FORMALIZED MATHEMATICS
Volume 5, Number 4, 1996
Warsaw University - Białystok

Functors for Alternative Categories

Andrzej Trybulec
Warsaw University
Białystok

Summary. An attempt to define the concept of a functor covering both cases (covariant and contravariant) resulted in a structure consisting of two fields: the object map and the morphism map, the first one mapping the Cartesian squares of the set of objects rather than the set of objects. We start with an auxiliary notion of *bifunction*, i.e. a function mapping the Cartesian square of a set A into the Cartesian square of a set B. A *bifunction* f is said to be *covariant* if there is a function g from A into B that f is the Cartesian square of g and f is *contravariant* if there is a function g such that $f(o_1, o_2) = \langle g(o_2), g(o_1) \rangle$. The term *transformation*, another auxiliary notion, might be misleading. It is not related to natural transformations. A transformation from a many sorted set A indexed by I into a many sorted set B indexed by J w.r.t. a function f from I into J is a (many sorted) function from A into $B \cdot f$. Eventually, the morphism map of a functor from C_1 into C_2 is a transformation from the arrows of the category C_1 into the composition of the object map of the functor and the arrows of C_2.

Several kinds of functor structures have been defined: one-to-one, faithful, onto, full and id-preserving. We were pressed to split property that the composition be preserved into two: comp-preserving (for covariant functors) and comp-reversing (for contravariant functors). We defined also some operation on functors, e.g. the composition, the inverse functor. In the last section it is defined what is meant that two categories are isomorphic (anti-isomorphic).

MML Identifier: FUNCTOR0.

The articles [15], [17], [6], [18], [16], [3], [4], [2], [10], [1], [5], [14], [9], [8], [13], [7], [11], and [12] provide the terminology and notation for this paper.

1. Preliminaries

The scheme *ValOnPair* concerns a non empty set $\mathcal{A}$, a function $\mathcal{B}$, elements $\mathcal{C}$, $\mathcal{D}$ of $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding arbitrary, and a binary predicate $\mathcal{P}$, and states that:

ISSN 1426–2630

$\mathcal{B}(\mathcal{C}, \mathcal{D}) = \mathcal{F}(\mathcal{C}, \mathcal{D})$

provided the following conditions are met:

- $\mathcal{B} = \{\langle\langle o, o'\rangle, \mathcal{F}(o, o')\rangle : o$ ranges over elements of $\mathcal{A}$, o' ranges over elements of $\mathcal{A}$, $\mathcal{P}[o, o']\}$,
- $\mathcal{P}[\mathcal{C}, \mathcal{D}]$.

One can prove the following propositions:

(1) For every set A holds $\emptyset$ is a function from A into $\emptyset$.

(2) For every set A and for every function f from A into $\emptyset$ holds $f = \emptyset$.

(3) For every set I and for every many sorted set M indexed by I holds $M \cdot \mathrm{id}_I = M$.

Let f be an empty function. Note that $\curvearrowleft f$ is empty. Let g be a function. One can verify that $[\!: f, g :\!]$ is empty and $[\!: g, f :\!]$ is empty.

The following propositions are true:

(4) For every set A and for every function f holds $f^{\circ}(\mathrm{id}_A) = (\curvearrowleft f)^{\circ}(\mathrm{id}_A)$.

(5) For all sets X, Y and for every function f from X into Y holds f is onto iff $[\!: f, f :\!]$ is onto.

Let I_1 be a set and let f, g be many sorted functions of I_1. Then $g \circ f$ is a many sorted function of I_1.

Let f be a function yielding function. One can verify that $\curvearrowleft f$ is function yielding.

One can prove the following propositions:

(6) For all sets A, B and for arbitrary a holds $\curvearrowleft([\!: A, B :\!] \longmapsto a) = [\!: B, A :\!] \longmapsto a$.

(7) For all functions f, g such that f is one-to-one and g is one-to-one holds $[\!: f, g :\!]^{-1} = [\!: f^{-1}, g^{-1} :\!]$.

(8) For every function f such that $[\!: f, f :\!]$ is one-to-one holds f is one-to-one.

(9) For every function f such that f is one-to-one holds $\curvearrowleft f$ is one-to-one.

(10) For all functions f, g such that $\curvearrowleft[\!: f, g :\!]$ is one-to-one holds $[\!: g, f :\!]$ is one-to-one.

(11) For all functions f, g such that f is one-to-one and g is one-to-one holds $(\curvearrowleft[\!: f, g :\!])^{-1} = \curvearrowleft([\!: g, f :\!]^{-1})$.

