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On the Category of Posets

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Summary. In the paper the construction of a category of partially ordered sets is shown: in the second section according to [8] and in the third section according to the definition given in [15]. Some of useful notions such as monotone map and the set of monotone maps between relational structures are given.

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The articles [18], [21], [9], [22], [24], [6], [1], [19], [3], [2], [7], [4], [13], [23], [14], [20], [8], [5], [16], [17], [10], [11], [12], and [15] provide the terminology and notation for this paper.

1. Preliminaries

Let I_1 be a relation structure. We say that I_1 is discrete if and only if:

(Def. 1) The internal relation of $I_1 = \triangle_{\text{the carrier of } I_1}$.

Let us mention that there exists a poset which is strict discrete and non empty and there exists a poset which is strict discrete and empty.

Let X be a set. Then $\triangle_X$ is an order in X.

Observe that $\langle \emptyset, \triangle_\emptyset \rangle$ is empty. Let P be an empty relation structure. One can check that the internal relation of P is empty.

Let us mention that every relation structure which is empty is also discrete.

Let P be a relation structure and let I_1 be a subset of P. We say that I_1 is disconnected if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exist subsets A, B of P such that

(i) $A \neq \emptyset$,

(ii) $B \neq \emptyset$,

(iii) $I_1 = A \cup B$,

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(iv) A misses B, and

(v) the internal relation of P = (the internal relation of P) $|^2$ (A) $\cup$ (the internal relation of P) $|^2$ (B).

We introduce I_1 is connected as an antonym of I_1 is disconnected.

Let I_1 be a non empty relation structure. We say that I_1 is disconnected if and only if:

(Def. 3) $\Omega_{(I_1)}$ is disconnected.

We introduce I_1 is connected as an antonym of I_1 is disconnected.

In the sequel T will denote a non empty relation structure and a will denote an element of T.

One can prove the following propositions:

(1) For every discrete non empty relation structure D_1 and for all elements x, y of D_1 holds $x \leq y$ iff $x = y$.

(2) For every binary relation R and for arbitrary a such that R is an order in $\{a\}$ holds $R = \triangle_{\{a\}}$.

(3) If T is reflexive and $\Omega_T = \{a\}$, then T is discrete.

In the sequel a will be arbitrary.

One can prove the following two propositions:

(4) If $\Omega_T = \{a\}$, then T is connected.

(5) For every discrete non empty poset D_1 such that there exist elements a, b of D_1 such that $a \neq b$ holds D_1 is disconnected.

One can check that there exists a non empty poset which is strict and connected and there exists a non empty poset which is strict disconnected and discrete.

2. On the Category of Posets

Let I_1 be a set. We say that I_1 is poset-membered if and only if:

(Def. 4) For arbitrary a such that $a \in I_1$ holds a is a non empty poset.

One can check that there exists a set which is non empty and poset-membered.

A set of posets is a poset-membered set.

Let P be a non empty set of posets. We see that the element of P is a non empty poset.

Let L_1, L_2 be relation structures and let f be a map from L_1 into L_2. We say that f is monotone if and only if:

(Def. 5) For all elements x, y of L_1 such that $x \leq y$ and for all elements a, b of L_2 such that $a = f(x)$ and $b = f(y)$ holds $a \leq b$.

In the sequel P will denote a non empty set of posets and A, B will denote elements of P.

Let A, B be relation structures. The functor $B^A_{\leq}$ is defined by the condition (Def. 6).

(Def. 6) $a \in B^{A}_{\leq}$ if and only if there exists a map f from A into B such that $a = f$ and $f \in (\text{the carrier of } B)^{\text{the carrier of } A}$ and f is monotone.

The following propositions are true:

(6) For all non empty relation structures A, B, C and for all functions f, g such that $f \in B^{A}_{\leq}$ and $g \in C^{B}_{\leq}$ holds $g \cdot f \in C^{A}_{\leq}$.

(7) $\mathrm{id}_{(\text{the carrier of } T)} \in T^{T}_{\leq}$.

Let us consider T. Observe that $T^{T}_{\leq}$ is non empty.

Let X be a set. The functor $\mathrm{Carr}(X)$ yields a set and is defined by:

(Def. 7) $a \in \mathrm{Carr}(X)$ iff there exists a 1-sorted structure s such that $s \in X$ and $a =$ the carrier of s.

Let us consider P. Observe that $\mathrm{Carr}(P)$ is non empty.

The following propositions are true:

(8) For every 1-sorted structure f holds $\mathrm{Carr}(\{f\}) = \{$the carrier of $f\}$.

(9) For all 1-sorted structures f, g holds $\mathrm{Carr}(\{f, g\}) = \{$the carrier of f, the carrier of $g\}$.

