 be relational structures. Suppose the relational structure of L_1 = the relational structure of L_2. Let X be a subset of the carrier of L_1 and Y be a subset of the carrier of L_2. If $X = Y$, then $\downarrow X = \downarrow Y$ and $\uparrow X = \uparrow Y$.

(14) Let L be a non empty relational structure and X be a subset of L. Then $\downarrow X = \{x : x$ ranges over elements of L, $\bigvee_{y\,:\,\text{element of } L}\ x \leq y\ \wedge\ y \in X\}$.

(15) Let L be a non empty relational structure and X be a subset of L. Then $\uparrow X = \{x : x$ ranges over elements of L, $\bigvee_{y\,:\,\text{element of } L}\ x \geq y\ \wedge\ y \in X\}$.

Let L be a non empty reflexive relational structure and let X be a non empty subset of the carrier of L. Note that $\downarrow X$ is non empty and $\uparrow X$ is non empty.

We now state the proposition

(16) For every reflexive relational structure L and for every subset X of the carrier of L holds $X \subseteq \downarrow X$ and $X \subseteq \uparrow X$.

Let L be a non empty relational structure and let x be an element of the carrier of L. The functor $\downarrow x$ yields a subset of L and is defined by:

(Def. 17) $\downarrow x = \downarrow \{x\}$.

The functor $\uparrow x$ yields a subset of L and is defined by:

(Def. 18) $\uparrow x = \uparrow \{x\}$.

Next we state several propositions:

(17) For every non empty relational structure L and for all elements x, y of L holds $y \in \downarrow x$ iff $y \leq x$.

(18) For every non empty relational structure L and for all elements x, y of L holds $y \in \uparrow x$ iff $x \leq y$.

(19) Let L be a non empty reflexive antisymmetric relational structure and x, y be elements of the carrier of L. If $\downarrow x = \downarrow y$, then $x = y$.

(20) Let L be a non empty reflexive antisymmetric relational structure and x, y be elements of the carrier of L. If $\uparrow x = \uparrow y$, then $x = y$.

(21) For every non empty transitive relational structure L and for all elements x, y of L such that $x \leq y$ holds $\downarrow x \subseteq \downarrow y$.

(22) For every non empty transitive relational structure L and for all elements x, y of L such that $x \leq y$ holds $\uparrow y \subseteq \uparrow x$.

Let L be a non empty reflexive relational structure and let x be an element of the carrier of L. Note that $\downarrow x$ is non empty and directed and $\uparrow x$ is non empty and filtered.

Let L be a relational structure and let X be a subset of L. We say that X is lower if and only if:

(Def. 19) For all elements x, y of L such that $x \in X$ and $y \leq x$ holds $y \in X$.

We say that X is upper if and only if:

(Def. 20) For all elements x, y of L such that $x \in X$ and $x \leq y$ holds $y \in X$.

Let L be a relational structure. One can check that there exists a subset of L which is lower and upper.

Next we state several propositions:

(23) For every relational structure L and for every subset X of L holds X is lower iff $\downarrow X \subseteq X$.

(24) For every relational structure L and for every subset X of L holds X is upper iff $\uparrow X \subseteq X$.

(25) Let L_1, L_2 be relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2. Let X_1 be a subset of L_1 and X_2 be a subset of L_2 such that $X_1 = X_2$. Then
  (i) if X_1 is lower, then X_2 is lower, and
  (ii) if X_1 is upper, then X_2 is upper.

(26) Let L be a relational structure and A be a subset of $2^{\text{the carrier of } L}$. Suppose that for every subset X of L such that $X \in A$ holds X is lower. Then $\bigcup A$ is a lower subset of L.

(27) Let L be a relational structure and X, Y be subsets of L. If X is lower and Y is lower, then $X \cap Y$ is lower and $X \cup Y$ is lower.

(28) Let L be a relational structure and A be a subset of $2^{\text{the carrier of } L}$. Suppose that for every subset X of L such that $X \in A$ holds X is upper. Then $\bigcup A$ is an upper subset of L.

(29) Let L be a relational structure and X, Y be subsets of L. If X is upper and Y is upper, then $X \cap Y$ is upper and $X \cup Y$ is upper.

