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Quantales

Grzegorz Bancerek
Institute of Mathematics
Polish Academy of Sciences

Summary. The concepts of Girard quantales (see [10] and [15]) and Blikle nets (see [5]) are introduced.

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The notation and terminology used in this paper are introduced in the following papers: [12], [11], [14], [7], [8], [6], [9], [16], [2], [3], [1], [13], and [4].

Let X be a set and let Y be a subset of 2^X. Then $\bigcup Y$ is a subset of X.

In this article we present several logical schemes. The scheme *DenestFraenkel* concerns a non empty set $\mathcal{A}$, a non empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding arbitrary, a unary functor $\mathcal{G}$ yielding an element of $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(a) : a$ ranges over elements of $\mathcal{B}$, $a \in \{\mathcal{G}(b) : b$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[b]\}\} = \{\mathcal{F}(\mathcal{G}(a)) : a$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[a]\}$

for all values of the parameters.

The scheme *EmptyFraenkel* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding arbitrary, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(a) : a$ ranges over elements of $\mathcal{A}$, $\mathcal{P}[a]\} = \emptyset$

provided the following requirement is met:

- It is not true that there exists an element a of $\mathcal{A}$ such that $\mathcal{P}[a]$.

We now state two propositions:

(1) Let L_1, L_2 be non empty lattice structures. Suppose the lattice structure of L_1 = the lattice structure of L_2. Let a_1, b_1 be elements of L_1, and let a_2, b_2 be elements of L_2, and let X be a set. Suppose $a_1 = a_2$ and $b_1 = b_2$. Then $a_1 \sqcup b_1 = a_2 \sqcup b_2$ and $a_1 \sqcap b_1 = a_2 \sqcap b_2$ and $a_1 \sqsubseteq b_1$ iff $a_2 \sqsubseteq b_2$.

(2) Let L_1, L_2 be non empty lattice structures. Suppose the lattice structure of L_1 = the lattice structure of L_2. Let a be an element of L_1, and

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let b be an element of L_2, and let X be a set. If $a = b$, then $a \sqsubseteq X$ iff $b \sqsubseteq X$ and $a \sqsupseteq X$ iff $b \sqsupseteq X$.

Let L be a 1-sorted structure. A binary operation on L is a binary operation on the carrier of L. A unary operation on L is a unary operation on the carrier of L.

Let L be a non empty lattice structure and let X be a subset of L. We say that X is directed if and only if:

(Def.1) For every finite subset Y of X there exists an element x of L such that $\bigsqcup_L Y \sqsubseteq x$ and $x \in X$.

The following proposition is true

(3) For every non empty lattice structure L and for every subset X of L such that X is directed holds X is non empty.

We introduce quantale structures which are extensions of lattice structure and half group structure and are systems

⟨ a carrier, a join operation, a meet operation, a multiplication ⟩,

where the carrier is a set and the join operation, the meet operation, and the multiplication are binary operations on the carrier.

Let us mention that there exists a quantale structure which is non empty.

We consider quasinet structures as extensions of quantale structure and multiplicative loop structure as systems

⟨ a carrier, a join operation, a meet operation, a multiplication, a unity ⟩,

where the carrier is a set, the join operation, the meet operation, and the multiplication are binary operations on the carrier, and the unity is an element of the carrier.

Let us note that there exists a quasinet structure which is non empty.

A non empty half group structure has left-zero if:

(Def.2) There exists an element a of it such that for every element b of it holds $a \cdot b = a$.

A non empty half group structure has right-zero if:

(Def.3) There exists an element b of it such that for every element a of it holds $a \cdot b = b$.

A non empty half group structure has zero if:

(Def.4) It has left-zero and right-zero.

One can verify that every non empty half group structure which has zero has also left-zero and right-zero and every non empty half group structure which has left-zero and right-zero has also zero.

Let us note that there exists a non empty half group structure has zero.

A non empty quantale structure is right-distributive if:

(Def.5) For every element a of it and for every set X holds $a \otimes \bigsqcup_{\rm it} X = \bigsqcup_{\rm it}\{a \otimes b :$ b ranges over elements of it, $b \in X\}$.

A non empty quantale structure is left-distributive if:

(Def.6) For every element a of it and for every set X holds $\bigsqcup_{\rm it} X \otimes a = \bigsqcup_{\rm it}\{b \otimes a :$ b ranges over elements of it, $b \in X\}$.