(12) For all sets A, B and for every function f from A into B such that f is onto holds $\mathrm{id}_B \subseteq [\!: f, f :\!]^{\circ}(\mathrm{id}_A)$.

(13) For all function yielding functions F, G and for every function f holds $(G \circ F) \cdot f = (G \cdot f) \circ (F \cdot f)$.

Let A, B, C be sets and let f be a function from $[\!: A, B :\!]$ into C. Then $\curvearrowleft f$ is a function from $[\!: B, A :\!]$ into C.

Next we state two propositions:

(14) For all sets A, B, C and for every function f from $[\!: A, B :\!]$ into C such that $\curvearrowleft f$ is onto holds f is onto.

(15) For every set A and for every non empty set B and for every function f from A into B holds $[\!: f, f :\!]^{\circ}(\mathrm{id}_A) \subseteq \mathrm{id}_B$.

2. Functions Between Cartesian Squares

Let A, B be sets.

(Def. 1) A function from $[\!: A, A :\!]$ into $[\!: B, B :\!]$ is called a bifunction from A into B.

Let A, B be sets and let f be a bifunction from A into B. We say that f is precovariant if and only if:

(Def. 2) There exists a function g from A into B such that $f = [\!: g, g :\!]$.

We say that f is precontravariant if and only if:

(Def. 3) There exists a function g from A into B such that $f = \curvearrowleft[\!: g, g :\!]$.

The following proposition is true

(16) Let A be a set, and let B be a non empty set, and let b be an element of B, and let f be a bifunction from A into B. If $f = [\!: A, A :\!] \longmapsto \langle b, b \rangle$, then f is precovariant and precontravariant.

Let A, B be sets. Note that there exists a bifunction from A into B which is precovariant and precontravariant.

Next we state the proposition

(17) Let A, B be non empty sets and let f be a precovariant precontravariant bifunction from A into B. Then there exists an element b of B such that $f = [\!: A, A :\!] \longmapsto \langle b, b \rangle$.

3. Unary Transformations

Let I_1, I_2 be sets, let f be a function from I_1 into I_2, let A be a many sorted set indexed by I_1, and let B be a many sorted set indexed by I_2. A many sorted set indexed by I_1 is called a f-transformation from A to B if:

(Def. 4) (i) There exists a non empty set I_2' and there exists a many sorted set B' indexed by I_2' and there exists a function f' from I_1 into I_2' such that $f = f'$ and $B = B'$ and it is a many sorted function from A into $B' \cdot f'$ if $I_2 \neq \emptyset$,

(ii) it $= \emptyset_{(I_1)}$, otherwise.

Let I_1 be a set, let I_2 be a non empty set, let f be a function from I_1 into I_2, let A be a many sorted set indexed by I_1, and let B be a many sorted set indexed by I_2. Let us note that the f-transformation from A to B can be characterized by the following (equivalent) condition:

(Def. 5) It is a many sorted function from A into $B \cdot f$.

Let I_1, I_2 be sets, let f be a function from I_1 into I_2, let A be a many sorted set indexed by I_1, and let B be a many sorted set indexed by I_2. Note that every f-transformation from A to B is function yielding.

We now state the proposition

(18) Let I_1 be a set, and let I_2, I_3 be non empty sets, and let f be a function from I_1 into I_2, and let g be a function from I_2 into I_3, and let B be a many sorted set indexed by I_2 and let C be a many sorted set indexed by I_3 and let G be a g-transformation from B to C. Then $G \cdot f$ is a $g \cdot f$-transformation from $B \cdot f$ to C.

Let I_1 be a set, let I_2 be a non empty set, let f be a function from I_1 into I_2, let A be a many sorted set indexed by $[\!: I_1, I_1 :\!]$, let B be a many sorted set indexed by $[\!: I_2, I_2 :\!]$, and let F be a $[\!: f, f :\!]$-transformation from A to B. Then $\curvearrowleft F$ is a $[\!: f, f :\!]$-transformation from $\curvearrowleft A$ to $\curvearrowleft B$.

One can prove the following two propositions:

(19) Let I_1, I_2 be non empty sets, and let A be a many sorted set indexed by I_1 and let B be a many sorted set indexed by I_2 and let o be an element of I_2. Suppose $B(o) \neq \emptyset$. Let m be an element of $B(o)$ and let f be a function from I_1 into I_2. Suppose $f = I_1 \longmapsto o$. Then $\{\langle o', A(o') \longmapsto m \rangle : o'$ ranges over elements of $I_1\}$ is a f-transformation from A to B.