Let L be a non empty transitive relational structure and let X be a subset of L. One can verify that $\downarrow X$ is lower and $\uparrow X$ is upper.

Let L be a non empty transitive relational structure and let x be an element of L. Observe that $\downarrow x$ is lower and $\uparrow x$ is upper.

Let L be a non empty relational structure. Observe that Ω_L is lower and upper.

Let L be a non empty relational structure. Note that there exists a subset of L which is non empty, lower, and upper.

Let L be a non empty reflexive transitive relational structure. Observe that there exists a subset of L which is non empty, lower, and directed and there exists a subset of L which is non empty, upper, and filtered.

Let L be a poset with g.l.b.'s and l.u.b.'s. One can verify that there exists a subset of L which is non empty, directed, filtered, lower, and upper.

Next we state the proposition

(30) Let L be a non empty transitive reflexive relational structure and X be a subset of L. Then X is directed if and only if $\downarrow X$ is directed.

Let L be a non empty transitive reflexive relational structure and let X be a directed subset of L. Note that $\downarrow X$ is directed.

We now state several propositions:

(31) Let L be a non empty transitive reflexive relational structure, X be a subset of L, and x be an element of L. Then $x \geq X$ if and only if $x \geq \downarrow X$.

(32) Let L be a non empty transitive reflexive relational structure and X be a subset of L. Then sup X exists in L if and only if sup $\downarrow X$ exists in L.

(33) Let L be a non empty transitive reflexive relational structure and X be a subset of L. If sup X exists in L, then $\sup X = \sup\downarrow X$.

(34) For every non empty poset L and for every element x of L holds sup $\downarrow x$ exists in L and $\sup\downarrow x = x$.

(35) Let L be a non empty transitive reflexive relational structure and X be a subset of L. Then X is filtered if and only if $\uparrow X$ is filtered.

Let L be a non empty transitive reflexive relational structure and let X be a filtered subset of L. Note that $\uparrow X$ is filtered.

One can prove the following four propositions:

(36) Let L be a non empty transitive reflexive relational structure, X be a subset of L, and x be an element of L. Then $x \leq X$ if and only if $x \leq \uparrow X$.

(37) Let L be a non empty transitive reflexive relational structure and X be a subset of L. Then inf X exists in L if and only if inf $\uparrow X$ exists in L.

(38) Let L be a non empty transitive reflexive relational structure and X be a subset of L. If inf X exists in L, then $\inf X = \inf\uparrow X$.

(39) For every non empty poset L and for every element x of L holds inf $\uparrow x$ exists in L and $\inf\uparrow x = x$.

4. Ideals and filters

Let L be a non empty reflexive transitive relational structure. An ideal of L is a directed lower non empty subset of L. A filter of L is a filtered upper non empty subset of L.

Next we state several propositions:

(40) Let L be an antisymmetric relational structure with l.u.b.'s and X be a lower subset of L. Then X is directed if and only if for all elements x, y of L such that $x \in X$ and $y \in X$ holds $x \sqcup y \in X$.

(41) Let L be an antisymmetric relational structure with g.l.b.'s and X be an upper subset of L. Then X is filtered if and only if for all elements x, y of L such that $x \in X$ and $y \in X$ holds $x \sqcap y \in X$.

(42) Let L be a poset with l.u.b.'s and X be a non empty lower subset of L. Then X is directed if and only if for every finite subset Y of X such that $Y \neq \emptyset$ holds $\bigsqcup_L Y \in X$.

(43) Let L be a poset with g.l.b.'s and X be a non empty upper subset of L. Then X is filtered if and only if for every finite subset Y of X such that $Y \neq \emptyset$ holds $\bigsqcap_L Y \in X$.

(44) Let L be a non empty antisymmetric relational structure. Suppose L has l.u.b.'s or g.l.b.'s. Let X, Y be subsets of L. Suppose X is lower directed and Y is lower directed. Then $X \cap Y$ is directed.

