FORMALIZED MATHEMATICS
Volume 5, Number 2, 1996
Warsaw University - Białystok

Preliminaries to Circuits, I [1]

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Summary. This article is the first in a series of four articles (continued in [24,23,22]) about modelling circuits by many-sorted algebras.

Here, we introduce some auxiliary notations and prove auxiliary facts about many sorted sets, many sorted functions and trees.

MML Identifier: PRE_CIRC.

The articles [29], [33], [18], [4], [30], [1], [34], [13], [17], [31], [28], [14], [25], [16], [15], [8], [5], [7], [9], [6], [3], [2], [27], [19], [20], [26], [21], [11], [10], [12], and [32] provide the terminology and notation for this paper.

1. Varia

One can prove the following proposition

(1) For all sets X, Y holds $X \setminus Y$ misses Y.

In this article we present several logical schemes. The scheme *Fraenkel Subset* deals with non empty sets $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(x) : x$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[x]\}$ is a subset of $\mathcal{B}$

for all values of the parameters.

The scheme *FraenkelFinIm* concerns a finite non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding arbitrary, and a unary predicate $\mathcal{P}$, and states that:

[1]This work was initiated while the second author visited Nagano (March-May 1994) and then continued when the third author visited Edmonton (May-June 1994). The work was finalized when the fourth author visited Bialystok (October-November 1994). Partial funding for this work has been provided by: Shinshu Endowment Fund for Information Science, NSERC Grant OGP9207, JSTF award 651-93-S009.

ISSN 1426–2630

$\{\mathcal{F}(x) : x$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[x]\}$ is finite
for all values of the parameters.

The following three propositions are true:

(2) For every function f and for arbitrary x, y such that $\operatorname{dom} f = \{x\}$ and $\operatorname{rng} f = \{y\}$ holds $f = \{\langle x, y\rangle\}$.

(3) For all functions f, g, h such that $f \subseteq g$ holds $f +\cdot h \subseteq g +\cdot h$.

(4) For all functions f, g, h such that $f \subseteq g$ and $\operatorname{dom} f$ misses $\operatorname{dom} h$ holds $f \subseteq g +\cdot h$.

Let X be a finite non empty subset of $\mathbb{R}$. The functor $\max X$ yields a real number and is defined as follows:

(Def.1) $\max X \in X$ and for every real number k such that $k \in X$ holds $k \leq \max X$.

Let X be a finite non empty subset of $\mathbb{N}$. The functor $\max X$ yielding a natural number is defined by:

(Def.2) There exists a finite non empty subset Y of $\mathbb{R}$ such that $Y = X$ and $\max X = \max Y$.

2. Many Sorted Sets and Functions

One can prove the following proposition

(5) For every set I and for every many sorted set M_1 indexed by I holds ${M_1}^{\#}(\varepsilon_I) = \{\varepsilon\}$.

The scheme *MSSLambda2Part* deals with a set $\mathcal{A}$, two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding arbitrary, and a unary predicate $\mathcal{P}$, and states that:

There exists a many sorted set f indexed by $\mathcal{A}$ such that for every element i of $\mathcal{A}$ holds if $i \in \mathcal{A}$, then if $\mathcal{P}[i]$, then $f(i) = \mathcal{F}(i)$ and if not $\mathcal{P}[i]$, then $f(i) = \mathcal{G}(i)$

for all values of the parameters.

Let I be a set. A many sorted set indexed by I is locally-finite if:

(Def.3) For arbitrary i such that $i \in I$ holds $\mathrm{it}(i)$ is finite.

Let I be a set. Observe that there exists a many sorted set indexed by I which is non-empty and locally-finite.

Let I, A be sets. Then $I \longmapsto A$ is a many sorted set indexed by I.

Let I be a set, let M be a many sorted set indexed by I, and let A be a subset of I. Then $M \restriction A$ is a many sorted set indexed by A.

Let M be a non-empty function and let A be a set. One can check that $M \restriction A$ is non-empty.

One can prove the following three propositions:

(6) For every non empty set I and for every non-empty many sorted set B indexed by I holds $\bigcup \operatorname{rng} B$ is non empty.

(7) For every set I holds $\operatorname{uncurry}(I \longmapsto \emptyset) = \emptyset$.

(8) Let I be a non empty set, and let A be a set, and let B be a non-empty many sorted set indexed by I, and let F be a many sorted function from $I \longmapsto A$ into B. Then $\operatorname{dom}\operatorname{commute}(F) = A$.

Now we present two schemes. The scheme *LambdaRecCorrD* concerns a non empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

(i) There exists a function f from $\mathbb{N}$ into $\mathcal{A}$ such that $f(0) = \mathcal{B}$ and for every natural number i and for every element x of $\mathcal{A}$ such that $x = f(i)$ holds $f(i+1) = \mathcal{F}(i, x)$, and

(ii) for all functions f_1, f_2 from $\mathbb{N}$ into $\mathcal{A}$ such that $f_1(0) = \mathcal{B}$ and for every natural number i and for every element x of $\mathcal{A}$ such that $x = f_1(i)$ holds $f_1(i+1) = \mathcal{F}(i, x)$ and $f_2(0) = \mathcal{B}$ and for every natural number i and for every element x of $\mathcal{A}$ such that $x = f_2(i)$ holds $f_2(i+1) = \mathcal{F}(i, x)$ holds $f_1 = f_2$

for all values of the parameters.