(20) Let I_1 be a set, and let I_2, I_3 be non empty sets, and let f be a function from I_1 into I_2, and let g be a function from I_2 into I_3, and let A be a many sorted set indexed by I_1 and let B be a many sorted set indexed by I_2 and let C be a many sorted set indexed by I_3 and let F be a f-transformation from A to B, and let G be a $g \cdot f$-transformation from $B \cdot f$ to C. Suppose that for arbitrary i_1 such that $i_1 \in I_1$ and $(B \cdot f)(i_1) = \emptyset$ holds $A(i_1) = \emptyset$ or $(C \cdot (g \cdot f))(i_1) = \emptyset$. Then $G \circ (F$ **qua** function yielding function) is a $g \cdot f$-transformation from A to C.

4. Functors

Let C_1, C_2 be 1-sorted structures. We introduce bimap structures from C_1 into C_2 which are systems

$\langle$ an object map $\rangle$,

where the object map is a bifunction from the carrier of C_1 into the carrier of C_2.

Let C_1, C_2 be non empty graphs, let F be a bimap structure from C_1 into C_2, and let o be an object of C_1. The functor $F(o)$ yields an object of C_2 and is defined as follows:

(Def. 6) $F(o) = (\text{the object map of } F)(o, o)_{\mathbf{1}}$.

Let A, B be 1-sorted structures and let F be a bimap structure from A into B. We say that F is one-to-one if and only if:

(Def. 7) The object map of F is one-to-one.

We say that F is onto if and only if:

(Def. 8) The object map of F is onto.

We say that F is reflexive if and only if:

(Def. 9) (The object map of F)$^\circ$($\mathrm{id}_{(\text{the carrier of } A)}$) $\subseteq \mathrm{id}_{(\text{the carrier of } B)}$.

We say that F is coreflexive if and only if:

(Def. 10) $\mathrm{id}_{(\text{the carrier of } B)} \subseteq$ (the object map of F)$^\circ$($\mathrm{id}_{(\text{the carrier of } A)}$).

Let A, B be non empty graphs and let F be a bimap structure from A into B. Let us observe that F is reflexive if and only if:

(Def. 11) For every object o of A holds (the object map of F)(o, o) $= \langle F(o), F(o)\rangle$.

We now state the proposition

(21) Let A, B be reflexive non empty graphs and let F be a bimap structure from A into B. Suppose F is coreflexive. Let o be an object of B. Then there exists an object o' of A such that $F(o') = o$.

Let C_1, C_2 be non empty graphs and let F be a bimap structure from C_1 into C_2. We say that F is feasible if and only if:

(Def. 12) For all objects o_1, o_2 of C_1 such that $\langle o_1, o_2\rangle \neq \emptyset$ holds (the arrows of C_2)((the object map of F)(o_1, o_2)) $\neq \emptyset$.

Let C_1, C_2 be graphs. We introduce functor structures from C_1 to C_2 which are extensions of bimap structure from C_1 into C_2 and are systems

⟨ an object map, a morphism map ⟩,

where the object map is a bifunction from the carrier of C_1 into the carrier of C_2 and the morphism map is a the object map-transformation from the arrows of C_1 to the arrows of C_2.

Let C_1, C_2 be 1-sorted structures and let I_4 be a bimap structure from C_1 into C_2. We say that I_4 is precovariant if and only if:

(Def. 13) The object map of I_4 is precovariant.

We say that I_4 is precontravariant if and only if:

(Def. 14) The object map of I_4 is precontravariant.

Let C_1, C_2 be graphs. One can verify that there exists a functor structure from C_1 to C_2 which is precovariant and there exists a functor structure from C_1 to C_2 which is precontravariant.

Let C_1, C_2 be graphs, let F be a functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. The functor $\text{Morph-Map}_F(o_1, o_2)$ is defined as follows:

(Def. 15) $\text{Morph-Map}_F(o_1, o_2) =$ (the morphism map of F)(o_1, o_2).

Let C_1, C_2 be graphs, let F be a functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. Observe that $\text{Morph-Map}_F(o_1, o_2)$ is relation-like and function-like.

Let C_1, C_2 be non empty graphs, let F be a precovariant functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. Then $\text{Morph-Map}_F(o_1, o_2)$ is a function from $\langle o_1, o_2\rangle$ into $\langle F(o_1), F(o_2)\rangle$.

