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Minimization of Finite State Machines [1]

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Summary. We have formalized deterministic finite state machines closely following the textbook [9], pp. 88–119 up to the minimization theorem. In places, we have changed the approach presented in the book as it turned out to be too specific and inconvenient. Our work also revealed several minor mistakes in the book. After defining a structure for an outputless finite state machine, we have derived the structures for the transition assigned output machine (Mealy) and state assigned output machine (Moore). The machines are then proved similar, in the sense that for any Mealy (Moore) machine there exists a Moore (Mealy) machine producing essentially the same response for the same input. The rest of work is then done for Mealy machines. Next, we define equivalence of machines, equivalence and k-equivalence of states, and characterize a process of constructing for a given Mealy machine, the machine equivalent to it in which no two states are equivalent. The final, minimization theorem states:

> **Theorem 4.5:** Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be reduced, connected finite-state machines. Then the state graphs of $\mathbf{M}_1$ and $\mathbf{M}_2$ are isomorphic if and only if $\mathbf{M}_1$ and $\mathbf{M}_2$ are equivalent.

and it is the last theorem in this article.

MML Identifier: `FSM_1`.

The papers [19], [23], [10], [2], [21], [13], [16], [8], [20], [18], [24], [5], [6], [7], [22], [3], [4], [1], [14], [17], [12], [11], and [15] provide the terminology and notation for this paper.

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1. Preliminaries

For simplicity we adopt the following convention: m, n, i, k will denote natural numbers, D will denote a non empty set, d will denote an element of D, and d_1, d_2 will denote finite sequences of elements of D.

Next we state several propositions:

(1) If $m < n$, then there exists a natural number p such that $n = m + p$ and $1 \leq p$.

(2) If $i \in \mathrm{Seg}\, n$, then $i + m \in \mathrm{Seg}(n + m)$.

(3) If $i > 0$ and $i + m \in \mathrm{Seg}(n + m)$, then $i \in \mathrm{Seg}\, n$ and $i \in \mathrm{Seg}(n + m)$.

(4) If $k < i$, then there exists a natural number j such that $j = i - k$ and $1 \leq j$.

(5) If $1 \leq \mathrm{len}\, d_1$, then there exist d, d_2 such that $d = d_1(1)$ and $d_1 = \langle d \rangle \frown d_2$.

(6) If $i \in \mathrm{dom}\, d_1$, then $(\langle d \rangle \frown d_1)(i + 1) = d_1(i)$.

For simplicity we adopt the following rules: S is a set, D_1, D_2 are non empty sets, f_1 is a function from S into D_1, and f_2 is a function from D_1 into D_2.

One can prove the following propositions:

(7) If f_1 is bijective and f_2 is bijective, then $f_2 \cdot f_1$ is bijective.

(8) For every set Y and for all equivalence relations E_1, E_2 of Y such that $\mathrm{Classes}\, E_1 = \mathrm{Classes}\, E_2$ holds $E_1 = E_2$.

(9) For every non empty set W holds every partition of W is non empty.

(10) For every finite set Z holds every partition of Z is finite.

Let W be a non empty set. Note that every partition of W is non empty.

Let Z be a finite set. Note that every partition of Z is finite.

Let X be a non empty finite set. Observe that there exists a partition of X which is non empty and finite.

We adopt the following rules: X, A will be non empty finite sets, P_1 will be a partition of X, and P_2, P_3 will be partitions of A.

We now state several propositions:

(11) For every set P_4 such that $P_4 \in P_1$ there exists an element x of X such that $x \in P_4$.

(12) $\mathrm{card}\, P_1 \leq \mathrm{card}\, X$.

(13) If P_2 is finer than P_3, then $\mathrm{card}\, P_3 \leq \mathrm{card}\, P_2$.

(14) If P_2 is finer than P_3, then for every element p_2 of P_3 there exists an element p_1 of P_2 such that $p_1 \subseteq p_2$.