A non empty quantale structure is $\otimes$-additive if:

(Def.7) For all elements a, b, c of it holds $(a \sqcup b) \otimes c = a \otimes c \sqcup b \otimes c$ and $c \otimes (a \sqcup b) = c \otimes a \sqcup c \otimes b$.

A non empty quantale structure is $\otimes$-continuous if it satisfies the condition (Def.8).

(Def.8) Let X_1, X_2 be subsets of it. Suppose X_1 is directed and X_2 is directed. Then $\bigsqcup X_1 \otimes \bigsqcup X_2 = \bigsqcup_{\rm it}\{a \otimes b : a$ ranges over elements of it, b ranges over elements of it, $a \in X_1 \ \wedge \ b \in X_2\}$.

The following proposition is true

(4) Let Q be a non empty quantale structure. Suppose the lattice structure of Q = the lattice of subsets of $\emptyset$. Then Q is associative commutative unital complete right-distributive left-distributive and lattice-like and has zero.

Let A be a non empty set and let b_1, b_2, b_3 be binary operations on A. Note that $\langle A, b_1, b_2, b_3 \rangle$ is non empty.

Let us observe that there exists a non empty quantale structure which is associative commutative unital left-distributive right-distributive complete and lattice-like and has zero.

The scheme *LUBFraenkelDistr* deals with a complete lattice-like non empty quantale structure $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and sets $\mathcal{B}$, $\mathcal{C}$, and states that:

$\bigsqcup_{\mathcal{A}}\{\bigsqcup_{\mathcal{A}}\{\mathcal{F}(a,b) : b$ ranges over elements of $\mathcal{A}$, $b \in \mathcal{C}\} : a$ ranges over elements of $\mathcal{A}$, $a \in \mathcal{B}\} = \bigsqcup_{\mathcal{A}}\{\mathcal{F}(a,b) : a$ ranges over elements of $\mathcal{A}$, b ranges over elements of $\mathcal{A}$, $a \in \mathcal{B} \ \wedge \ b \in \mathcal{C}\}$

for all values of the parameters.

In the sequel Q denotes a left-distributive right-distributive complete lattice-like non empty quantale structure and a, b, c denote elements of Q.

Next we state two propositions:

(5) For every Q and for all sets X, Y holds $\bigsqcup_Q X \otimes \bigsqcup_Q Y = \bigsqcup_Q\{a \otimes b : a \in X \ \wedge \ b \in Y\}$.

(6) $(a \sqcup b) \otimes c = a \otimes c \sqcup b \otimes c$ and $c \otimes (a \sqcup b) = c \otimes a \sqcup c \otimes b$.

Let A be a non empty set, let b_1, b_2, b_3 be binary operations on A, and let e be an element of A. Observe that $\langle A, b_1, b_2, b_3, e \rangle$ is non empty.

One can verify that there exists a non empty quasinet structure which is complete and lattice-like.

Let us note that every complete lattice-like non empty quasinet structure which is left-distributive and right-distributive is also $\otimes$-continuous and $\otimes$-additive.

Let us observe that there exists a non empty quasinet structure which is associative commutative well unital left-distributive right-distributive complete and lattice-like and has zero and left-zero.

A quantale is an associative left-distributive right-distributive complete lattice-like non empty quantale structure. A quasinet is a well unital associative $\otimes$-continuous $\otimes$-additive complete lattice-like non empty quasinet structure with left-zero.

A Blikle net is a non empty quasinet with zero.

The following proposition is true

(7) For every well unital non empty quasinet structure Q such that Q is a quantale holds Q is a Blikle net.

We adopt the following rules: Q will be a quantale and a, b, c, d, D will be elements of Q.

The following propositions are true:

(8) If $a \sqsubseteq b$, then $a \otimes c \sqsubseteq b \otimes c$ and $c \otimes a \sqsubseteq c \otimes b$.

(9) If $a \sqsubseteq b$ and $c \sqsubseteq d$, then $a \otimes c \sqsubseteq b \otimes d$.

Let A be a non empty set. A unary operation on A is idempotent if:

(Def.9) For every element a of A holds $\mathrm{it}(\mathrm{it}(a)) = \mathrm{it}(a)$.

Let L be a non empty lattice structure. A unary operation on L is inflationary if:

(Def.10) For every element p of L holds $p \sqsubseteq \mathrm{it}(p)$.