Let C_1, C_2 be non empty graphs, let F be a precovariant functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. Let us assume that $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle F(o_1), F(o_2)\rangle \neq \emptyset$. Let m be a morphism from o_1 to o_2. The functor $F(m)$ yielding a morphism from $F(o_1)$ to $F(o_2)$ is defined as follows:

(Def. 16) $F(m) = (\text{Morph-Map}_F(o_1, o_2))(m)$.

Let C_1, C_2 be non empty graphs, let F be a precontravariant functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. Then $\text{Morph-Map}_F(o_1, o_2)$ is a function from $\langle o_1, o_2\rangle$ into $\langle F(o_2), F(o_1)\rangle$.

Let C_1, C_2 be non empty graphs, let F be a precontravariant functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1. Let us assume that $\langle o_1, o_2\rangle \neq \emptyset$ and $\langle F(o_2), F(o_1)\rangle \neq \emptyset$. Let m be a morphism from o_1 to o_2. The functor $F(m)$ yielding a morphism from $F(o_2)$ to $F(o_1)$ is defined as follows:

(Def. 17) $F(m) = (\text{Morph-Map}_F(o_1, o_2))(m)$.

Let C_1, C_2 be non empty graphs and let o be an object of C_2. Let us assume that $\langle o, o\rangle \neq \emptyset$. Let m be a morphism from o to o. The functor $C_1 \longmapsto m$ yields a strict functor structure from C_1 to C_2 and is defined by the conditions (Def. 18).

(Def. 18) (i) The object map of $C_1 \longmapsto m = [\!:$ the carrier of C_1, the carrier of $C_1 :\!] \longmapsto \langle o, o\rangle$, and

(ii) the morphism map of $C_1 \longmapsto m = \{\langle\langle o_1, o_2\rangle, (\langle o_1, o_2\rangle) \longmapsto m\rangle : o_1$ ranges over objects of C_1, o_2 ranges over objects of $C_1\}$.

We now state the proposition

(22) Let C_1, C_2 be non empty graphs and let o_2 be an object of C_2. Suppose $\langle o_2, o_2\rangle \neq \emptyset$. Let m be a morphism from o_2 to o_2 and let o_1 be an object of C_1. Then $(C_1 \longmapsto m)(o_1) = o_2$.

Let C_1 be a non empty graph, let C_2 be a non empty reflexive graph, let o be an object of C_2, and let m be a morphism from o to o. One can verify that $C_1 \longmapsto m$ is precovariant precontravariant and feasible.

Let C_1 be a non empty graph and let C_2 be a non empty reflexive graph. One can check that there exists a functor structure from C_1 to C_2 which is feasible precovariant and precontravariant.

The following proposition is true

(23) Let C_1, C_2 be non empty graphs, and let F be a precovariant functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1 Then (the object map of F)$(o_1, o_2) = \langle F(o_1), F(o_2)\rangle$.

Let C_1, C_2 be non empty graphs and let F be a precovariant functor structure from C_1 to C_2. Let us observe that F is feasible if and only if:

(Def. 19) For all objects o_1, o_2 of C_1 such that $\langle o_1, o_2\rangle \neq \emptyset$ holds $\langle F(o_1), F(o_2)\rangle \neq \emptyset$.

One can prove the following proposition

(24) Let C_1, C_2 be non empty graphs, and let F be a precontravariant functor structure from C_1 to C_2, and let o_1, o_2 be objects of C_1 Then (the object map of F)$(o_1, o_2) = \langle F(o_2), F(o_1)\rangle$.

Let C_1, C_2 be non empty graphs and let F be a precontravariant functor structure from C_1 to C_2. Let us observe that F is feasible if and only if:

(Def. 20) For all objects o_1, o_2 of C_1 such that $\langle o_1, o_2 \rangle \neq \emptyset$ holds $\langle F(o_2), F(o_1) \rangle \neq \emptyset$.

Let C_1, C_2 be graphs and let F be a functor structure from C_1 to C_2. Observe that the morphism map of F is function yielding.

Let us note that there exists a category structure which is non empty and reflexive.

Let C_1, C_2 be non empty category structures with units and let F be a functor structure from C_1 to C_2. We say that F is id-preserving if and only if:

(Def. 21) For every object o of C_1 holds $(\text{Morph-Map}_F(o, o))(\text{id}_o) = \text{id}_{F(o)}$.

We now state the proposition

(25) Let C_1, C_2 be non empty graphs and let o_2 be an object of C_2. Suppose $\langle o_2, o_2 \rangle \neq \emptyset$. Let m be a morphism from o_2 to o_2, and let o, o' be objects of C_1 and let f be a morphism from o to o'. If $\langle o, o' \rangle \neq \emptyset$, then $(\text{Morph-Map}_{C_1 \longmapsto m}(o, o'))(f) = m$.