(45) Let L be a non empty antisymmetric relational structure. Suppose L has l.u.b.'s or g.l.b.'s. Let X, Y be subsets of L. Suppose X is upper filtered and Y is upper filtered. Then $X \cap Y$ is filtered.

(46) Let L be a relational structure and A be a subset of $2^{\text{the carrier of } L}$. Suppose that

(i) for every subset X of L such that $X \in A$ holds X is directed, and

(ii) for all subsets X, Y of L such that $X \in A$ and $Y \in A$ there exists a subset Z of L such that $Z \in A$ and $X \cup Y \subseteq Z$.

Let X be a subset of L. If $X = \bigcup A$, then X is directed.

(47) Let L be a relational structure and A be a subset of $2^{\text{the carrier of } L}$. Suppose that

(i) for every subset X of L such that $X \in A$ holds X is filtered, and

(ii) for all subsets X, Y of L such that $X \in A$ and $Y \in A$ there exists a subset Z of L such that $Z \in A$ and $X \cup Y \subseteq Z$.

Let X be a subset of L. If $X = \bigcup A$, then X is filtered.

Let L be a non empty reflexive transitive relational structure and let I be an ideal of L. We say that I is principal if and only if:

(Def. 21) There exists an element x of L such that $x \in I$ and $x \geq I$.

Let L be a non empty reflexive transitive relational structure and let F be a filter of L. We say that F is principal if and only if:

(Def. 22) There exists an element x of L such that $x \in F$ and $x \leq F$.

Next we state two propositions:

(48) Let L be a non empty reflexive transitive relational structure and I be an ideal of L. Then I is principal if and only if there exists an element x of L such that $I = \downarrow x$.

(49) Let L be a non empty reflexive transitive relational structure and F be a filter of L. Then F is principal if and only if there exists an element x of L such that $F = \uparrow x$.

Let L be a non empty reflexive transitive relational structure. The functor $\mathrm{Ids}(L)$ yields a set and is defined by:

(Def. 23) $\mathrm{Ids}(L) = \{X : X \text{ ranges over ideals of } L\}$.

The functor $\mathrm{Filt}(L)$ yields a set and is defined as follows:

(Def. 24) $\mathrm{Filt}(L) = \{X : X \text{ ranges over filters of } L\}$.

Let L be a non empty reflexive transitive relational structure. The functor $\mathrm{Ids}_0(L)$ yielding a set is defined by:

(Def. 25) $\mathrm{Ids}_0(L) = \mathrm{Ids}(L) \cup \{\emptyset\}$.

The functor $\mathrm{Filt}_0(L)$ yielding a set is defined as follows:

(Def. 26) $\mathrm{Filt}_0(L) = \mathrm{Filt}(L) \cup \{\emptyset\}$.

Let L be a non empty relational structure and let X be a subset of the carrier of L. The functor finsups(X) yielding a subset of L is defined as follows:

(Def. 27) $\mathrm{finsups}(X) = \{\bigsqcup_L Y : Y$ ranges over finite subsets of X, $\sup Y$ exists in $L\}$.

The functor fininfs(X) yielding a subset of L is defined as follows:

(Def. 28) $\mathrm{fininfs}(X) = \{\bigsqcap_L Y : Y$ ranges over finite subsets of X, $\inf Y$ exists in $L\}$.

Let L be a non empty antisymmetric lower-bounded relational structure and let X be a subset of the carrier of L. Note that finsups(X) is non empty.

Let L be a non empty antisymmetric upper-bounded relational structure and let X be a subset of the carrier of L. Note that fininfs(X) is non empty.

Let L be a non empty reflexive antisymmetric relational structure and let X be a non empty subset of the carrier of L. Note that finsups(X) is non empty and fininfs(X) is non empty.

One can prove the following two propositions:

(50) Let L be a non empty reflexive antisymmetric relational structure and X be a subset of the carrier of L. Then $X \subseteq \mathrm{finsups}(X)$ and $X \subseteq \mathrm{fininfs}(X)$.

