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On the Many Sorted Closure Operator and the Many Sorted Closure System

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The papers [20], [21], [7], [16], [22], [4], [5], [3], [8], [6], [1], [19], [18], [2], [12], [13], [14], [15], [11], [17], [10], and [9] provide the notation and terminology for this paper.

1. Preliminaries

For simplicity we follow a convention: I is a set, i, x are arbitrary, A, M are many sorted sets indexed by I, f is a function, and F is a many sorted function of I.

The scheme *MSSUBSET* concerns a set $\mathcal{A}$, a non-empty many sorted set $\mathcal{B}$ indexed by $\mathcal{A}$, a many sorted set $\mathcal{C}$ indexed by $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

If for every many sorted set X indexed by $\mathcal{A}$ holds $X \in \mathcal{B}$ iff $X \in \mathcal{C}$ and $\mathcal{P}[X]$, then $\mathcal{B} \subseteq \mathcal{C}$

for all values of the parameters.

The following two propositions are true:

(1) Let X be a non empty set and let x, y be arbitrary. If $x \subseteq y$, then $\{t : t$ ranges over elements of X, $y \subseteq t\} \subseteq \{z : z$ ranges over elements of X, $x \subseteq z\}$.

(2) If there exists A such that $A \in M$, then M is non-empty.

Let us consider I, F, A. Then $F \multimap A$ is a many sorted set indexed by I.

Let us consider I, let A, B be non-empty many sorted sets indexed by I, let F be a many sorted function from A into B, and let X be an element of A. Then $F \multimap X$ is an element of B.

One can prove the following propositions:

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(3) Let A, X be many sorted sets indexed by I, and let B be a non-empty many sorted set indexed by I and let F be a many sorted function from A into B. If $X \in A$, then $F \looparrowleft X \in B$.

(4) Let F, G be many sorted functions of I and let A be a many sorted set indexed by I. If $A \in \mathrm{dom}_\kappa\, G(\kappa)$, then $F \looparrowleft (G \looparrowleft A) = (F \circ G) \looparrowleft A$.

(5) If F is "1-1", then for all many sorted sets A, B indexed by I such that $A \in \mathrm{dom}_\kappa\, F(\kappa)$ and $B \in \mathrm{dom}_\kappa\, F(\kappa)$ and $F \looparrowleft A = F \looparrowleft B$ holds $A = B$.

(6) Suppose $\mathrm{dom}_\kappa\, F(\kappa)$ is non-empty and for all many sorted sets A, B indexed by I such that $A \in \mathrm{dom}_\kappa\, F(\kappa)$ and $B \in \mathrm{dom}_\kappa\, F(\kappa)$ and $F \looparrowleft A = F \looparrowleft B$ holds $A = B$. Then F is "1-1".

(7) Let A, B be non-empty many sorted sets indexed by I and let F, G be many sorted functions from A into B. If for every M such that $M \in A$ holds $F \looparrowleft M = G \looparrowleft M$, then $F = G$.

Let us consider I, M. One can verify that there exists an element of 2^M which is empty yielding and locally-finite.

2. Properties of Many Sorted Closure Operators

Let us consider I, M.

(Def. 1) A many sorted function from 2^M into 2^M is called a set many sorted operation in M.

Let us consider I, M, let O be a set many sorted operation in M, and let X be an element of 2^M. Then $O \looparrowleft X$ is an element of 2^M.

Let us consider I, M and let I_1 be a set many sorted operation in M. We say that I_1 is reflexive if and only if:

(Def. 2) For every element X of 2^M holds $X \subseteq I_1 \looparrowleft X$.

We say that I_1 is monotonic if and only if:

(Def. 3) For all elements X, Y of 2^M such that $X \subseteq Y$ holds $I_1 \looparrowleft X \subseteq I_1 \looparrowleft Y$.

We say that I_1 is idempotent if and only if:

(Def. 4) For every element X of 2^M holds $I_1 \looparrowleft X = I_1 \looparrowleft (I_1 \looparrowleft X)$.

We say that I_1 is topological if and only if:

(Def. 5) For all elements X, Y of 2^M holds $I_1 \looparrowleft (X \cup Y) = I_1 \looparrowleft X \cup I_1 \looparrowleft Y$.

One can prove the following propositions:

(8) For every non-empty many sorted set M indexed by I and for every element X of M holds $X = \mathrm{id}_M \looparrowleft X$.

(9) Let M be a non-empty many sorted set indexed by I and let X, Y be elements of M. If $X \subseteq Y$, then $\mathrm{id}_M \looparrowleft X \subseteq \mathrm{id}_M \looparrowleft Y$.