(15) If P_2 is finer than P_3 and $\mathrm{card}\, P_2 = \mathrm{card}\, P_3$, then $P_2 = P_3$.

2. Definitions and Terminology

Let I_1 be a non empty set. We consider FSM over I_1 as systems
⟨ states, a Tran, a InitS ⟩,
where the states constitute a finite non empty set, the Tran is a function from $[\!:$ the states, $I_1 :\!]$ into the states, and the InitS is an element of the states.

Let I_1 be a non empty set and let f_3 be a FSM over I_1. A state of f_3 is an element of the states of f_3.

For simplicity we follow a convention: I_1, O_1 are non empty sets, f_3 is a FSM over I_1, s is an element of I_1, w, w_1, w_2 are finite sequences of elements of I_1, q, q', q_1, q_2 are states of f_3, and q_3 is a finite sequence of elements of the states of f_3.

Let us consider I_1, f_3, s, q. The functor s-succ(q) yielding a state of f_3 is defined by:

(Def.1) $s\text{-succ}(q) = (\text{the Tran of } f_3)(\langle q, s\rangle)$.

Let us consider I_1, f_3, q, w. The functor (q, w)-admissible yields a finite sequence of elements of the states of f_3 and is defined by the conditions (Def.2).

(Def.2) (i) $(q, w)\text{-admissible}(1) = q$,

(ii) $\text{len}((q, w)\text{-admissible}) = \text{len}\, w + 1$, and

(iii) for every i such that $1 \leq i$ and $i \leq \text{len}\, w$ there exists an element w_3 of I_1 and there exist states q_4, q_5 of f_3 such that $w_3 = w(i)$ and $q_4 = (q, w)\text{-admissible}(i)$ and $q_5 = (q, w)\text{-admissible}(i + 1)$ and $w_3\text{-succ}(q_4) = q_5$.

The following proposition is true

(16) $(q, \varepsilon_{(I_1)})\text{-admissible} = \langle q\rangle$.

Let us consider I_1, f_3, w, q_1, q_2. The predicate $q_1 \xrightarrow{w} q_2$ is defined as follows:

(Def.3) $(q_1, w)\text{-admissible}(\text{len}\, w + 1) = q_2$.

We now state the proposition

(17) $q \xrightarrow{\varepsilon_{(I_1)}} q$.

Let us consider I_1, f_3, w, q_3. We say that q_3 is admissible for w if and only if:

(Def.4) There exists q_1 such that $q_1 = q_3(1)$ and $(q_1, w)\text{-admissible} = q_3$.

We now state the proposition

(18) $\langle q\rangle$ is admissible for $\varepsilon_{(I_1)}$.

Let us consider I_1, f_3, q, w. The functor w-succ(q) yields a state of f_3 and is defined by:

(Def.5) $q \xrightarrow{w} w\text{-succ}(q)$.

One can prove the following propositions:

(19) $(q, w)\text{-admissible}(\text{len}((q, w)\text{-admissible})) = q'$ iff $q \xrightarrow{w} q'$.

(20) For every k such that $1 \leq k$ and $k \leq \text{len}\, w_1$ holds $(q_1, w_1 \frown w_2)\text{-admissible}(k) = (q_1, w_1)\text{-admissible}(k)$.

(21) If $q_1 \xrightarrow{w_1} q_2$, then $(q_1, w_1 \frown w_2)$-admissible$(\mathrm{len}\, w_1 + 1) = q_2$.

(22) If $q_1 \xrightarrow{w_1} q_2$, then for every k such that $1 \leq k$ and $k \leq \mathrm{len}\, w_2 + 1$ holds $(q_1, w_1 \frown w_2)$-admissible$(\mathrm{len}\, w_1 + k) = (q_2, w_2)$-admissible$(k)$.