A unary operation on L is deflationary if:

(Def.11) For every element p of L holds $\mathrm{it}(p) \sqsubseteq p$.

A unary operation on L is monotone if:

(Def.12) For all elements p, q of L such that $p \sqsubseteq q$ holds $\mathrm{it}(p) \sqsubseteq \mathrm{it}(q)$.

A unary operation on L is $\bigsqcup$ -distributive if:

(Def.13) For every subset X of L holds $\mathrm{it}(\bigsqcup X) \sqsubseteq \bigsqcup_L \{\mathrm{it}(a) : a$ ranges over elements of L, $a \in X\}$.

We now state the proposition

(10) Let L be a complete lattice and let j be a unary operation on L. Suppose j is monotone. Then j is $\bigsqcup$ -distributive if and only if for every subset X of L holds $j(\bigsqcup X) = \bigsqcup_L \{j(a) : a$ ranges over elements of L, $a \in X\}$.

Let Q be a non empty quantale structure. A unary operation on Q is $\otimes$-monotone if:

(Def.14) For all elements a, b of Q holds $\mathrm{it}(a) \otimes \mathrm{it}(b) \sqsubseteq \mathrm{it}(a \otimes b)$.

Let Q be a non empty quantale structure and let a, b be elements of Q. The functor $a \to_r b$ yields an element of Q and is defined by:

(Def.15) $a \to_r b = \bigsqcup_Q \{c : c$ ranges over elements of Q, $c \otimes a \sqsubseteq b\}$.

The functor $a \to_l b$ yields an element of Q and is defined by:

(Def.16) $a \to_l b = \bigsqcup_Q \{c : c$ ranges over elements of Q, $a \otimes c \sqsubseteq b\}$.

One can prove the following propositions:

(11) $a \otimes b \sqsubseteq c$ iff $b \sqsubseteq a \to_l c$.

(12) $a \otimes b \sqsubseteq c$ iff $a \sqsubseteq b \to_r c$.

(13) For every quantale Q and for all elements s, a, b of Q such that $a \sqsubseteq b$ holds $b \to_r s \sqsubseteq a \to_r s$ and $b \to_l s \sqsubseteq a \to_l s$.

(14) Let Q be a quantale, and let s be an element of Q, and let j be a unary operation on Q. If for every element a of Q holds $j(a) = (a \to_r s) \to_r s$, then j is monotone.

Let Q be a non empty quantale structure. An element of Q is dualizing if:

(Def.17) For every element a of Q holds $(a \to_r \text{it}) \to_l \text{it} = a$ and $(a \to_l \text{it}) \to_r \text{it} = a$.

An element of Q is cyclic if:

(Def.18) For every element a of Q holds $a \to_r \text{it} = a \to_l \text{it}$.

We now state several propositions:

(15) c is cyclic iff for all a, b such that $a \otimes b \sqsubseteq c$ holds $b \otimes a \sqsubseteq c$.

(16) For every quantale Q and for all elements s, a of Q such that s is cyclic holds $a \sqsubseteq (a \to_r s) \to_r s$ and $a \sqsubseteq (a \to_l s) \to_l s$.

(17) For every quantale Q and for all elements s, a of Q such that s is cyclic holds $a \to_r s = ((a \to_r s) \to_r s) \to_r s$ and $a \to_l s = ((a \to_l s) \to_l s) \to_l s$.

(18) For every quantale Q and for all elements s, a, b of Q such that s is cyclic holds $((a \to_r s) \to_r s) \otimes ((b \to_r s) \to_r s) \sqsubseteq (a \otimes b \to_r s) \to_r s$.

(19) If D is dualizing, then Q is unital and $\mathbf{1}_{\text{the multiplication of } Q} = D \to_r D$ and $\mathbf{1}_{\text{the multiplication of } Q} = D \to_l D$.

(20) If a is dualizing, then $b \to_r c = b \otimes (c \to_l a) \to_r a$ and $b \to_l c = (c \to_r a) \otimes b \to_l a$.

We introduce Girard quantale structures which are extensions of quasinet structure and are systems

⟨ a carrier, a join operation, a meet operation, a multiplication, a unity, absurd ⟩,

where the carrier is a set, the join operation, the meet operation, and the multiplication are binary operations on the carrier, and the unity and the absurd constitute elements of the carrier.

One can check that there exists a Girard quantale structure which is non empty.