One can check that every non empty category structure which has units is reflexive.

Let C_1, C_2 be non empty category structures with units and let o_2 be an object of C_2. Note that $C_1 \longmapsto \text{id}_{(o_2)}$ is id-preserving.

Let C_1 be a non empty graph, let C_2 be a non empty reflexive graph, let o_2 be an object of C_2, and let m be a morphism from o_2 to o_2. Observe that $C_1 \longmapsto m$ is reflexive.

Let C_1 be a non empty graph and let C_2 be a non empty reflexive graph. Observe that there exists a functor structure from C_1 to C_2 which is feasible and reflexive.

Let C_1, C_2 be non empty category structures with units. Note that there exists a functor structure from C_1 to C_2 which is id-preserving feasible reflexive and strict.

Let C_1, C_2 be non empty category structures and let F be a functor structure from C_1 to C_2. We say that F is comp-preserving if and only if the condition (Def. 22) is satisfied.

(Def. 22) Let o_1, o_2, o_3 be objects of C_1 Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and let g be a morphism from o_2 to o_3. Then there exists a morphism f' from $F(o_1)$ to $F(o_2)$ and there exists a morphism g' from $F(o_2)$ to $F(o_3)$ such that $f' = (\text{Morph-Map}_F(o_1, o_2))(f)$ and $g' = (\text{Morph-Map}_F(o_2, o_3))(g)$ and $(\text{Morph-Map}_F(o_1, o_3))(g \cdot f) = g' \cdot f'$.

Let C_1, C_2 be non empty category structures and let F be a functor structure from C_1 to C_2. We say that F is comp-reversing if and only if the condition (Def. 23) is satisfied.

(Def. 23) Let o_1, o_2, o_3 be objects of C_1 Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and let g be a morphism from o_2 to o_3.

Then there exists a morphism f' from $F(o_2)$ to $F(o_1)$ and there exists a morphism g' from $F(o_3)$ to $F(o_2)$ such that $f' = (\text{Morph-Map}_F(o_1, o_2))(f)$ and $g' = (\text{Morph-Map}_F(o_2, o_3))(g)$ and $(\text{Morph-Map}_F(o_1, o_3))(g \cdot f) = f' \cdot g'$.

Let C_1 be a non empty transitive category structure, let C_2 be a non empty reflexive category structure, and let F be a precovariant feasible functor structure from C_1 to C_2. Let us observe that F is comp-preserving if and only if the condition (Def. 24) is satisfied.

(Def. 24) Let o_1, o_2, o_3 be objects of C_1 Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and let g be a morphism from o_2 to o_3. Then $F(g \cdot f) = F(g) \cdot F(f)$.

Let C_1 be a non empty transitive category structure, let C_2 be a non empty reflexive category structure, and let F be a precontravariant feasible functor structure from C_1 to C_2. Let us observe that F is comp-reversing if and only if the condition (Def. 25) is satisfied.

(Def. 25) Let o_1, o_2, o_3 be objects of C_1 Suppose $\langle o_1, o_2 \rangle \neq \emptyset$ and $\langle o_2, o_3 \rangle \neq \emptyset$. Let f be a morphism from o_1 to o_2 and let g be a morphism from o_2 to o_3. Then $F(g \cdot f) = F(f) \cdot F(g)$.

The following two propositions are true:

(26) Let C_1 be a non empty graph, and let C_2 be a non empty reflexive graph, and let o_2 be an object of C_2, and let m be a morphism from o_2 to o_2, and let F be a precovariant feasible functor structure from C_1 to C_2. Suppose $F = C_1 \longmapsto m$. Let o, o' be objects of C_1 and let f be a morphism from o to o'. If $\langle o, o' \rangle \neq \emptyset$, then $F(f) = m$.

(27) Let C_1 be a non empty graph, and let C_2 be a non empty reflexive graph, and let o_2 be an object of C_2, and let m be a morphism from o_2 to o_2, and let o, o' be objects of C_1 and let f be a morphism from o to o'. If $\langle o, o' \rangle \neq \emptyset$, then $(C_1 \longmapsto m)(f) = m$.

Let C_1 be a non empty transitive category structure, let C_2 be a non empty category structure with units, and let o be an object of C_2. Note that $C_1 \longmapsto \mathrm{id}_o$ is comp-preserving and comp-reversing.