The scheme *LambdaMSFD* concerns a non empty set $\mathcal{A}$, a subset $\mathcal{B}$ of $\mathcal{A}$, many sorted sets $\mathcal{C}$, $\mathcal{D}$ indexed by $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding arbitrary, and states that:

There exists a many sorted function f from $\mathcal{C}$ into $\mathcal{D}$ such that for every element i of $\mathcal{A}$ such that $i \in \mathcal{B}$ holds $f(i) = \mathcal{F}(i)$

provided the following requirement is met:

- For every element i of $\mathcal{A}$ such that $i \in \mathcal{B}$ holds $\mathcal{F}(i)$ is a function from $\mathcal{C}(i)$ into $\mathcal{D}(i)$.

Let F be a non-empty function and let f be a function. Observe that $F \cdot f$ is non-empty.

Let I be a set and let M_1 be a non-empty many sorted set indexed by I. Note that every element of $\prod M_1$ is function-like and relation-like.

One can prove the following propositions:

(9) Let I be a set, and let f be a non-empty many sorted set indexed by I, and let g be a function, and let s be an element of $\prod f$. Suppose $\operatorname{dom} g \subseteq \operatorname{dom} f$ and for arbitrary x such that $x \in \operatorname{dom} g$ holds $g(x) \in f(x)$. Then $s +\cdot g$ is an element of $\prod f$.

(10) Let A, B be non empty sets, and let C be a non-empty many sorted set indexed by A, and let I_1 be a many sorted function from $A \longmapsto B$ into C, and let b be an element of B. Then there exists a many sorted set c indexed by A such that $c = (\operatorname{commute}(I_1))(b)$ and $c \in C$.

(11) Let I be a set, and let M be a many sorted set indexed by I, and let x, g be functions. If $x \in \prod M$, then $x \cdot g \in \prod (M \cdot g)$.

(12) For every natural number n and for arbitrary a holds $\prod (n \mapsto \{a\}) = \{n \mapsto a\}$.

3. Trees

We follow the rules: T, T_1 will denote finite trees, t, p will denote elements of T, and t_1 will denote an element of T_1.

Let D be a non empty set. Note that every element of FinTrees(D) is finite.

Let T be a finite decorated tree and let t be an element of dom T. Observe that $T \upharpoonright t$ is finite.

We now state the proposition

(13) $T \upharpoonright p \approx \{t : p \preceq t\}$.

Let T be a finite decorated tree, let t be an element of dom T, and let T_1 be a finite decorated tree. Note that $T(t/T_1)$ is finite.

Next we state a number of propositions:

(14) $T(p/T_1) = \{t : p \not\preceq t\} \cup \{p \mathbin{^\frown} t_1\}$.

(15) For every finite sequence f of elements of $\mathbb{N}$ such that $f \in T(p/T_1)$ and $p \preceq f$ there exists t_1 such that $f = p \mathbin{^\frown} t_1$.

(16) For every tree yielding finite sequence p and for every natural number k such that $k+1 \in \operatorname{dom} p$ holds $\overbrace{p} \upharpoonright \langle k \rangle = p(k+1)$.

(17) Let q be a decorated tree yielding finite sequence and let k be a natural number. If $k+1 \in \operatorname{dom} q$, then $\langle k \rangle \in \overbrace{\operatorname*{dom}_{\kappa} q(\kappa)}$.

(18) Let p, q be tree yielding finite sequences and let k be a natural number. Suppose $\operatorname{len} p = \operatorname{len} q$ and $k+1 \in \operatorname{dom} p$ and for every natural number i such that $i \in \operatorname{dom} p$ and $i \neq k+1$ holds $p(i) = q(i)$. Let t be a tree. If $q(k+1) = t$, then $\overbrace{q} = \overbrace{p}(\langle k \rangle / t)$.

(19) Let e_1, e_2 be finite decorated trees, and let x be arbitrary, and let k be a natural number, and let p be a decorated tree yielding finite sequence. Suppose $\langle k \rangle \in \operatorname{dom} e_1$ and $e_1 = x\text{-tree}(p)$. Then there exists a decorated tree yielding finite sequence q such that $e_1(\langle k \rangle / e_2) = x\text{-tree}(q)$ and $\operatorname{len} q = \operatorname{len} p$ and $q(k+1) = e_2$ and for every natural number i such that $i \in \operatorname{dom} p$ and $i \neq k+1$ holds $q(i) = p(i)$.

(20) For every finite tree T and for every element p of T such that $p \neq \varepsilon$ holds $\operatorname{card}(T \upharpoonright p) < \operatorname{card} T$.

(21) For every finite function f holds $\operatorname{card} f = \operatorname{card} \operatorname{dom} f$.

(22) For all finite trees T, T_1 and for every element p of T holds $\operatorname{card}(T(p/T_1)) + \operatorname{card}(T \upharpoonright p) = \operatorname{card} T + \operatorname{card} T_1$.

(23) For all finite decorated trees T, T_1 and for every element p of dom T holds $\operatorname{card}(T(p/T_1)) + \operatorname{card}(T \upharpoonright p) = \operatorname{card} T + \operatorname{card} T_1$.

Let x be arbitrary. One can check that the root tree of x is finite.

We now state the proposition

(24) For arbitrary x holds card (the root tree of x) $= 1$.

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Received November 17, 1994