(10) $B^{A}_{\leq} \subseteq \mathrm{Funcs}\,\mathrm{Carr}(P)$.

(11) For all relation structures A, B holds $B^{A}_{\leq} \subseteq (\text{the carrier of } B)^{\text{the carrier of } A}$.

Let A, B be non empty poset. Observe that $B^{A}_{\leq}$ is functional.

Let P be a non empty set of posets. The functor $\mathrm{POSCat}(P)$ yielding a strict category with triple-like morphisms is defined by the conditions (Def. 8).

(Def. 8) (i) The objects of $\mathrm{POSCat}(P) = P$,

(ii) for all elements a, b of P and for every element f of $\mathrm{Funcs}\,\mathrm{Carr}(P)$ such that $f \in b^{a}_{\leq}$ holds $\langle\langle a, b\rangle, f\rangle$ is a morphism of $\mathrm{POSCat}(P)$,

(iii) for every morphism m of $\mathrm{POSCat}(P)$ there exist elements a, b of P and there exists an element f of $\mathrm{Funcs}\,\mathrm{Carr}(P)$ such that $m = \langle\langle a, b\rangle, f\rangle$ and $f \in b^{a}_{\leq}$, and

(iv) for all morphisms m_1, m_2 of $\mathrm{POSCat}(P)$ and for all elements a_1, a_2, a_3 of P and for all elements f_1, f_2 of $\mathrm{Funcs}\,\mathrm{Carr}(P)$ such that $m_1 = \langle\langle a_1, a_2\rangle, f_1\rangle$ and $m_2 = \langle\langle a_2, a_3\rangle, f_2\rangle$ holds $m_2 \cdot m_1 = \langle\langle a_1, a_3\rangle, f_2 \cdot f_1\rangle$.

3. On the Alternative Category of Posets

In this article we present several logical schemes. The scheme *AltCatEx* concerns a non empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a functional set, and states that:

There exists a strict category structure C such that

(i) the carrier of $C = \mathcal{A}$, and

(ii) for all elements i, j of $\mathcal{A}$ holds (the arrows of C)$(i, j) = \mathcal{F}(i, j)$ and for all elements i, j, k of $\mathcal{A}$ holds (the composition of C)$(i, j, k) = \mathrm{FuncComp}(\mathcal{F}(i, j), \mathcal{F}(j, k))$

provided the following condition is met:

- For all elements i, j, k of $\mathcal{A}$ and for all functions f, g such that $f \in \mathcal{F}(i,j)$ and $g \in \mathcal{F}(j,k)$ holds $g \cdot f \in \mathcal{F}(i,k)$.

The scheme *AltCatUniq* deals with a non empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a functional set, and states that:

Let C_1, C_2 be strict category structures. Suppose that

(i) the carrier of $C_1 = \mathcal{A}$,

(ii) for all elements i, j of $\mathcal{A}$ holds (the arrows of $C_1)(i, j) = \mathcal{F}(i,j)$ and for all elements i, j, k of $\mathcal{A}$ holds (the composition of $C_1)(i, j, k) = \mathrm{FuncComp}(\mathcal{F}(i,j), \mathcal{F}(j,k))$,

(iii) the carrier of $C_2 = \mathcal{A}$, and

(iv) for all elements i, j of $\mathcal{A}$ holds (the arrows of $C_2)(i, j) = \mathcal{F}(i,j)$ and for all elements i, j, k of $\mathcal{A}$ holds (the composition of $C_2)(i, j, k) = \mathrm{FuncComp}(\mathcal{F}(i,j), \mathcal{F}(j,k))$.

Then $C_1 = C_2$

for all values of the parameters.

Let P be a non empty set of posets. The functor POSAltCat(P) yielding a strict category structure is defined by the conditions (Def. 9).

(Def. 9) (i) The carrier of POSAltCat(P) = P, and

(ii) for all elements i, j of P holds (the arrows of POSAltCat(P))(i, j) = $j^{i}_{\leq}$ and for all elements i, j, k of P holds (the composition of POSAltCat(P))(i, j, k) = $\mathrm{FuncComp}(j^{i}_{\leq}, k^{j}_{\leq})$.

Let P be a non empty set of posets. One can verify that POSAltCat(P) is transitive and non empty.

Let P be a non empty set of posets. Observe that POSAltCat(P) is associative and has units.

One can prove the following proposition

(12) Let o_1, o_2 be objects of POSAltCat(P) and let A, B be elements of P. If $o_1 = A$ and $o_2 = B$, then $\langle o_1, o_2 \rangle \subseteq$ (the carrier of $B)^{\text{the carrier of } A}$.

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