(23) If $q_1 \xrightarrow{w_1} q_2$, then $(q_1, w_1 \frown w_2)$-admissible $= ((q_1, w_1)$-admissible$_{\restriction \mathrm{len}\, w_1 + 1}) \frown$ (q_2, w_2)-admissible.

3. Mealy and Moore Machines

Let I_1, O_1 be non empty sets. We consider Mealy-FSM over I_1, O_1 as extensions of FSM over I_1 as systems

⟨ states, a Tran, a OFun, a InitS ⟩,

where the states constitute a finite non empty set, the Tran is a function from $[\!:$ the states, $I_1 :\!]$ into the states, the OFun is a function from $[\!:$ the states, $I_1 :\!]$ into O_1, and the InitS is an element of the states. We introduce Moore-FSM over I_1, O_1 which are extensions of FSM over I_1 and are systems

⟨ states, a Tran, a OFun, a InitS ⟩,

where the states constitute a finite non empty set, the Tran is a function from $[\!:$ the states, $I_1 :\!]$ into the states, the OFun is a function from the states into O_1, and the InitS is an element of the states.

For simplicity we adopt the following convention: t_1, t_2, t_3, t_4 will denote Mealy-FSM over I_1, O_1, s_1 will denote a Moore-FSM over I_1, O_1, q_6 will denote a state of s_1, q, q_1, q_2, q_7, q_8, q_9, q_{10}, q'_1, q_{11}, q_{12}, q_{13} will denote states of t_1, q_{14}, q_{15} will denote states of t_2, and q_{21}, q_{22} will denote states of t_3.

Let us consider I_1, O_1, t_1, q_{11}, w. The functor (q_{11}, w)-response yields a finite sequence of elements of O_1 and is defined as follows:

(Def.6) $\mathrm{len}((q_{11}, w)$-response$) = \mathrm{len}\, w$ and for every i such that $i \in \mathrm{dom}\, w$ holds (q_{11}, w)-response$(i) =$ (the OFun of $t_1)(\langle\!\langle (q_{11}, w)$-admissible$(i)$, $w(i)\rangle\!\rangle)$.

The following proposition is true

(24) $(q_{11}, \varepsilon_{(I_1)})$-response $= \varepsilon_{(O_1)}$.

Let us consider I_1, O_1, s_1, q_6, w. The functor (q_6, w)-response yields a finite sequence of elements of O_1 and is defined by:

(Def.7) $\mathrm{len}((q_6, w)$-response$) = \mathrm{len}\, w + 1$ and for every i such that $i \in \mathrm{Seg}(\mathrm{len}\, w + 1)$ holds (q_6, w)-response$(i) =$ (the OFun of $s_1)((q_6, w)$-admissible$(i))$.

One can prove the following propositions:

(25) (q_6, w)-response$(1) =$ (the OFun of $s_1)(q_6)$.

(26) If $q_{12} \xrightarrow{w_1} q_{13}$, then $(q_{12}, w_1 \frown w_2)$-response $= (q_{12}, w_1)$-response $\frown$ (q_{13}, w_2)-response.

(27) If $q_{14} \xrightarrow{w_1} q_{15}$ and $q_{21} \xrightarrow{w_1} q_{22}$ and (q_{15}, w_2)-response $\neq$ (q_{22}, w_2)-response, then $(q_{14}, w_1 \frown w_2)$-response $\neq (q_{21}, w_1 \frown w_2)$-response.

In the sequel O_2 is a finite non empty set, t_5 is a Mealy-FSM over I_1, O_2, and s_2 is a Moore-FSM over I_1, O_2.

Let us consider I_1, O_1, t_1, s_1. We say that t_1 is similar to s_1 if and only if the condition (Def.8) is satisfied.

(Def.8) Let t_6 be a finite sequence of elements of I_1. Then $\langle$(the OFun of s_1)(the InitS of s_1)$\rangle \frown$(the InitS of t_1, t_6)-response = (the InitS of s_1, t_6)-response.

The following propositions are true:

(28) There exists t_1 which is similar to s_1.