(51) Let L be a non empty transitive relational structure and X, F be subsets of L. Suppose that
- (i) for every finite subset Y of X such that $Y \neq \emptyset$ holds $\sup Y$ exists in L,
- (ii) for every element x of L such that $x \in F$ there exists a finite subset Y of X such that $\sup Y$ exists in L and $x = \bigsqcup_L Y$, and
- (iii) for every finite subset Y of X such that $Y \neq \emptyset$ holds $\bigsqcup_L Y \in F$.

Then F is directed.

Let L be a poset with l.u.b.'s and let X be a subset of the carrier of L. Note that finsups(X) is directed.

The following propositions are true:

(52) Let L be a non empty transitive reflexive relational structure and X, F be subsets of L. Suppose that
- (i) for every finite subset Y of X such that $Y \neq \emptyset$ holds $\sup Y$ exists in L,
- (ii) for every element x of L such that $x \in F$ there exists a finite subset Y of X such that $\sup Y$ exists in L and $x = \bigsqcup_L Y$, and
- (iii) for every finite subset Y of X such that $Y \neq \emptyset$ holds $\bigsqcup_L Y \in F$.

Let x be an element of L. Then $x \geq X$ if and only if $x \geq F$.

(53) Let L be a non empty transitive reflexive relational structure and X, F be subsets of L. Suppose that

(i) for every finite subset Y of X such that $Y \neq \emptyset$ holds sup Y exists in L,
(ii) for every element x of L such that $x \in F$ there exists a finite subset Y of X such that sup Y exists in L and $x = \bigsqcup_L Y$, and
(iii) for every finite subset Y of X such that $Y \neq \emptyset$ holds $\bigsqcup_L Y \in F$.
Then sup X exists in L if and only if sup F exists in L.

(54) Let L be a non empty transitive reflexive relational structure and X, F be subsets of L. Suppose that
(i) for every finite subset Y of X such that $Y \neq \emptyset$ holds sup Y exists in L,
(ii) for every element x of L such that $x \in F$ there exists a finite subset Y of X such that sup Y exists in L and $x = \bigsqcup_L Y$,
(iii) for every finite subset Y of X such that $Y \neq \emptyset$ holds $\bigsqcup_L Y \in F$, and
(iv) sup X exists in L.
Then $\sup F = \sup X$.

(55) Let L be a poset with l.u.b.'s and X be a subset of L. If sup X exists in L or L is complete, then $\sup X = \sup \mathrm{finsups}(X)$.

(56) Let L be a non empty transitive relational structure and X, F be subsets of L. Suppose that
(i) for every finite subset Y of X such that $Y \neq \emptyset$ holds inf Y exists in L,
(ii) for every element x of L such that $x \in F$ there exists a finite subset Y of X such that inf Y exists in L and $x = \bigsqcap_L Y$, and
(iii) for every finite subset Y of X such that $Y \neq \emptyset$ holds $\bigsqcap_L Y \in F$.
Then F is filtered.

Let L be a poset with g.l.b.'s and let X be a subset of the carrier of L. One can check that fininfs(X) is filtered.

The following propositions are true:

(57) Let L be a non empty transitive reflexive relational structure and X, F be subsets of L. Suppose that
(i) for every finite subset Y of X such that $Y \neq \emptyset$ holds inf Y exists in L,
(ii) for every element x of L such that $x \in F$ there exists a finite subset Y of X such that inf Y exists in L and $x = \bigsqcap_L Y$, and
(iii) for every finite subset Y of X such that $Y \neq \emptyset$ holds $\bigsqcap_L Y \in F$.
Let x be an element of L. Then $x \leq X$ if and only if $x \leq F$.

(58) Let L be a non empty transitive reflexive relational structure and X, F be subsets of L. Suppose that
(i) for every finite subset Y of X such that $Y \neq \emptyset$ holds inf Y exists in L,
(ii) for every element x of L such that $x \in F$ there exists a finite subset Y of X such that inf Y exists in L and $x = \bigsqcap_L Y$, and
(iii) for every finite subset Y of X such that $Y \neq \emptyset$ holds $\bigsqcap_L Y \in F$.
Then inf X exists in L if and only if inf F exists in L.