(10) Let M be a non-empty many sorted set indexed by I and let X, Y be elements of M. If $X \cup Y$ is an element of M, then $\mathrm{id}_M \looparrowleft (X \cup Y) = \mathrm{id}_M \looparrowleft X \cup \mathrm{id}_M \looparrowleft Y$.

(11) Let X be an element of 2^M and let i, x be arbitrary. Suppose $i \in I$ and $x \in (\mathrm{id}_{2^M} \looparrowleft X)(i)$. Then there exists a locally-finite element Y of 2^M such that $Y \subseteq X$ and $x \in (\mathrm{id}_{2^M} \looparrowleft Y)(i)$.

Let us consider I, M. Note that there exists a set many sorted operation in M which is reflexive monotonic idempotent and topological.

Next we state four propositions:

(12) id_{2^A} is a reflexive set many sorted operation in A.

(13) id_{2^A} is a monotonic set many sorted operation in A.

(14) id_{2^A} is an idempotent set many sorted operation in A.

(15) id_{2^A} is a topological set many sorted operation in A.

In the sequel P, R will denote set many sorted operations in M and E, T will denote elements of 2^M.

One can prove the following three propositions:

(16) If $E = M$ and P is reflexive, then $E = P \looparrowleft E$.

(17) If P is reflexive and for every element X of 2^M holds $P \looparrowleft X \subseteq X$, then P is idempotent.

(18) If P is monotonic, then $P \looparrowleft (E \cap T) \subseteq P \looparrowleft E \cap P \looparrowleft T$.

Let us consider I, M. Observe that every set many sorted operation in M which is topological is also monotonic.

One can prove the following proposition

(19) If P is topological, then $P \looparrowleft E \setminus P \looparrowleft T \subseteq P \looparrowleft (E \setminus T)$.

Let us consider I, M, R, P. Then $P \circ R$ is a set many sorted operation in M.

One can prove the following propositions:

(20) If P is reflexive and R is reflexive, then $P \circ R$ is reflexive.

(21) If P is monotonic and R is monotonic, then $P \circ R$ is monotonic.

(22) If P is idempotent and R is idempotent and $P \circ R = R \circ P$, then $P \circ R$ is idempotent.

(23) If P is topological and R is topological, then $P \circ R$ is topological.

(24) If P is reflexive and $i \in I$ and $f = P(i)$, then for every element x of $2^{M(i)}$ holds $x \subseteq f(x)$.

(25) If P is monotonic and $i \in I$ and $f = P(i)$, then for all elements x, y of $2^{M(i)}$ such that $x \subseteq y$ holds $f(x) \subseteq f(y)$.

(26) If P is idempotent and $i \in I$ and $f = P(i)$, then for every element x of $2^{M(i)}$ holds $f(x) = f(f(x))$.

(27) If P is topological and $i \in I$ and $f = P(i)$, then for all elements x, y of $2^{M(i)}$ holds $f(x \cup y) = f(x) \cup f(y)$.

3. On the Many Sorted Closure Operator and the Many Sorted Closure System

In the sequel S will be a 1-sorted structure.

Let us consider S. We consider many sorted closure system structures over S as extensions of many-sorted structure over S as systems

⟨ sorts, a family ⟩,

where the sorts constitute a many sorted set indexed by the carrier of S and the family is a subset family of the sorts.

In the sequel M_1 will be a many-sorted structure over S.

Let us consider S and let I_1 be a many sorted closure system structure over S. We say that I_1 is additive if and only if:

(Def. 6) The family of I_1 is additive.

We say that I_1 is absolutely-additive if and only if:

(Def. 7) The family of I_1 is absolutely-additive.

We say that I_1 is multiplicative if and only if:

(Def. 8) The family of I_1 is multiplicative.

We say that I_1 is absolutely-multiplicative if and only if:

(Def. 9) The family of I_1 is absolutely-multiplicative.

We say that I_1 is properly upper bound if and only if:

(Def. 10) The family of I_1 is properly upper bound.

We say that I_1 is properly lower bound if and only if:

(Def. 11) The family of I_1 is properly lower bound.

Let us consider S, M_1. The functor $\mathrm{MSFull}(M_1)$ yields a many sorted closure system structure over S and is defined as follows:

(Def. 12) $\mathrm{MSFull}(M_1) = \langle \text{the sorts of } M_1, 2^{\text{the sorts of } M_1} \rangle$.