(29) There exists s_2 such that t_5 is similar to s_2.

4. Equivalence of States and Machines

Let us consider I_1, O_1, t_2, t_3. We say that t_2 and t_3 are equivalent if and only if:

(Def.9) For every w holds (the InitS of t_2, w)-response = (the InitS of t_3, w)-response.

Let us observe that the predicate introduced above is reflexive and symmetric.

We now state the proposition

(30) If t_2 and t_3 are equivalent and t_3 and t_4 are equivalent, then t_2 and t_4 are equivalent.

Let us consider I_1, O_1, t_1, q_8, q_9. We say that q_8 and q_9 are equivalent if and only if:

(Def.10) For every w holds (q_8, w)-response = (q_9, w)-response.

We now state several propositions:

(31) q and q are equivalent.

(32) If q_1 and q_2 are equivalent, then q_2 and q_1 are equivalent.

(33) If q_1 and q_2 are equivalent and q_2 and q_7 are equivalent, then q_1 and q_7 are equivalent.

(34) If q'_1 = (the Tran of t_1)($\langle q_8, s\rangle$), then for every i such that $i \in \mathrm{Seg}(\mathrm{len}\, w+1)$ holds $(q_8, \langle s\rangle \frown w)$-admissible$(i+1) = (q'_1, w)$-admissible$(i)$.

(35) If q'_1 = (the Tran of t_1)($\langle q_8, s\rangle$), then $(q_8, \langle s\rangle \frown w)$-response = $\langle$(the OFun of t_1)($\langle q_8, s\rangle$)$\rangle \frown (q'_1, w)$-response.

(36) q_8 and q_9 are equivalent if and only if for every s holds (the OFun of t_1)($\langle q_8, s\rangle$) = (the OFun of t_1)($\langle q_9, s\rangle$) and (the Tran of t_1)($\langle q_8, s\rangle$) and (the Tran of t_1)($\langle q_9, s\rangle$) are equivalent.

(37) Suppose q_8 and q_9 are equivalent. Given w, i. Suppose $i \in \mathrm{dom}\, w$. Then there exist states q_{16}, q_{17} of t_1 such that $q_{16} = (q_8, w)$-admissible(i) and $q_{17} = (q_9, w)$-admissible(i) and q_{16} and q_{17} are equivalent.

Let us consider I_1, O_1, t_1, q_8, q_9, k. We say that q_8 and q_9 are k-equivalent if and only if:

(Def.11) For every w such that $\operatorname{len} w \leq k$ holds (q_8, w)-response $=$ (q_9, w)-response.

One can prove the following propositions:

(38) q_8 and q_8 are k-equivalent.

(39) If q_8 and q_9 are k-equivalent, then q_9 and q_8 are k-equivalent.

(40) If q_8 and q_9 are k-equivalent and q_9 and q_{10} are k-equivalent, then q_8 and q_{10} are k-equivalent.

(41) If q_8 and q_9 are equivalent, then q_8 and q_9 are k-equivalent.

(42) q_8 and q_9 are 0-equivalent.

(43) If q_8 and q_9 are $k+m$-equivalent, then q_8 and q_9 are k-equivalent.

(44) Suppose $1 \leq k$. Then q_8 and q_9 are k-equivalent if and only if the following conditions are satisfied:

(i) q_8 and q_9 are 1-equivalent, and

(ii) for every element s of I_1 and for every natural number k_1 such that $k_1 = k - 1$ holds (the Tran of t_1)($\langle q_8, s \rangle$) and (the Tran of t_1)($\langle q_9, s \rangle$) are k_1-equivalent.

Let us consider I_1, O_1, t_1, i. The functor i-EqS-Rel(t_1) yielding an equivalence relation of the states of t_1 is defined as follows:

(Def.12) For all q_8, q_9 holds $\langle q_8, q_9 \rangle \in i$-EqS-Rel($t_1$) iff q_8 and q_9 are i-equivalent.