Let C_1 be a transitive non empty category structure with units and let C_2 be a non empty category structure with units. A functor structure from C_1 to C_2 is said to be a functor from C_1 to C_2 if:

(Def. 26) It is feasible and id-preserving but it is precovariant and comp-preserving or it is precontravariant and comp-reversing.

Let C_1 be a transitive non empty category structure with units, let C_2 be a non empty category structure with units, and let F be a functor from C_1 to C_2. We say that F is covariant if and only if:

(Def. 27) F is precovariant and comp-preserving.

We say that F is contravariant if and only if:

(Def. 28) F is precontravariant and comp-reversing.

Let A be a category structure and let B be a substructure of A. The functor $\overset{B}{\hookrightarrow}$ yields a strict functor structure from B to A and is defined by the conditions (Def. 29).

(Def. 29) (i) The object map of $\overset{B}{\hookrightarrow} = \mathrm{id}_{[\!:\text{the carrier of } B,\ \text{the carrier of } B\,:\!]}$, and

(ii) the morphism map of $\overset{B}{\hookrightarrow} = \mathrm{id}_{(\text{the arrows of } B)}$.

Let A be a graph. The functor id_A yielding a strict functor structure from A to A is defined by the conditions (Def. 30).

(Def. 30) (i) The object map of $\mathrm{id}_A = \mathrm{id}_{[\!:\text{the carrier of } A,\ \text{the carrier of } A\,:\!]}$, and

(ii) the morphism map of $\mathrm{id}_A = \mathrm{id}_{(\text{the arrows of } A)}$.

Let A be a category structure and let B be a substructure of A. Note that $\overset{B}{\hookrightarrow}$ is precovariant.

One can prove the following propositions:

(28) Let A be a non empty category structure, and let B be a non empty substructure of A, and let o be an object of B. Then $(\overset{B}{\hookrightarrow})(o) = o$.

(29) Let A be a non empty category structure, and let B be a non empty substructure of A, and let o_1, o_2 be objects of B Then $\langle o_1, o_2\rangle \subseteq \langle (\overset{B}{\hookrightarrow})(o_1), (\overset{B}{\hookrightarrow})(o_2)\rangle$.

Let A be a non empty category structure and let B be a non empty substructure of A. Observe that $\overset{B}{\hookrightarrow}$ is feasible.

Let A, B be graphs and let F be a functor structure from A to B. We say that F is faithful if and only if:

(Def. 31) The morphism map of F is "1-1".

Let A, B be graphs and let F be a functor structure from A to B. We say that F is full if and only if the condition (Def. 32) is satisfied.

(Def. 32) There exists a many sorted set B' indexed by $[\!:$the carrier of A, the carrier of $A\,:\!]$ and there exists a many sorted function f from the arrows of A into B' such that $B' =$ (the arrows of B) $\cdot$ (the object map of F) and $f =$ the morphism map of F and f is onto.

Let A be a graph, let B be a non empty graph, and let F be a functor structure from A to B. Let us observe that F is full if and only if the condition (Def. 33) is satisfied.

(Def. 33) There exists a many sorted function f from the arrows of A into (the arrows of B) $\cdot$ (the object map of F) such that $f =$ the morphism map of F and f is onto.

Let A, B be graphs and let F be a functor structure from A to B. We say that F is injective if and only if:

(Def. 34) F is one-to-one and faithful.

We say that F is surjective if and only if:

(Def. 35) F is full and onto.

Let A, B be graphs and let F be a functor structure from A to B. We say that F is bijective if and only if:

(Def. 36) F is injective and surjective.

Let A, B be transitive non empty category structures with units. One can check that there exists a functor from A to B which is strict covariant contravariant and feasible.

The following two propositions are true:

(30) For every non empty graph A and for every object o of A holds $\mathrm{id}_A(o) = o$.

(31) Let A be a non empty graph and let o_1, o_2 be objects of A If $\langle o_1, o_2 \rangle \neq \emptyset$, then for every morphism m from o_1 to o_2 holds $(\text{Morph-Map}_{\mathrm{id}_A}(o_1, o_2))(m) = m$.

Let A be a non empty graph. Note that id_A is feasible and precovariant.

Let A be a non empty graph. Note that there exists a functor structure from A to A which is precovariant and feasible.

One can prove the following proposition

(32) Let A be a non empty graph and let o_1, o_2 be objects of A Suppose $\langle o_1, o_2 \rangle \neq \emptyset$. Let F be a precovariant feasible functor structure from A to A. If $F = \mathrm{id}_A$, then for every morphism m from o_1 to o_2 holds $F(m) = m$.