Let us consider S, M_1. One can check that $\mathrm{MSFull}(M_1)$ is strict additive absolutely-additive multiplicative absolutely-multiplicative properly upper bound and properly lower bound.

Let us consider S and let M_1 be a non-empty many-sorted structure over S. One can check that $\mathrm{MSFull}(M_1)$ is non-empty.

Let us consider S. Observe that there exists a many sorted closure system structure over S which is strict non-empty additive absolutely-additive multiplicative absolutely-multiplicative properly upper bound and properly lower bound.

Let us consider S and let C_1 be an additive many sorted closure system structure over S. Note that the family of C_1 is additive.

Let us consider S and let C_1 be an absolutely-additive many sorted closure system structure over S. Observe that the family of C_1 is absolutely-additive.

Let us consider S and let C_1 be a multiplicative many sorted closure system structure over S. One can verify that the family of C_1 is multiplicative.

Let us consider S and let C_1 be an absolutely-multiplicative many sorted closure system structure over S. One can check that the family of C_1 is absolutely-multiplicative.

Let us consider S and let C_1 be a properly upper bound many sorted closure system structure over S. One can check that the family of C_1 is properly upper bound.

Let us consider S and let C_1 be a properly lower bound many sorted closure system structure over S. Note that the family of C_1 is properly lower bound.

Let us consider S, let M be a non-empty many sorted set indexed by the carrier of S, and let F be a subset family of M. Observe that $\langle M, F\rangle$ is non-empty.

Let us consider S, M_1 and let F be an additive subset family of the sorts of M_1. Observe that $\langle$the sorts of M_1, $F\rangle$ is additive.

Let us consider S, M_1 and let F be an absolutely-additive subset family of the sorts of M_1. One can check that $\langle$the sorts of M_1, $F\rangle$ is absolutely-additive.

Let us consider S, M_1 and let F be a multiplicative subset family of the sorts of M_1. Note that $\langle$the sorts of M_1, $F\rangle$ is multiplicative.

Let us consider S, M_1 and let F be an absolutely-multiplicative subset family of the sorts of M_1. Observe that $\langle$the sorts of M_1, $F\rangle$ is absolutely-multiplicative.

Let us consider S, M_1 and let F be a properly upper bound subset family of the sorts of M_1. One can verify that $\langle$the sorts of M_1, $F\rangle$ is properly upper bound.

Let us consider S, M_1 and let F be a properly lower bound subset family of the sorts of M_1. Observe that $\langle$the sorts of M_1, $F\rangle$ is properly lower bound.

Let us consider S. Observe that every many sorted closure system structure over S which is absolutely-additive is also additive.

Let us consider S. One can check that every many sorted closure system structure over S which is absolutely-multiplicative is also multiplicative.

Let us consider S. Observe that every many sorted closure system structure over S which is absolutely-multiplicative is also properly upper bound.

Let us consider S. One can verify that every many sorted closure system structure over S which is absolutely-additive is also properly lower bound.

Let us consider S. A many sorted closure system of S is an absolutely-multiplicative many sorted closure system structure over S.

Let us consider I, M. A many sorted closure operator of M is a reflexive monotonic idempotent set many sorted operation in M.

Let us consider I, M and let F be a many sorted function from M into M. The functor FixPoints(F) yielding a many sorted subset of M is defined by:

(Def. 13) For every i such that $i \in I$ holds $x \in (\mathrm{FixPoints}(F))(i)$ iff there exists a function f such that $f = F(i)$ and $x \in \mathrm{dom}\, f$ and $f(x) = x$.

Let us consider I, let M be an empty yielding many sorted set indexed by I, and let F be a many sorted function from M into M. One can verify that FixPoints(F) is empty yielding.

Next we state a number of propositions:

(28) For every many sorted function F from M into M holds $A \in M$ and $F \looparrowleft A = A$ iff $A \in \mathrm{FixPoints}(F)$.

(29) $\mathrm{FixPoints}(\mathrm{id}_A) = A$.

(30) Let A be a many sorted set indexed by the carrier of S, and let J be a reflexive monotonic set many sorted operation in A, and let D be a subset family of A. If $D = \mathrm{FixPoints}(J)$, then $\langle A, D\rangle$ is a many sorted closure system of S.

(31) Let D be a properly upper bound subset family of M and let X be an element of 2^M. Then there exists a non-empty subset family S_1 of M such that for every many sorted set Y indexed by I holds $Y \in S_1$ if and only if the following conditions are satisfied:
  (i) $Y \in D$, and
  (ii) $X \subseteq Y$.