Let us consider I_1, O_1, t_1, i. The functor i-EqS-Part(t_1) yields a non empty finite partition of the states of t_1 and is defined by:

(Def.13) i-EqS-Part(t_1) = Classes(i-EqS-Rel(t_1)).

One can prove the following propositions:

(45) $(k+1)$-EqS-Part(t_1) is finer than k-EqS-Part(t_1).

(46) If Classes(k-EqS-Rel(t_1)) = Classes($(k+1)$-EqS-Rel(t_1)), then for every m holds Classes($(k+m)$-EqS-Rel(t_1)) = Classes(k-EqS-Rel(t_1)).

(47) If k-EqS-Part(t_1) = $(k+1)$-EqS-Part(t_1), then for every m holds $(k+m)$-EqS-Part(t_1) = k-EqS-Part(t_1).

(48) If $(k+1)$-EqS-Part(t_1) $\neq$ k-EqS-Part(t_1), then for every i such that $i \leq k$ holds $(i+1)$-EqS-Part(t_1) $\neq$ i-EqS-Part(t_1).

(49) k-EqS-Part(t_1) = $(k+1)$-EqS-Part(t_1) or card(k-EqS-Part(t_1)) $<$ card($(k+1)$-EqS-Part(t_1)).

(50) $[q]_{0\text{-EqS-Rel}(t_1)}$ = the states of t_1.

(51) 0-EqS-Part(t_1) = {the states of t_1}.

(52) If $n + 1$ = card (the states of t_1), then $(n+1)$-EqS-Part(t_1) = n-EqS-Part(t_1).

Let us consider I_1, O_1, t_1. A partition of the states of t_1 is final if:

(Def.14) For all q_8, q_9 holds q_8 and q_9 are equivalent iff there exists an element X of it such that $q_8 \in X$ and $q_9 \in X$.

Next we state three propositions:

(53) If k-EqS-Part(t_1) is final, then $(k+1)$-EqS-Rel(t_1) = k-EqS-Rel(t_1).

(54) k-EqS-Part$(t_1) = (k+1)$-EqS-Part(t_1) iff k-EqS-Part(t_1) is final.

(55) If $n + 1 = \mathrm{card}$ (the states of t_1), then there exists a natural number k such that $k \leq n$ and k-EqS-Part(t_1) is final.

Let us consider I_1, O_1, t_1. The functor final-Partition(t_1) yields a partition of the states of t_1 and is defined by:

(Def.15) final-Partition(t_1) is final.

We now state the proposition

(56) If $n + 1 = \mathrm{card}$ (the states of t_1), then final-Partition(t_1) = n-EqS-Part(t_1).

5. The Reduction of a Mealy Machine

In the sequel r_1 will be a Mealy-FSM over I_1, O_1, q_{18} will be a state of r_1, and q_{19} will be an element of final-Partition(t_1).

Let us consider I_1, O_1, t_1, q_{19}, s. The functor (s, q_{19})-C-succ yields an element of final-Partition(t_1) and is defined by:

(Def.16) There exist q, n such that $q \in q_{19}$ and $n + 1 = \mathrm{card}$ (the states of t_1) and (s, q_{19})-C-succ $= [(\text{the Tran of } t_1)(\langle q, s\rangle)]_{n\text{-EqS-Rel}(t_1)}$.

Let us consider I_1, O_1, t_1, q_{19}, s. The functor (q_{19}, s)-C-response yielding an element of O_1 is defined by:

(Def.17) There exists q such that $q \in q_{19}$ and (q_{19}, s)-C-response = (the OFun of t_1)$(\langle q, s\rangle)$.

Let us consider I_1, O_1, t_1. The reduction of t_1 yielding a strict Mealy-FSM over I_1, O_1 is defined by the conditions (Def.18).