Let A be a transitive non empty category structure with units. One can check that id_A is id-preserving and comp-preserving.

Let A be a transitive non empty category structure with units. Then id_A is a strict covariant functor from A to A.

Let A be a graph. One can verify that id_A is bijective.

5. The Composition of Functors

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a feasible functor structure from C_1 to C_2, and let G be a functor structure from C_2 to C_3. The functor $G \cdot F$ yielding a strict functor structure from C_1 to C_3 is defined by the conditions (Def. 37).

(Def. 37) (i) The object map of $G \cdot F =$ (the object map of G) $\cdot$ (the object map of F), and

(ii) the morphism map of $G \cdot F =$ ((the morphism map of G) $\cdot$ (the object map of F)) $\circ$ (the morphism map of F).

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a precovariant feasible functor structure from C_1 to C_2, and let G be a precovariant functor structure from C_2 to C_3. Observe that $G \cdot F$ is precovariant.

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a precontravariant feasible functor structure from C_1 to C_2, and let G be a precovariant functor structure from C_2 to C_3. Observe that $G \cdot F$ is precontravariant.

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a precovariant feasible functor structure from C_1 to C_2, and let G be a precontravariant functor structure from C_2 to C_3. Observe that $G \cdot F$ is precontravariant.

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a precontravariant feasible functor structure from C_1 to C_2, and let G be a precontravariant functor structure from C_2 to C_3. Observe that $G \cdot F$ is precovariant.

Let C_1 be a non empty graph, let C_2, C_3 be non empty reflexive graphs, let F be a feasible functor structure from C_1 to C_2, and let G be a feasible functor structure from C_2 to C_3. Note that $G \cdot F$ is feasible.

The following three propositions are true:

(33) Let C_1 be a non empty graph, and let C_2, C_3, C_4 be non empty reflexive graphs, and let F be a feasible functor structure from C_1 to C_2, and let G be a feasible functor structure from C_2 to C_3, and let H be a functor structure from C_3 to C_4. Then $(H \cdot G) \cdot F = H \cdot (G \cdot F)$.

(34) Let C_1 be a non empty category structure, and let C_2, C_3 be non empty reflexive category structures, and let F be a feasible reflexive functor structure from C_1 to C_2, and let G be a functor structure from C_2 to C_3, and let o be an object of C_1. Then $(G \cdot F)(o) = G(F(o))$.

(35) Let C_1 be a non empty graph, and let C_2, C_3 be non empty reflexive graphs, and let F be a feasible reflexive functor structure from C_1 to C_2, and let G be a functor structure from C_2 to C_3, and let o be an object of C_1. Then $\text{Morph-Map}_{G \cdot F}(o, o) = \text{Morph-Map}_G(F(o), F(o)) \cdot \text{Morph-Map}_F(o, o)$.

Let C_1, C_2, C_3 be non empty category structures with units, let F be an id-preserving feasible reflexive functor structure from C_1 to C_2, and let G be an id-preserving functor structure from C_2 to C_3. Note that $G \cdot F$ is id-preserving.

Let A, C be non empty reflexive category structures, let B be a non empty substructure of A, and let F be a functor structure from A to C. The functor $F \restriction B$ yielding a functor structure from B to C is defined as follows:

(Def. 38) $F \restriction B = F \cdot (\overset{B}{\hookrightarrow})$.

6. The Inverse Functor

Let A, B be non empty graphs and let F be a functor structure from A to B. Let us assume that F is bijective. The functor F^{-1} yielding a strict functor structure from B to A is defined by the conditions (Def. 39).

(Def. 39) (i) The object map of F^{-1} = (the object map of $F)^{-1}$, and

(ii) there exists a many sorted function f from the arrows of A into (the arrows of $B) \cdot$ (the object map of F) such that f = the morphism map of F and the morphism map of $F^{-1} = f^{-1} \cdot$ (the object map of $F)^{-1}$.

One can prove the following propositions:

(36) Let A, B be transitive non empty category structures with units and let F be a feasible functor structure from A to B. If F is bijective, then F^{-1} is bijective and feasible.

(37) Let A, B be transitive non empty category structures with units and let F be a feasible reflexive functor structure from A to B If F is bijective and coreflexive, then F^{-1} is reflexive.

(38) Let A, B be transitive non empty category structures with units and let F be a feasible reflexive id-preserving functor structure from A to B If F is bijective and coreflexive, then F^{-1} is id-preserving.

(39) Let A, B be transitive non empty category structures with units and let F be a feasible functor structure from A to B If F is bijective and precovariant, then F^{-1} is precovariant.