(32) Let D be a properly upper bound subset family of M, and let X be an element of 2^M, and let S_1 be a non-empty subset family of M. Suppose that for every many sorted set Y indexed by I holds $Y \in S_1$ iff $Y \in D$ and $X \subseteq Y$. Let i be arbitrary and let D_1 be a non empty set. If $i \in I$ and $D_1 = D(i)$, then $S_1(i) = \{z : z$ ranges over elements of D_1, $X(i) \subseteq z\}$.

(33) Let D be a properly upper bound subset family of M. Then there exists a set many sorted operation J in M such that for every element X of 2^M and for every non-empty subset family S_1 of M if for every many sorted set Y indexed by I holds $Y \in S_1$ iff $Y \in D$ and $X \subseteq Y$, then $J \looparrowleft X = \bigcap S_1$.

(34) Let D be a properly upper bound subset family of M, and let A be an element of 2^M, and let J be a set many sorted operation in M. Suppose that
  (i) $A \in D$, and
  (ii) for every element X of 2^M and for every non-empty subset family S_1 of M such that for every many sorted set Y indexed by I holds $Y \in S_1$ iff $Y \in D$ and $X \subseteq Y$ holds $J \looparrowleft X = \bigcap S_1$.

  Then $J \looparrowleft A = A$.

(35) Let D be an absolutely-multiplicative subset family of M, and let A be an element of 2^M, and let J be a set many sorted operation in M. Suppose that
  (i) $J \looparrowleft A = A$, and
  (ii) for every element X of 2^M and for every non-empty subset family S_1 of M such that for every many sorted set Y indexed by I holds $Y \in S_1$ iff $Y \in D$ and $X \subseteq Y$ holds $J \looparrowleft X = \bigcap S_1$.

  Then $A \in D$.

(36) Let D be a properly upper bound subset family of M and let J be a set many sorted operation in M. Suppose that for every element X of 2^M and for every non-empty subset family S_1 of M such that for every many sorted set Y indexed by I holds $Y \in S_1$ iff $Y \in D$ and $X \subseteq Y$ holds

$J \leftrightarrow X = \bigcap S_1$. Then J is reflexive and monotonic.

(37) Let D be an absolutely-multiplicative subset family of M and let J be a set many sorted operation in M. Suppose that for every element X of 2^M and for every non-empty subset family S_1 of M such that for every many sorted set Y indexed by I holds $Y \in S_1$ iff $Y \in D$ and $X \subseteq Y$ holds $J \leftrightarrow X = \bigcap S_1$. Then J is idempotent.

(38) Let D be a many sorted closure system of S and let J be a set many sorted operation in the sorts of D. Suppose that for every element X of $2^{\text{the sorts of } D}$ and for every non-empty subset family S_1 of the sorts of D such that for every many sorted set Y indexed by the carrier of S holds $Y \in S_1$ iff $Y \in$ the family of D and $X \subseteq Y$ holds $J \leftrightarrow X = \bigcap S_1$. Then J is a many sorted closure operator of the sorts of D.

Let us consider S, let A be a many sorted set indexed by the carrier of S, and let C be a many sorted closure operator of A. The functor ClSys(C) yielding a many sorted closure system of S is defined as follows:

(Def. 14) There exists a subset family D of A such that $D = \text{FixPoints}(C)$ and $\text{ClSys}(C) = \langle A, D \rangle$.

Let us consider S, let A be a many sorted set indexed by the carrier of S, and let C be a many sorted closure operator of A. One can verify that ClSys(C) is strict.

Let us consider S, let A be a non-empty many sorted set indexed by the carrier of S, and let C be a many sorted closure operator of A. Note that ClSys(C) is non-empty.

Let us consider S and let D be a many sorted closure system of S. The functor ClOp(D) yielding a many sorted closure operator of the sorts of D is defined by the condition (Def. 15).

(Def. 15) Let X be an element of $2^{\text{the sorts of } D}$ and let S_1 be a non-empty subset family of the sorts of D. Suppose that for every many sorted set Y indexed by the carrier of S holds $Y \in S_1$ iff $Y \in$ the family of D and $X \subseteq Y$. Then $(\text{ClOp}(D)) \leftrightarrow X = \bigcap S_1$.

The following two propositions are true:

(39) Let A be a many sorted set indexed by the carrier of S and let J be a many sorted closure operator of A. Then $\text{ClOp}(\text{ClSys}(J)) = J$.

(40) For every many sorted closure system D of S holds $\text{ClSys}(\text{ClOp}(D)) =$ the many sorted closure system structure of D.

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