(Def.18) (i) The states of the reduction of t_1 = final-Partition(t_1),

(ii) for every state Q of the reduction of t_1 and for all s, q such that $q \in Q$ holds (the Tran of t_1)$(\langle q, s\rangle) \in$ (the Tran of the reduction of t_1)$(\langle Q, s\rangle)$ and (the OFun of t_1)$(\langle q, s\rangle)$ = (the OFun of the reduction of t_1)$(\langle Q, s\rangle)$, and

(iii) the InitS of $t_1 \in$ the InitS of the reduction of t_1.

The following two propositions are true:

(57) If r_1 = the reduction of t_1 and $q \in q_{18}$, then for every k such that $k \in \mathrm{Seg}(\mathrm{len}\, w + 1)$ holds (q, w)-admissible$(k) \in (q_{18}, w)$-admissible(k).

(58) t_1 and the reduction of t_1 are equivalent.

6. Machine Isomorphism

In the sequel q_{20}, q_{23} will denote states of r_1 and T_1 will denote a function from the states of t_2 into the states of t_3.

Let us consider I_1, O_1, t_2, t_3. We say that t_2 and t_3 are isomorphic if and only if the condition (Def.19) is satisfied.

(Def.19) There exists T_1 such that
(i) T_1 is bijective,
(ii) T_1(the InitS of t_2) = the InitS of t_3, and
(iii) for all q_{14}, s holds T_1((the Tran of t_2)($\langle q_{14}, s\rangle$)) = (the Tran of t_3)($\langle T_1(q_{14}), s\rangle$) and (the OFun of t_2)($\langle q_{14}, s\rangle$) = (the OFun of t_3)($\langle T_1(q_{14}), s\rangle$).

Let us observe that the predicate introduced above is reflexive and symmetric.

We now state four propositions:

(59) If t_2 and t_3 are isomorphic and t_3 and t_4 are isomorphic, then t_2 and t_4 are isomorphic.

(60) Suppose that for every state q of t_2 and for every s holds T_1((the Tran of t_2)($\langle q, s\rangle$)) = (the Tran of t_3)($\langle T_1(q), s\rangle$). Given k. If $1 \leq k$ and $k \leq \operatorname{len} w + 1$, then $T_1((q_{14}, w)\text{-admissible}(k)) = (T_1(q_{14}), w)\text{-admissible}(k)$.

(61) Suppose that
(i) T_1(the InitS of t_2) = the InitS of t_3, and
(ii) for every state q of t_2 and for every s holds T_1((the Tran of t_2)($\langle q, s\rangle$)) = (the Tran of t_3)($\langle T_1(q), s\rangle$) and (the OFun of t_2)($\langle q, s\rangle$) = (the OFun of t_3)($\langle T_1(q), s\rangle$).

Then q_{14} and q_{15} are equivalent if and only if $T_1(q_{14})$ and $T_1(q_{15})$ are equivalent.

(62) If r_1 = the reduction of t_1 and $q_{20} \neq q_{23}$, then q_{20} and q_{23} are not equivalent.

7. Reduced and Connected Machines

Let I_1, O_1 be non empty sets. A Mealy-FSM over I_1, O_1 is reduced if:

(Def.20) For all states q_8, q_9 of it such that $q_8 \neq q_9$ holds q_8 and q_9 are not equivalent.

One can prove the following proposition

(63) The reduction of t_1 is reduced.

Let us consider I_1, O_1. Note that there exists a Mealy-FSM over I_1, O_1 which is reduced.

In the sequel R_1 will denote a reduced Mealy-FSM over I_1, O_1.

Next we state two propositions:

(64) R_1 and the reduction of R_1 are isomorphic.

(65) t_1 is reduced iff there exists a Mealy-FSM M over I_1, O_1 such that t_1 and the reduction of M are isomorphic.

Let us consider I_1, O_1, t_1. A state of t_1 is accessible if:

(Def.21) There exists w such that the InitS of $t_1 \xrightarrow{w}$ it.