(40) Let A, B be transitive non empty category structures with units and let F be a feasible functor structure from A to B If F is bijective and precontravariant, then F^{-1} is precontravariant.

(41) Let A, B be transitive non empty category structures with units and let F be a feasible reflexive functor structure from A to B Suppose F is bijective coreflexive and precovariant. Let o_1, o_2 be objects of B and let m be a morphism from o_1 to o_2. If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(\text{Morph-Map}_F(F^{-1}(o_1), F^{-1}(o_2)))((\text{Morph-Map}_{F^{-1}}(o_1, o_2))(m)) = m$.

(42) Let A, B be transitive non empty category structures with units and let F be a feasible reflexive functor structure from A to B Suppose F is bijective coreflexive and precontravariant. Let o_1, o_2 be objects of B and let m be a morphism from o_1 to o_2. If $\langle o_1, o_2 \rangle \neq \emptyset$, then $(\text{Morph-Map}_F(F^{-1}(o_2), F^{-1}(o_1)))((\text{Morph-Map}_{F^{-1}}(o_1, o_2))(m)) = m$.

(43) Let A, B be transitive non empty category structures with units and let F be a feasible reflexive functor structure from A to B Suppose F is bijective comp-preserving precovariant and coreflexive. Then F^{-1} is comp-preserving.

(44) Let A, B be transitive non empty category structures with units and let F be a feasible reflexive functor structure from A to B Suppose F is bijective comp-reversing precontravariant and coreflexive. Then F^{-1} is comp-reversing.

Let C_1 be a 1-sorted structure and let C_2 be a non empty 1-sorted structure. One can verify that every bimap structure from C_1 into C_2 which is precovariant is also reflexive.

Let C_1 be a 1-sorted structure and let C_2 be a non empty 1-sorted structure. One can verify that every bimap structure from C_1 into C_2 which is precontravariant is also reflexive.

Next we state two propositions:

(45) Let C_1, C_2 be 1-sorted structures and let M be a bimap structure from C_1 into C_2. If M is precovariant and onto, then M is coreflexive.

(46) Let C_1, C_2 be 1-sorted structures and let M be a bimap structure from C_1 into C_2. If M is precontravariant and onto, then M is coreflexive.

Let C_1 be a transitive non empty category structure with units and let C_2 be a non empty category structure with units. Note that every functor from C_1

to C_2 which is covariant is also reflexive.

Let C_1 be a transitive non empty category structure with units and let C_2 be a non empty category structure with units. One can verify that every functor from C_1 to C_2 which is contravariant is also reflexive.

The following propositions are true:

(47) Let C_1 be a transitive non empty category structure with units, and let C_2 be a non empty category structure with units, and let F be a functor from C_1 to C_2. If F is covariant and onto, then F is coreflexive.

(48) Let C_1 be a transitive non empty category structure with units, and let C_2 be a non empty category structure with units, and let F be a functor from C_1 to C_2. If F is contravariant and onto, then F is coreflexive.

(49) Let A, B be transitive non empty category structures with units and let F be a covariant functor from A to B. Suppose F is bijective. Then there exists a functor G from B to A such that $G = F^{-1}$ and G is bijective and covariant.

(50) Let A, B be transitive non empty category structures with units and let F be a contravariant functor from A to B. Suppose F is bijective. Then there exists a functor G from B to A such that $G = F^{-1}$ and G is bijective and contravariant.

Let A, B be transitive non empty category structures with units. We say that A and B are isomorphic if and only if:

(Def. 40) There exists functor from A to B which is bijective and covariant.

Let us observe that this predicate is reflexive and symmetric. We say that A, B are anti-isomorphic if and only if:

(Def. 41) There exists functor from A to B which is bijective and contravariant.

Let us note that the predicate introduced above is symmetric.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. The modification of a function by a function and the iteration of the composition of a function. *Formalized Mathematics*, 1(**3**):521–527, 1990.

[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[7] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Formalized Mathematics*, 5(**1**):61–65, 1996.

[8] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.

[9] Yozo Toda. The formalization of simple graphs. *Formalized Mathematics*, 5(**1**):137–144, 1996.

[10] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[11] Andrzej Trybulec. Categories without uniqueness of **cod** and **dom**. *Formalized Mathematics*, 5(**2**):259–267, 1996.

[12] Andrzej Trybulec. Examples of category structures. *Formalized Mathematics*, 5(**4**):493–500, 1996.

[13] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.

[14] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.

[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[16] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[17] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received April 24, 1996