(59) Let L be a non empty transitive reflexive relational structure and X, F be subsets of L. Suppose that
(i) for every finite subset Y of X such that $Y \neq \emptyset$ holds inf Y exists in L,

(ii) for every element x of L such that $x \in F$ there exists a finite subset Y of X such that inf Y exists in L and $x = \bigsqcap_L Y$,

(iii) for every finite subset Y of X such that $Y \neq \emptyset$ holds $\bigsqcap_L Y \in F$, and

(iv) inf X exists in L.

Then $\inf F = \inf X$.

(60) Let L be a poset with g.l.b.'s and X be a subset of L. If inf X exists in L or L is complete, then $\inf X = \inf \mathrm{fininfs}(X)$.

(61) Let L be a poset with l.u.b.'s and X be a subset of the carrier of L. Then $X \subseteq \downarrow\mathrm{finsups}(X)$ and for every ideal I of L such that $X \subseteq I$ holds $\downarrow\mathrm{finsups}(X) \subseteq I$.

(62) Let L be a poset with g.l.b.'s and X be a subset of the carrier of L. Then $X \subseteq \uparrow\mathrm{fininfs}(X)$ and for every filter F of L such that $X \subseteq F$ holds $\uparrow\mathrm{fininfs}(X) \subseteq F$.

5. Chains

Let L be a non empty relational structure. We say that L is connected if and only if:

(Def. 29) For all elements x, y of L holds $x \leq y$ or $y \leq x$.

Let us observe that every non empty reflexive relational structure which is trivial is also connected.

Let us observe that there exists a non empty poset which is connected.

A chain is a connected non empty poset.

Let L be a chain. Observe that $L^{\smile}$ is connected.

6. Semilattices

A semilattice is a poset with g.l.b.'s. A sup-semilattice is a poset with l.u.b.'s. A lattice is a poset with g.l.b.'s and l.u.b.'s.

The following two propositions are true:

(63) Let L be a semilattice and X be an upper non empty subset of L. Then X is a filter of L if and only if $\mathrm{sub}(X)$ is meet-inheriting.

(64) Let L be a sup-semilattice and X be a lower non empty subset of L. Then X is an ideal of L if and only if $\mathrm{sub}(X)$ is join-inheriting.

7. Maps

Let S, T be non empty relational structures, let f be a map from S into T, and let X be a subset of S. We say that f preserves inf of X if and only if:

(Def. 30) If $\inf X$ exists in S, then $\inf f^{\circ}X$ exists in T and $\inf(f^{\circ}X) = f(\inf X)$.

We say that f preserves sup of X if and only if:

(Def. 31) If $\sup X$ exists in S, then $\sup f^{\circ}X$ exists in T and $\sup(f^{\circ}X) = f(\sup X)$.

We now state the proposition

(65) Let S_1, S_2, T_1, T_2 be non empty relational structures. Suppose that
 (i) the relational structure of S_1 = the relational structure of T_1, and
 (ii) the relational structure of S_2 = the relational structure of T_2.

 Let f be a map from S_1 into S_2 and g be a map from T_1 into T_2. Suppose $f = g$. Let X be a subset of S_1 and Y be a subset of T_1 such that $X = Y$. Then
 (iii) if f preserves sup of X, then g preserves sup of Y, and
 (iv) if f preserves inf of X, then g preserves inf of Y.

Let L_1, L_2 be non empty relational structures and let f be a map from L_1 into L_2. We say that f is infs-preserving if and only if:

(Def. 32) For every subset X of L_1 holds f preserves inf of X.

We say that f is sups-preserving if and only if:

(Def. 33) For every subset X of L_1 holds f preserves sup of X.

We say that f is meet-preserving if and only if:

(Def. 34) For all elements x, y of L_1 holds f preserves inf of $\{x, y\}$.

We say that f is join-preserving if and only if:

(Def. 35) For all elements x, y of L_1 holds f preserves sup of $\{x, y\}$.

We say that f is filtered-infs-preserving if and only if:

(Def. 36) For every subset X of L_1 such that X is non empty filtered holds f preserves inf of X.

We say that f is directed-sups-preserving if and only if:

(Def. 37) For every subset X of L_1 such that X is non empty directed holds f preserves sup of X.