Let us consider I_1, O_1. A Mealy-FSM over I_1, O_1 is connected if:

(Def.22) Every state of it is accessible.

Let us consider I_1, O_1. One can check that there exists a Mealy-FSM over I_1, O_1 which is connected.

In the sequel C_1, C_2, C_3 will be connected Mealy-FSM over I_1, O_1.

We now state the proposition

(66) The reduction of C_1 is connected.

Let us consider I_1, O_1. Note that there exists a Mealy-FSM over I_1, O_1 which is connected and reduced.

Let us consider I_1, O_1, t_1. The functor accessible-States(t_1) yields a finite non empty set and is defined as follows:

(Def.23) accessible-States(t_1) $= \{q : q$ ranges over states of t_1, q is accessible$\}$.

The following propositions are true:

(67) accessible-States(t_1) $\subseteq$ the states of t_1 and for every q holds $q \in$ accessible-States(t_1) iff q is accessible.

(68) (The Tran of t_1) $\restriction$ [: accessible-States(t_1), I_1 :] is a function from [: accessible-States(t_1), I_1 :] into accessible-States(t_1).

(69) Let c_1 be a function from [: accessible-States(t_1), I_1 :] into accessible-States(t_1), and let c_2 be a function from [: accessible-States(t_1), I_1 :] into O_1, and let c_3 be an element of accessible-States(t_1). Suppose c_1 = (the Tran of t_1) $\restriction$ [: accessible-States(t_1), I_1 :] and c_2 = (the OFun of t_1) $\restriction$ [: accessible-States(t_1), I_1 :] and c_3 = the InitS of t_1. Then t_1 and Mealy-FSM$\langle$accessible-States$(t_1), c_1, c_2, c_3\rangle$ are equivalent.

(70) There exists C_1 such that
- (i) the Tran of C_1 = (the Tran of t_1) $\restriction$ [: accessible-States(t_1), I_1 :],
- (ii) the OFun of C_1 = (the OFun of t_1) $\restriction$ [: accessible-States(t_1), I_1 :],
- (iii) the InitS of C_1 = the InitS of t_1, and
- (iv) t_1 and C_1 are equivalent.

8. Machine Union

Let us consider I_1, O_1, t_2, t_3. The functor Mealy-U(t_2, t_3) yields a strict Mealy-FSM over I_1, O_1 and is defined by the conditions (Def.24).

(Def.24) (i) The states of Mealy-U(t_2, t_3) = (the states of t_2) $\cup$ (the states of t_3),
- (ii) the Tran of Mealy-U(t_2, t_3) = (the Tran of t_2) $+\cdot$ (the Tran of t_3),
- (iii) the OFun of Mealy-U(t_2, t_3) = (the OFun of t_2) $+\cdot$ (the OFun of t_3), and
- (iv) the InitS of Mealy-U(t_2, t_3) = the InitS of t_2.

One can prove the following propositions:

(71) If t_1 = Mealy-U(t_2, t_3) and (the states of t_2) $\cap$ (the states of t_3) $= \emptyset$ and $q_{14} = q$, then (q_{14}, w)-admissible $= (q, w)$-admissible.

(72) If $t_1 = \text{Mealy-U}(t_2, t_3)$ and (the states of t_2) $\cap$ (the states of t_3) $= \emptyset$ and $q_{14} = q$, then (q_{14}, w)-response $= (q, w)$-response.

(73) If $t_1 = \text{Mealy-U}(t_2, t_3)$ and (the states of t_2) $\cap$ (the states of t_3) $= \emptyset$ and $q_{21} = q$, then (q_{21}, w)-admissible $= (q, w)$-admissible.

(74) If $t_1 = \text{Mealy-U}(t_2, t_3)$ and (the states of t_2) $\cap$ (the states of t_3) $= \emptyset$ and $q_{21} = q$, then (q_{21}, w)-response $= (q, w)$-response.