A non empty Girard quantale structure is cyclic if:

(Def.19) The absurd of it is cyclic.

A non empty Girard quantale structure is dualized if:

(Def.20) The absurd of it is dualizing.

The following proposition is true

(21) Let Q be a non empty Girard quantale structure. Suppose the lattice structure of Q = the lattice of subsets of $\emptyset$. Then Q is cyclic and dualized.

Let A be a non empty set, let b_1, b_2, b_3 be binary operations on A, and let e_1, e_2 be elements of A. One can verify that $\langle A, b_1, b_2, b_3, e_1, e_2 \rangle$ is non empty.

Let us note that there exists a non empty Girard quantale structure which is associative commutative well unital left-distributive right-distributive complete lattice-like cyclic dualized and strict.

A Girard quantale is an associative well unital left-distributive right-distributive complete lattice-like cyclic dualized non empty Girard quantale structure.

Let G be a Girard quantale structure. The functor $\bot_G$ yielding an element of G is defined as follows:

(Def.21) $\bot_G =$ the absurd of G.

Let G be a non empty Girard quantale structure. The functor $\top_G$ yielding an element of G is defined by:

(Def.22) $\top_G = \bot_G \to_r \bot_G$.

Let a be an element of G. The functor $\bot_a$ yielding an element of G is defined by:

(Def.23) $\bot_a = a \to_r \bot_G$.

Let G be a non empty Girard quantale structure. The functor Negation(G) yields a unary operation on G and is defined as follows:

(Def.24) For every element a of G holds $(\mathrm{Negation}(G))(a) = \bot_a$.

Let G be a non empty Girard quantale structure and let u be a unary operation on G. The functor $\bot_u$ yielding a unary operation on G is defined by:

(Def.25) $\bot_u = \mathrm{Negation}(G) \cdot u$.

Let G be a non empty Girard quantale structure and let o be a binary operation on G. The functor $\bot_o$ yields a binary operation on G and is defined as follows:

(Def.26) $\bot_o = \mathrm{Negation}(G) \cdot o$.

We adopt the following convention: Q denotes a Girard quantale, a, a_1, a_2, b, b_1, b_2, c denote elements of Q, and X denotes a set.

We now state several propositions:

(22) $\bot_{\bot_a} = a$.

(23) If $a \sqsubseteq b$, then $\bot_b \sqsubseteq \bot_a$.

(24) $\bot_{\bigsqcup_Q X} = \bigsqcap_Q \{\bot_a : a \in X\}$.

(25) $\bot_{\bigsqcap_Q X} = \bigsqcup_Q \{\bot_a : a \in X\}$.

(26) $\bot_{a \sqcup b} = \bot_a \sqcap \bot_b$ and $\bot_{a \sqcap b} = \bot_a \sqcup \bot_b$.

Let us consider Q, a, b. The functor $a \wp b$ yields an element of Q and is defined as follows:

(Def.27) $a \wp b = \bot_{\bot_a \otimes \bot_b}$.

We now state several propositions:

(27) $a \otimes \bigsqcup_Q X = \bigsqcup_Q \{a \otimes b : b \in X\}$ and $a \wp \bigsqcap_Q X = \bigsqcap_Q \{a \wp c : c \in X\}$.

(28) $\bigsqcup_Q X \otimes a = \bigsqcup_Q \{b \otimes a : b \in X\}$ and $\bigsqcap_Q X \wp a = \bigsqcap_Q \{c \wp a : c \in X\}$.

(29) $a \wp b \sqcap c = (a \wp b) \sqcap (a \wp c)$ and $b \sqcap c \wp a = (b \wp a) \sqcap (c \wp a)$.

(30) If $a_1 \sqsubseteq b_1$ and $a_2 \sqsubseteq b_2$, then $a_1 \wp a_2 \sqsubseteq b_1 \wp b_2$.

(31) $(a \wp b) \wp c = a \wp (b \wp c)$.

(32) $a \otimes \top_Q = a$ and $\top_Q \otimes a = a$.

(33) $a \wp \bot_Q = a$ and $\bot_Q \wp a = a$.

(34) Let Q be a quantale and let j be a unary operation on Q. Suppose j is monotone idempotent and $\bigsqcup$-distributive. Then there exists a complete lattice L such that the carrier of $L = \operatorname{rng} j$ and for every subset X of L holds $\bigsqcup X = j(\bigsqcup_Q X)$.

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