Let L_1, L_2 be non empty relational structures. Note that every map from L_1 into L_2 which is infs-preserving is also filtered-infs-preserving and meet-preserving and every map from L_1 into L_2 which is sups-preserving is also directed-sups-preserving and join-preserving.

Let S, T be relational structures and let f be a map from S into T. We say that f is isomorphic if and only if:

(Def. 38) (i) f is one-to-one monotone and there exists a map g from T into S such that $g = f^{-1}$ and g is monotone if S is non empty and T is non empty,

(ii) S is empty and T is empty, otherwise.

The following proposition is true

(66) Let S, T be non empty relational structures and f be a map from S into T. Then f is isomorphic if and only if the following conditions are satisfied:

(i) f is one-to-one,

(ii) $\mathrm{rng}\, f =$ the carrier of T, and

(iii) for all elements x, y of S holds $x \leq y$ iff $f(x) \leq f(y)$.

Let S, T be non empty relational structures. Note that every map from S into T which is isomorphic is also one-to-one and monotone.

We now state several propositions:

(67) Let S, T be non empty relational structures and f be a map from S into T. Suppose f is isomorphic. Then f^{-1} is a map from T into S and $\mathrm{rng}(f^{-1}) =$ the carrier of S.

(68) Let S, T be non empty relational structures and f be a map from S into T. Suppose f is isomorphic. Let g be a map from T into S. If $g = f^{-1}$, then g is isomorphic.

(69) Let S, T be non empty posets and f be a map from S into T. Suppose that for every filter X of S holds f preserves inf of X. Then f is monotone.

(70) Let S, T be non empty posets and f be a map from S into T. Suppose that for every filter X of S holds f preserves inf of X. Then f is filtered-infs-preserving.

(71) Let S be a semilattice, T be a non empty poset, and f be a map from S into T. Suppose that

(i) for every finite subset X of S holds f preserves inf of X, and

(ii) for every non empty filtered subset X of S holds f preserves inf of X.

Then f is infs-preserving.

(72) Let S, T be non empty posets and f be a map from S into T. Suppose that for every ideal X of S holds f preserves sup of X. Then f is monotone.

(73) Let S, T be non empty posets and f be a map from S into T. Suppose that for every ideal X of S holds f preserves sup of X. Then f is directed-sups-preserving.

(74) Let S be a sup-semilattice, T be a non empty poset, and f be a map from S into T. Suppose that

(i) for every finite subset X of S holds f preserves sup of X, and

(ii) for every non empty directed subset X of S holds f preserves sup of X.

Then f is sups-preserving.

8. Completeness wrt directed sets

Let L be a non empty reflexive relational structure. We say that L is up-complete if and only if the condition (Def. 39) is satisfied.

(Def. 39) Let X be a non empty directed subset of L. Then there exists an element x of L such that $x \geq X$ and for every element y of L such that $y \geq X$ holds $x \leq y$.

One can verify that every reflexive relational structure with l.u.b.'s which is up-complete is also upper-bounded.

The following proposition is true

(75) Let L be a non empty reflexive antisymmetric relational structure. Then L is up-complete if and only if for every non empty directed subset X of L holds sup X exists in L.

Let L be a non empty reflexive relational structure. We say that L is inf-complete if and only if the condition (Def. 40) is satisfied.

(Def. 40) Let X be a non empty subset of L. Then there exists an element x of L such that $x \leq X$ and for every element y of L such that $y \leq X$ holds $x \geq y$.

Next we state the proposition

(76) Let L be a non empty reflexive antisymmetric relational structure. Then L is inf-complete if and only if for every non empty subset X of L holds inf X exists in L.

One can check the following observations:

* every non empty reflexive relational structure which is complete is also up-complete and inf-complete,
* every non empty reflexive relational structure which is inf-complete is also lower-bounded, and
* every non empty poset which is up-complete and lower-bounded and has l.u.b.'s is also complete.

Let us note that every non empty reflexive antisymmetric relational structure which is inf-complete has g.l.b.'s.

Let us note that every non empty reflexive antisymmetric upper-bounded relational structure which is inf-complete has l.u.b.'s.

One can check that there exists a lattice which is complete and strict.

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