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The Correspondence Between Homomorphisms of Universal Algebra & Many Sorted Algebra

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Summary. The aim of the article is to check the compatibility of the homomorphism of universal algebras introduced in [13] and the corresponding concept for many sorted algebras introduced in [14].

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The articles [22], [25], [26], [28], [8], [9], [11], [21], [23], [3], [12], [10], [1], [19], [6], [27], [18], [15], [2], [5], [4], [16], [7], [24], [13], [14], [17], and [20] provide the notation and terminology for this paper.

For simplicity we follow the rules: U_1, U_2, U_3 denote universal algebras, n denotes a natural number, A denotes a non empty set, and h denotes a function from U_1 into U_2.

The following propositions are true:

(1) For all functions f, g and for every set C such that $\operatorname{rng} f \subseteq C$ holds $(g \restriction C) \cdot f = g \cdot f$.

(2) For every set I and for every subset C of I holds $C^* \subseteq I^*$.

(3) For every function f and for every set C such that f is function yielding holds $f \restriction C$ is function yielding.

(4) For every set I and for every subset C of I and for every many sorted set M indexed by I holds $(M \restriction C)^{\#} = M^{\#} \restriction C^*$.

Let us consider A, n and let a be an element of A. Then $n \mapsto a$ is a finite sequence of elements of A.

Let S, S' be non empty many sorted signatures. The predicate $S \leq S'$ is defined by the conditions (Def.1).

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(Def.1) (i) The carrier of $S \subseteq$ the carrier of S',

(ii) the operation symbols of $S \subseteq$ the operation symbols of S',

(iii) (the arity of S') $\restriction$ (the operation symbols of S) = the arity of S, and

(iv) (the result sort of S') $\restriction$ (the operation symbols of S) = the result sort of S.

Let us note that this predicate is reflexive.

Next we state four propositions:

(5) For all non empty many sorted signatures S, S', S'' such that $S \leq S'$ and $S' \leq S''$ holds $S \leq S''$.

(6) For all strict non empty many sorted signatures S, S' such that $S \leq S'$ and $S' \leq S$ holds $S = S'$.

(7) Let g be a function, and let a be an element of A, and let k be a natural number. If $1 \leq k$ and $k \leq n$, then $(a \dot{\longmapsto} g)(\pi_k(n \mapsto a)) = g$.

(8) Let I be a set, and let I_0 be a subset of I, and let A, B be many sorted sets indexed by I, and let F be a many sorted function from A into B, and let A_0, B_0 be many sorted sets indexed by I_0. Suppose $A_0 = A \restriction I_0$ and $B_0 = B \restriction I_0$. Then $F \restriction I_0$ is a many sorted function from A_0 into B_0.

Let S, S' be strict non void non empty many sorted signatures and let A be a non-empty strict algebra over S'. Let us assume that $S \leq S'$. The functor $(A \operatorname{over} S)$ yielding a non-empty strict algebra over S is defined by the conditions (Def.2).

(Def.2) (i) The sorts of $(A \operatorname{over} S)$ = (the sorts of A) $\restriction$ (the carrier of S), and

(ii) the characteristics of $(A \operatorname{over} S)$ = (the characteristics of A) $\restriction$ (the operation symbols of S).

We now state two propositions:

(9) For every strict non void non empty many sorted signature S and for every non-empty strict algebra A over S holds $A = (A \operatorname{over} S)$.

(10) For all U_1, U_2 such that U_1 and U_2 are similar holds $\operatorname{MSSign}(U_1) = \operatorname{MSSign}(U_2)$.

Let U_1, U_2 be universal algebras and let h be a function from U_1 into U_2. Let us assume that $\operatorname{MSSign}(U_1) = \operatorname{MSSign}(U_2)$. The functor $\operatorname{MSAlg}(h)$ yielding a many sorted function from $\operatorname{MSAlg}(U_1)$ into $(\operatorname{MSAlg}(U_2) \operatorname{over} \operatorname{MSSign}(U_1))$ is defined by:

(Def.3) $\operatorname{MSAlg}(h) = \{0\} \longmapsto h$.

The following propositions are true:

(11) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Let o be an operation symbol of $\operatorname{MSSign}(U_1)$. Then $(\operatorname{MSAlg}(h))$(the result sort of o) $= h$.

(12) For every operation symbol o of $\operatorname{MSSign}(U_1)$ holds $\operatorname{Den}(o, \operatorname{MSAlg}(U_1)) =$ (the characteristic of U_1)(o).

(13) For every operation symbol o of $\operatorname{MSSign}(U_1)$ holds $\operatorname{Den}(o, \operatorname{MSAlg}(U_1))$ is an operation of U_1.

(14) For every operation symbol o of MSSign(U_1) holds every element of Args(o, MSAlg(U_1)) is a finite sequence of elements of the carrier of U_1.

(15) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Let o be an operation symbol of MSSign(U_1) and let y be an element of Args(o, MSAlg(U_1)). Then $\mathrm{MSAlg}(h)\#y = h \cdot y$.

(16) If h is a homomorphism of U_1 into U_2, then MSAlg(h) is a homomorphism of MSAlg(U_1) into (MSAlg(U_2) over MSSign(U_1)).

(17) If U_1 and U_2 are similar, then MSAlg(h) is a many sorted set indexed by $\{0\}$.

(18) If h is an epimorphism of U_1 onto U_2, then MSAlg(h) is an epimorphism of MSAlg(U_1) onto (MSAlg(U_2) over MSSign(U_1)).

(19) If h is a monomorphism of U_1 into U_2, then MSAlg(h) is a monomorphism of MSAlg(U_1) into (MSAlg(U_2) over MSSign(U_1)).

(20) If h is an isomorphism of U_1 and U_2, then MSAlg(h) is an isomorphism of MSAlg(U_1) and (MSAlg(U_2) over MSSign(U_1)).

(21) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose MSAlg(h) is a homomorphism of MSAlg(U_1) into (MSAlg(U_2) over MSSign(U_1)). Then h is a homomorphism of U_1 into U_2.

(22) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose MSAlg(h) is an epimorphism of MSAlg(U_1) onto (MSAlg(U_2) over MSSign(U_1)). Then h is an epimorphism of U_1 onto U_2.

(23) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose MSAlg(h) is a monomorphism of MSAlg(U_1) into (MSAlg(U_2) over MSSign(U_1)). Then h is a monomorphism of U_1 into U_2.

(24) Given U_1, U_2, h. Suppose U_1 and U_2 are similar. Suppose MSAlg(h) is an isomorphism of MSAlg(U_1) and (MSAlg(U_2) over MSSign(U_1)). Then h is an isomorphism of U_1 and U_2.

(25) $\mathrm{MSAlg}(\mathrm{id}_{(\text{the carrier of } U_1)}) = \mathrm{id}_{(\text{the sorts of } \mathrm{MSAlg}(U_1))}$.

(26) Given U_1, U_2, U_3. Suppose U_1 and U_2 are similar and U_2 and U_3 are similar. Let h_1 be a function from U_1 into U_2 and let h_2 be a function from U_2 into U_3. Then $\mathrm{MSAlg}(h_2) \circ \mathrm{MSAlg}(h_1) = \mathrm{MSAlg}(h_2 \cdot h_1)$.

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