In the sequel R_2, R_3 will be reduced Mealy-FSM over I_1, O_1.

The following proposition is true

(75) Suppose $t_1 = \text{Mealy-U}(R_2, R_3)$ and (the states of R_2) $\cap$ (the states of R_3) $= \emptyset$ and R_2 and R_3 are equivalent. Then there exists a state Q of the reduction of t_1 such that the InitS of $R_2 \in Q$ and the InitS of $R_3 \in Q$ and $Q =$ the InitS of the reduction of t_1.

For simplicity we follow a convention: C_4, C_5 will denote connected reduced Mealy-FSM over I_1, O_1, c_{11}, c_{12} will denote states of C_4, c_{21}, c_{22} will denote states of C_5, and q_{24}, q_{25} will denote states of t_1.

The following propositions are true:

(76) Suppose that
- (i) $c_{11} = q_{24}$,
- (ii) $c_{12} = q_{25}$,
- (iii) (the states of C_4) $\cap$ (the states of C_5) $= \emptyset$,
- (iv) C_4 and C_5 are equivalent,
- (v) $t_1 = \text{Mealy-U}(C_4, C_5)$, and
- (vi) c_{11} and c_{12} are not equivalent.

Then q_{24} and q_{25} are not equivalent.

(77) Suppose that
- (i) $c_{21} = q_{24}$,
- (ii) $c_{22} = q_{25}$,
- (iii) (the states of C_4) $\cap$ (the states of C_5) $= \emptyset$,
- (iv) C_4 and C_5 are equivalent,
- (v) $t_1 = \text{Mealy-U}(C_4, C_5)$, and
- (vi) c_{21} and c_{22} are not equivalent.

Then q_{24} and q_{25} are not equivalent.

(78) Suppose (the states of C_4) $\cap$ (the states of C_5) $= \emptyset$ and C_4 and C_5 are equivalent and $t_1 = \text{Mealy-U}(C_4, C_5)$. Let Q be a state of the reduction of t_1. Then there do not exist elements q_1, q_2 of Q such that $q_1 \in$ the states of C_4 and $q_2 \in$ the states of C_4 and $q_1 \neq q_2$.

(79) Suppose (the states of C_4) $\cap$ (the states of C_5) $= \emptyset$ and C_4 and C_5 are equivalent and $t_1 = \text{Mealy-U}(C_4, C_5)$. Let Q be a state of the reduction of t_1. Then there do not exist elements q_1, q_2 of Q such that $q_1 \in$ the states of C_5 and $q_2 \in$ the states of C_5 and $q_1 \neq q_2$.

(80) Suppose (the states of C_4) $\cap$ (the states of C_5) $= \emptyset$ and C_4 and C_5 are equivalent and $t_1 = \text{Mealy-U}(C_4, C_5)$. Let Q be a state of the reduction of t_1. Then there exist elements q_1, q_2 of Q such that $q_1 \in$ the states of

C_4 and $q_2 \in$ the states of C_5 and for every element q of Q holds $q = q_1$ or $q = q_2$.

9. The Minimization Theorem

We now state several propositions:

(81) There exist Mealy-FSM f_4, f_5 over I_1, O_1 such that (the states of f_4) $\cap$ (the states of f_5) $= \emptyset$ and f_4 and t_2 are isomorphic and f_5 and t_3 are isomorphic.

(82) If t_2 and t_3 are isomorphic, then t_2 and t_3 are equivalent.

(83) If (the states of C_4)$\cap$(the states of C_5) $= \emptyset$ and C_4 and C_5 are equivalent, then C_4 and C_5 are isomorphic.

(84) If C_2 and C_3 are equivalent, then the reduction of C_2 and the reduction of C_3 are isomorphic.

(85) Let M_1, M_2 be connected reduced Mealy-FSM over I_1, O_1. Then M_1 and M_2 are isomorphic if and only if M_1 and M_2 are equivalent.

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